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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Some surjectivity results for operators of generalized monotone type via a topological degree

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ABSTRACT. We introduce a topological degree for a class of operators of generalized monotone type in reflexive Banach spaces, based on the recent Berkovits degree. Using the degree theory, we give some surjectivity results for operators of generalized monotone type in reflexive Banach spaces. In the Hilbert space case, this reduces to the celebrated Browder-Minty theorem for monotone operators.

1. INTRODUCTION

Topological degree theory is one of the most powerful and effective tools for solving nonlinear equations. Browder [4, 5] introduced a degree theory for monotone type operators in reflexive Banach spaces. Berkovits [1] constructed the Browder degree in a different way, using a compact embedding due to Browder and Ton [6] instead of the Galerkin method, so that the domain remains untouched in the approximation procedure. Berkovits [2] defined a topological degree for operators of generalized (S_+) type as an extension of the Leray-Schauder degree in order to solve abstract Hammerstein equation.

Let X be an infinite-dimensional reflexive Banach space. Many partial differential equations can be transformed to the following operator equation:

$$Au = b, \quad u \in X, \tag{1.1}$$

where A acts from X to the dual space X^*. At this time, the monotone nature of the operators plays an important role in investigating the solvability of equation (1.1). Earlier, Browder [3] and Minty [9] studied the existence of a solution of equation (1.1) in the case where A is monotone, demicontinuous, and coercive; see also [12].

In this paper, we consider operator equations of the form:

$$Fu = h, \quad u \in X, \tag{1.2}$$

where $F : X \to X$ is an operator of generalized monotone type. Our goal is to study the solvability of equation (1.2). To do this, we will first define a topological degree for a class of operators of generalized monotone type. Actually, this class may be wider than that in [2] and the degree extends the Berkovits degree in [2] in some sense. Using the degree theory, we will prove some surjectivity results for operators of generalized monotone type. In the case where X is a Hilbert space, our surjectivity result can be expressed as the famous Browder-Minty theorem mentioned above. The advantage of our approach is to use the degree theory, which is more elegant than the Galerkin method of proving the Browder-Minty theorem.

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This paper is organized as follows. In Section 2, we introduce a class of operators of generalized monotone type we shall be dealing with, and several examples show that the newly defined class is wider than the other known classes. Section 3 is devoted to developing a topological degree theory for this class. In Section 4, we establish some surjectivity results for operators of generalized monotone type in reflexive Banach spaces which include earlier results.

2. Operators of monotone type

Let X and Y be two real Banach spaces. Given a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

An operator $F : \Omega \subset X \to Y$ is said to be *bounded* if it takes any bounded set into a bounded set; F is said to be *locally bounded* if for each $u \in \Omega$ there exists a neighborhood U of u such that $F(U)$ is bounded. F is said to be *demicontinuous* if for each $u \in \Omega$ and any sequence (u_k) in Ω, $u_k \to u$ implies $Fu_k \rightharpoonup Fu$; F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

Let X be a real reflexive Banach space with dual space X^*. An operator $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+) if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$; F is said to be *pseudomonotone* if for any sequence (u_k) in Ω with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$, we have $\lim\langle Fu_n, u_n - u\rangle = 0$ and if $u \in \Omega$ then $Fu_n \rightharpoonup Fu$; F is said to be *quasimonotone* if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, we have $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.

Definition 2.1. Given a bounded operator $T : \Omega \subset X \to X^*$, an operator $F : \Omega \subset X \to X$ is said to be :

1. *T-monotone*, if $\langle Fu - Fv, Tu - Tv\rangle \geq 0$ for all $u, v \in \Omega$;
2. *of class* $(S_+)_T$, written $F \in (S_+)_T$, if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$, $Tu_n \rightharpoonup y$, and $\limsup_{n\to\infty} \langle Fu_n, Tu_n - y\rangle \leq 0$, we have $u_n \to u$;
3. *T-pseudomonotone*, written $F \in (PM)_T$, if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$, $Tu_n \rightharpoonup y$, and $\limsup_{n\to\infty} \langle Fu_n, Tu_n - y\rangle \leq 0$, we have $\lim_{n\to\infty} \langle Fu_n, Tu_n - y\rangle = 0$ and if $u \in \Omega$ and $Fu_k \rightharpoonup w$ for some subsequence (Fu_k) then $Fu = w$;
4. *T-quasimonotone*, written $F \in (QM)_T$, if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$, $Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty} \langle Fu_n, Tu_n - y\rangle \geq 0$.

We present a simple example of an operator F which is of class $(QM)_T$ but not of classes (QM) and $(S_+)_T$.

Example 2.1. Let E be a closed subspace of a real separable Hilbert space X. Let $P : X \to E$ and $Q : X \to E^\perp$ be the orthogonal projections, respectively. Then $F := P - Q$ is P-monotone and P-quasimonotone. If $\dim E^\perp = \infty$, then F is not quasimonotone and is not of class $(S_+)_P$.

We give another example of an operator F which is of class $(QM)_T$ but not of class $(S_+)_T$.

Example 2.2. Let X be an infinite-dimensional real separable Hilbert space. Suppose that $F : X \to X$ is compact and $T : X \to X$ is bounded. Then $F : X \to X$ is T-quasimonotone but not of class $(S_+)_T$.

The next example of an operator which is of class $(S_+)_T$ but not of class (S_+) is taken from [8, Example 2.4]; see also [2, Example 3.3].

Example 2.3. Let X be an infinite-dimensional real separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. If we define a linear operator $T : X \to X$ by setting

$$Te_n = e_n + (-1)^n e_{n+(-1)^n} \quad \text{for } n \in \mathbb{N},$$

then the operator $F := T \circ T$ is of class $(S_+)_T$ but not of class (S_+).

3. Degree theory

In this section, we introduce a topological degree theory for operators of generalized monotone type, based on the Berkovits degree in [2, 8].

For a real Banach space X with its dual X^*, the *duality operator* $J\colon X \to 2^{X^*}$ is defined by $Ju := \{u^* \in X^* : \langle u^*, u\rangle = \|u\|^2 \text{ and } \|u^*\| = \|u\|\}$ for $u \in X$. If X^* is strictly convex, then J is single-valued. See *e.g.*, [2, 4, 10, 12].

Proposition 3.1. *Let X be a real reflexive Banach space such that X and X^* are locally uniformly convex. Then the duality operator $J\colon X \to X^*$ has the following properties:*

(a) *J is bounded, continuous, of class (S_+), and coercive.*
(b) *J^{-1} is the duality operator from X^* to $X^{**} \simeq X$, and is of class (S_+).*
(c) *J is linear if and only if X is a Hilbert space.*

It is known in [11] that in every reflexive Banach space X, an equivalent norm can be introduced so that both X and X^* are locally uniformly convex.

Throughout this paper, let X be a real reflexive Banach space such that X and X^* are locally uniformly convex.

Let D_F denote the domain of F. For any $\Omega \subset D_F$, we consider some classes of operators:

$$\mathcal{F}_0(\Omega) := \{F : \Omega \subset X \to X^* |\, F \text{ is bounded, demicontinuous, and of class } (S_+)\},$$
$$\mathcal{F}_T(\Omega) := \{F : \Omega \subset X \to X |\, F \text{ is demicontinuous and of class } (S_+)_T\},$$
$$\mathcal{F}_{Q,T}(\Omega) := \{F : \Omega \subset X \to X |\, F \text{ is demicontinuous and } T\text{-quasimonotone}\}.$$

Here $T \in \mathcal{F}_0(\Omega)$ is called an *essential inner map* to F. To define a degree, let

$$\mathcal{F}_Q(X) := \{F \in \mathcal{F}_{Q,T}(\overline{G}) |\, G \in \mathcal{O}, T \in \mathcal{F}_0(\overline{G})\},$$

where $\mathcal{O}$ denotes the collection of all bounded open sets in X.

Berkovits [2] defined a topological degree for bounded demicontinuous operators of class $(S_+)_T$, based on the (S_+)-degree. For our aim, we need the $(S_+)_T$-degree stated in [8], where the boundedness condition was relaxed.

Theorem 3.1. *There is a unique degree function*

$$d : \{\, (F, G, h) \,|\, G \in \mathcal{O},\ T \in \mathcal{F}_0(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ h \notin F(\partial G) \,\} \to \mathbb{Z}$$

which has the following properties:

(a) (Existence) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
(b) (Additivity) *Let $F \in \mathcal{F}_T(\overline{G})$. If G_1 and G_2 are disjoint open subsets of G such that $h \notin F(\overline{G}\backslash(G_1 \cup G_2))$, then $d(F, G, h) = d(F, G_1, h) + d(F, G_2, h)$.*
(c) (Homotopy invariance) *Let $H : [0, 1] \times \overline{G} \to X$ be an affine homotopy given by $H(t, u) := (1 - t)Fu + tSu$, where $F, S \in \mathcal{F}_T(\overline{G})$ have a common continuous essential inner map $T \in \mathcal{F}_0(\overline{G})$. If $h : [0, 1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0, 1]$, then $d(H(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.*
(d) (Normalization) *For any $h \in G$, we have $d(I, G, h) = +1$.*

In the normalization, the identity operator I can be decomposed by $I = J^{-1} \circ J \in \mathcal{F}_J(\overline{G})$, in view of Proposition 3.1. For more details, we refer to [8, Lemma 2.3]. More generally, the decomposition $I = T^{-1} \circ T \in \mathcal{F}_T(\overline{G})$ was observed in [2, 7] in the following sense. See [7, Lemma 5.2] for the proof.

Lemma 3.1. *Assume that $T\colon X \to X^*$ is bijective, bounded, continuous, of class (S_+), and weakly coercive, that is, $\|T(u)\| \to \infty$ as $\|u\| \to \infty$. Then the inverse operator $T^{-1}\colon X^* \to X$ is bounded, continuous, of class (S_+), and weakly coercive. In this case, we have $I = T^{-1} \circ T \in \mathcal{F}_T(\overline{G})$ for all $G \in \mathcal{O}$.*

Let $\mathcal{H}_{\text{aff}}$ denote the class of affine homotopies H given by

$$H(t, \cdot) = (1 - t)F + tS,$$

where $F, S \in (QM)_T$, $G \in \mathcal{O}$, and $T \in \mathcal{F}_0(\overline{G})$.

In what follows, the common continuous essential inner map T will always be considered in the sense of Lemma 3.1 whenever the identity operator I appears in the (constant) homotopy.

For $H \in \mathcal{H}_{\text{aff}}$ and $\varepsilon > 0$, let $H_\varepsilon : [0, 1] \times \overline{G} \to X$ be defined by

$$H_\varepsilon(t, u) := H(t, u) + \varepsilon I u \qquad \text{for } (t, u) \in [0, 1] \times \overline{G}.$$

Then H_ε is a homotopy of class $(S_+)_T$.

Now we give an essential result to the construction of our degree and its properties.

Lemma 3.2. *Let G be any bounded open set in X and A a closed subset of $\overline{G}$. Let $H \in \mathcal{H}_{\text{aff}}$ and suppose that $h(t) \notin \overline{H([0, 1] \times A)}$ for all $t \in [0, 1]$, where $h : [0, 1] \to X$ is a continuous path in X. Then there exists a positive number ε_0 such that*

$$h(t) \notin H_\varepsilon(s, A) \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and all } s, t \in [0, 1].$$

Proof. Assume to the contrary that there are sequences (ε_n) in $(0, \infty)$ with $\varepsilon_n \to 0$, (s_n), (t_n) in $[0, 1]$, and (u_n) in A such that $H(s_n, u_n) + \varepsilon_n u_n = h(t_n)$ for all $n \in \mathbb{N}$. We may suppose that $t_n \to t$ for some $t \in [0, 1]$. Then $H(s_n, u_n) \to h(t)$, which contradicts the hypothesis. □

Lemma 3.3. *Let $F : \overline{G} \to X$ be a demicontinuous operator of class $(QM)_T$. If $h \notin \overline{F(\partial G)}$, then there is a positive number ε_0 such that $h \notin Fu + \varepsilon I u$ for all $\varepsilon \in (0, \varepsilon_0)$ and all $u \in \partial G$ and $d(F + \varepsilon I, G, h)$ is constant for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Applying Lemma 3.2 with $H(t, \cdot) = F$ for all $t \in [0, 1]$ and $A = \partial G$, there exists a positive number ε_0 such that

$$h \notin Fu + \varepsilon I u \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and all } u \in \partial G.$$

Let $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ with $\varepsilon_1 < \varepsilon_2$ be arbitrary. Define $H : [0, 1] \times \overline{G} \to X$ by

$$H(t, u) := (1 - t)(Fu + \varepsilon_1 I u) + t(Fu + \varepsilon_2 I u) = Fu + \varepsilon(t) I u \quad \text{for } (t, u) \in [0, 1] \times \overline{G},$$

where $\varepsilon(t) = (1 - t)\varepsilon_1 + t\varepsilon_2 \in [\varepsilon_1, \varepsilon_2] \subset (0, \varepsilon_0)$. Then H is an affine homotopy of class $(S_+)_T$ and $h \notin H(t, \partial G)$ for all $t \in [0, 1]$. Hence it follows from Theorem 3.1(c) that

$$d(F + \varepsilon_1 I, G, h) = d(F + \varepsilon_2 I, G, h).$$

Since $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ were arbitrary, $d(F + \varepsilon I, G, h)$ is constant for all $\varepsilon \in (0, \varepsilon_0)$. □

In view of Lemma 3.3, we can now define a topological degree for the class $\mathcal{F}_Q(X)$.

Definition 3.2. Let G be any bounded open set in X. If $F : \overline{G} \to X$ is a demicontinuous operator of class $(QM)_T$ and $h \notin \overline{F(\partial G)}$, then a degree function d_Q is defined as follows:

$$d_Q(F, G, h) := \lim_{\varepsilon \to 0} d(F + \varepsilon I, G, h).$$

The following theorem shows that this degree satisfies some of basic properties.

Theorem 3.2. *Let $F : \overline{G} \to X$ be a demicontinuous operator of class $(QM)_T$ such that $h \notin \overline{F(\partial G)}$, where G is any bounded open set in X, then the degree d_Q has the following properties:*

(a) (Existence) *If $d_Q(F, G, h) \neq 0$, then $h \in \overline{F(G)}$.*
(b) (Additivity) *If G_1 and G_2 are disjoint open subsets of G such that $h \notin \overline{F(\overline{G} \backslash (G_1 \cup G_2))}$, then we have $d_Q(F, G, h) = d_Q(F, G_1, h) + d_Q(F, G_2, h)$.*
(c) (Homotopy invariance) *Let $H \in \mathcal{H}_{\text{aff}}$ be an affine homotopy with a common continuous essential inner map. Suppose that $h : [0, 1] \to X$ is a continuous path in X such that $h(t) \notin \overline{H([0, 1] \times \partial G)}$ for all $t \in [0, 1]$. Then the value of $d_Q(H(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.*
(d) (Normalization) *For any $h \in G$, we have $d_Q(I, G, h) = +1$.*

Proof. Statements (a), (b), and (d) are deduced from Definition 3.2, Lemma 3.2, and Theorem 3.1.

(c) By Lemma 3.2, there exists a positive number ε_0 such that

$$h(t) \notin H_\varepsilon(t, \partial G) \qquad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and all } t \in [0, 1].$$

Theorem 3.1(c) implies that for every $\varepsilon \in (0, \varepsilon_0)$, $d(H_\varepsilon(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$. It suffices to show that for a fixed $t \in [0, 1]$, $d(H_\varepsilon(t, \cdot), G, h(t))$ is constant for all $\varepsilon \in (0, \varepsilon_0)$. Now fix $t \in [0, 1]$ and write $h = h(t)$. Let $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ with $\varepsilon_1 < \varepsilon_2$ be arbitrary but fixed. Then $\Pi : [0, 1] \times \overline{G} \to X$ given by

$$\Pi(\alpha, u) := H_{\varepsilon(\alpha)}(t, u) \quad \text{for } (\alpha, u) \in [0, 1] \times \overline{G},$$

where $\varepsilon(\alpha) = (1 - \alpha)\varepsilon_1 + \alpha\varepsilon_2 \in (0, \varepsilon_0)$, defines an affine $(S_+)_T$-homotopy. Since $h \notin \Pi(\alpha, \partial G)$ for all $\alpha \in [0, 1]$, it follows from Theorem 3.1(c) that

$$d(H_{\varepsilon_1}(t, \cdot), G, h(t)) = d(H_{\varepsilon_2}(t, \cdot), G, h(t)).$$

Since $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ were arbitrary, $d(H_\varepsilon(t, \cdot), G, h(t))$ is constant for all $\varepsilon \in (0, \varepsilon_0)$. We conclude that the value of $d_Q(H(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.

□

4. Surjectivity

In this section, we establish some surjectivity results for demicontinuous operators of class $(QM)_T$ in reflexive Banach spaces, based on the degree theory stated in Section 3. Moreover, we discuss the Dirichlet boundary value problem including the Laplacian.

When we are dealing with operators from the reflexive Banach space X to $X \simeq X^{**}$ rather than from X to X^*, we inevitably need the notion of J-coercivity.

Definition 4.3. We say that $F : D_F \subset X \to X$ *has the property* (B) if for every $v \in X$ there exists a neighborhood V of v such that $F^{-1}(V)$ is bounded; F is said to be *J-coercive* if $\langle Fu, Ju\rangle / \|Ju\| \to \infty$ as $\|u\| \to \infty$.

We first prove an existence result for the equation $Fu = 0$ concerning class $(QM)_T$ by using the degree theory.

Lemma 4.4. *Let $F : \overline{G} \subset X \to X$ be a demicontinuous operator of class $(S_+)_T$, where G is any bounded open set in X, and assume that there exists a point $\overline{u} \in G$ such that*

$$Fu \neq -\alpha(u - \overline{u}) \quad \text{for all } u \in \partial G \quad \text{and all } \alpha \geq 0. \tag{4.3}$$

Then $d(F, G, 0) = +1$.

Proof. It follows from condition (4.3) that $0 \notin F(\partial G)$ and therefore $d(F, G, 0)$ is defined. Let $\overline{I} : X \to X$ be defined by $\overline{I}u := Iu - \overline{u}$ for $u \in X$. We consider an affine homotopy $H : [0, 1] \times \overline{G} \to X$ given by

$$H(t, u) := (1 - t)Fu + t\overline{I}u \quad \text{for } (t, u) \in [0, 1] \times \overline{G}.$$

If $t = 1$, then $H(1, u) = u - \overline{u} \neq 0$ for $u \in \partial G$. If $t \in [0, 1)$, then using condition (4.3) gives $H(t, u) \neq 0$ for all $t \in [0, 1] \times \partial G$. Hence it follows from Theorem 3.1 that

$$d(F, G, 0) = d(\overline{I}, G, 0) = d(I, G, \overline{u}) = +1.$$

□

Theorem 4.3. *Let $F : \overline{G} \subset X \to X$ be a demicontinuous operator of class $(QM)_T$ such that $F(\overline{G})$ is closed in X and condition (4.3) holds. Then there exists at least one point $u_0 \in G$ such that $F(u_0) = 0$. Moreover, if $0 \notin \overline{F(\partial G)}$, then $d_Q(F, G, 0) = +1$.*

Proof. First, assume that $0 \in \overline{F(\partial G)}$. Since $F(\overline{G})$ is closed, we have $0 \in F(\overline{G})$. It follows from (4.3) that $0 \in F(G)$, as claimed. Next, assume that $0 \notin \overline{F(\partial G)}$, which implies $d_Q(F, G, 0)$ to be defined. For any $\varepsilon > 0$, we consider an operator $\overline{F}_\varepsilon : \overline{G} \to X$ given by

$$\overline{F}_\varepsilon u := Fu + \varepsilon\overline{I}u \quad \text{for } u \in \overline{G}.$$

Then we have $\overline{F}_\varepsilon \in (S_+)_T$ and $\overline{F}_\varepsilon u \neq -\beta(u - \overline{u})$ for all $u \in \partial G$ and all $\beta \geq 0$. Indeed, if $\overline{F}_\varepsilon u = -\beta(u - \overline{u})$ for some $u \in \partial G$ and $\beta \geq 0$, then $Fu = -(\beta + \varepsilon)(u - \overline{u})$, in contradiction to condition (4.3). By Lemma 4.4, we have $d(\overline{F}_\varepsilon, G, 0) = +1$ for any $\varepsilon > 0$. We now show that there exists a positive number ε_0 such that

$$(1 - t)(Fu + \varepsilon Iu) + t\overline{F}_\varepsilon u \neq 0 \quad \text{for all } (t, u) \in [0, 1] \times \partial G \text{ and all } \varepsilon \in (0, \varepsilon_0).$$

Suppose the contrary, then we can choose sequences (t_n) in $[0, 1]$, (u_n) in ∂G and (ε_n) in $(0, \infty)$ with $\varepsilon_n \to 0$ such that $(1 - t_n)(Fu_n + \varepsilon_n u_n) + t_n(Fu_n + \varepsilon_n(u_n - \overline{u})) = 0$. It follows that $Fu_n \to 0$ and hence $0 \in \overline{F(\partial G)}$, which contradicts our assumption that $0 \notin \overline{F(\partial G)}$. By Theorem 3.1(c), we get

$$d(F + \varepsilon I, G, 0) = d(\overline{F}_\varepsilon, G, 0) = +1 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

This implies that

$$d_Q(F, G, 0) = \lim_{\varepsilon \to 0} d(F + \varepsilon I, G, 0) = +1.$$

Thus, we have $0 \in \overline{F(G)}$. Since $0 \notin F(\partial G)$ and $F(\overline{G})$ is closed, we obtain that $F(u_0) = 0$ for some $u_0 \in G$. This completes the proof. □

Next, we give our main surjectivity result for demicontinuous operators of class $(QM)_T$.

Theorem 4.4. *Let $F : X \to X$ be a demicontinuous operator of class $(QM)_T$ such that F has the property (B). Suppose there exists a positive number R_0 such that*

$$Fu \neq -\alpha u \quad \text{for all } u \in X \text{ with } \|u\| \geq R_0 \text{ and all } \alpha \geq 0. \tag{4.4}$$

If $F(\overline{B_r})$ is closed for each $r \geq R_0$, where $B_r := \{u \in X : \|u\| < r\}$. Then the operator F is surjective.

Proof. Let y be any element of X. Then there are positive numbers k and $R \geq R_0$ such that

$$\|Fu - ty\| \geq k \quad \text{for all } t \in [0,1] \text{ and all } u \in X \ \text{ with } \|u\| \geq R. \tag{4.5}$$

Indeed, assume that there exist sequences (u_n) in X with $\|u_n\| \to \infty$ and (t_n) in $[0,1]$ such that $\|Fu_n - t_n y\| \to 0$. We may suppose that $t_n \to t_0$ for some $t_0 \in [0,1]$. Then $Fu_n \to t_0 y$. Since F has the property (B), the sequence (u_n) is bounded, which contradicts our assumption that $\|u_n\| \to \infty$. Applying Theorem 3.2(c) with $h(t) = ty$ for $t \in [0,1]$, we get $d_Q(F, B_R, y) = d_Q(F, B_R, 0)$. It follows from (4.4) that $Fu \neq -\alpha u$ for all $u \in \partial B_R$ and all $\alpha \geq 0$. Note that $0 \notin \overline{F(\partial B_R)}$. Otherwise, if we take a sequence (u_n) in ∂B_R such that $Fu_n \to 0$, then we have by (4.5) $\lim_{n\to\infty} \|Fu_n\| \geq k > 0$, a contradiction. By Theorem 4.3, we have $d_Q(F, B_R, 0) = +1$. Since $d_Q(F, B_R, y) = +1$, this implies that $y \in \overline{F(B_R)}$. As $F(\overline{B_R})$ is closed and $y \notin F(\partial B_R)$ by (4.5), we conclude that $y \in F(B_R) \subset F(X)$. This completes the proof. □

Corollary 4.1. *If $F : X \to X$ is demicontinuous, T-pseudomonotone, and J-coercive, then F is surjective.*

Proof. Since F is J-coercive, it is obvious that F has the property (B) and there is a positive number R such that $Fu \neq -\alpha u$ for all $\alpha \geq 0$ and all $u \in X$ with $\|u\| \geq R$. It remains to show that $F(\overline{B_r})$ is closed for each $r \geq R$. Let (u_n) be any sequence in $\overline{B_r}$ such that $Fu_n \to w$ for some $w \in X$. We may suppose that $u_n \rightharpoonup u$ in X and $Tu_n \rightharpoonup y$ in X^*. Then $\lim_{n\to\infty}\langle Fu_n, Tu_n - y\rangle = 0$. It follows from $F \in (PM)_T$ that $w = Fu \in F(\overline{B_r})$. Thus, $F(\overline{B_r})$ is closed. Since all of the conditions in Theorem 4.4 are satisfied, Theorem 4.4 implies that F is surjective. □

Corollary 4.2. *Let X be a real Hilbert space. If $F : X \to X$ is demicontinuous, pseudomonotone, and coercive, then it is surjective.*

Proof. In the case where X is a real Hilbert space, the duality map J is linear, bijective and $\|Ju\| = \|u\|$. Hence we may identify J with I and T with I. Apply Corollary 4.1. □

Finally, we illustrate Corollary 4.2 by an example taken from [12, Proposition 27.11].

Let G be a bounded domain in $\mathbb{R}^N$ with $N = 1, 2, 3$. We consider the following boundary value problem

$$\begin{cases} -\Delta u + \alpha \displaystyle\sum_{i=1}^{N} (\sin u) D_i u = f & \text{on } G, \\ u = 0 & \text{on } \partial G, \end{cases} \tag{4.6}$$

where α is a real number. Let $X = W_0^{1,2}(G)$ be the Sobolev space and $f \in X$ be given.

Definition 4.4. A point $u \in X$ is said to be a *weak solution* of (4.6) if

$$\int_G \sum_{i=1}^{N} D_i u D_i v dx + \alpha \int_G \sum_{i=1}^{N} (\sin u)(D_i u) v dx = \int_G f v dx \quad \text{for all } v \in X. \tag{4.7}$$

Theorem 4.5. *There is a positive number c such that problem* (4.6) *has a weak solution for all $\alpha \in [-c, c]$ and all $f \in X$.*

Proof. Define $A_1, A_2 : X \to X^*$ by setting

$$\langle A_1 u, v\rangle = \int_G \sum_{i=1}^{N} D_i u D_i v dx \qquad \text{and} \qquad \langle A_2 u, v\rangle = \alpha \int_G \sum_{i=1}^{N} (\sin u)(D_i u) v dx.$$

Then it is obvious that A_1 is linear, continuous and strongly monotone, and A_2 is strongly continuous. Equation (4.7) can be written as the operator equation

$$A_1u + A_2u = b, \quad \text{where } b(v) = \int_G fv dx. \tag{4.8}$$

By the Poincaré-Friedrichs inequality, there is a positive number c such that $A_1 + A_2$ is coercive if $|\alpha| < c$. Note that $A_1 + A_2$ is pseudomonotone and continuous. Since X is a real Hilbert space, we identify the dual space X^* with X. Let $\alpha \in [-c, c]$ be arbitrary. According to Corollary 4.2, equation (4.8) has a solution for every $f \in X$. We conclude that equation (4.6) has a weak solution for every $f \in X$. □

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