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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

On existence of solution of a class of quadratic-integral equations using contraction defined by simulation functions and measure of noncompactness

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ABSTRACT. In this paper we have introduced a new type of contraction condition using a class of simulation functions, in the sequel using the new contraction definition, involving measure of noncompactness; we establish few results on existence of fixed points of continuous functions defined on a subset of Banach space. This result also generalizes other related results obtained in Arab [Arab, R., *Some generalizations of Darbo fixed point theorem and its application*, Miskolc Math. Notes, **18** (2017), No. 2, 595–610], Banaś [Banaś, J. and Goebel, K., *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, 60 (1980)]. The obtained results are used in establishing existence theorems for a class of nonlinear quadratic equation (which generalizes several types of fractional-quadratic integral equations such as Abel's integral equation) defined on a closed and bounded subset of $\mathbb{R}$. The existence of solution is established with the aid of a measure of noncompactness defined on function space $C(I)$ introduced in [Banaś, J. and Olszowy, L., *Measures of Noncompactness related to monotonicity*, Comment. Math., **41** (2001), 13–23].

1. INTRODUCTION

Integral equation create a very important and significant part of the mathematical analysis and has various applications into real world problems. The technique of measures of noncompactness is often used in several branches of nonlinear analysis. Especially, that technique turns out to be very useful tool in the existence theory for several types of integral equations[2, 3, 6, 7, 8, 9, 17]. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations [10, 14]. In our investigations, we apply the method associated with the technique of measures of noncompactness to generalize the Darbo fixed point theorem [13] and to extend some recent results of Arab [4]. Moreover, as an application, we study the problem of existence of solutions for the following nonlinear integral equation

$$x(t) = a(t) + h(t, x(t) + F\Big(t, \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}k_1(f_1(t,s))x(s)\,ds, \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}k_2(f_2(t,s))x(s)\,ds\Big), \tag{1.1}$$

for $t \in I = [0, 1],\ 0 < \alpha, \beta \leq 1,\ m, n > 0$, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ and $H : C(I) \to C(I)$ is operator which satisfy special assumptions (see Section 4). Let us recall that the function $h = h(t, x)$ involved in Eq.(1.1) generates the superposition operator H, defined by $(Hx)(t) = h(t, x(t))$, where $x = x(t)$ is an arbitrary function defined on I, see [5]. We show that Eq. (1.1) has solutions in $C(I)$.

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The rest of the paper is organized as follows. In Section 2, we present some definitions and preliminary results about the concept of measure of noncompactness. In Section 3, using the new contraction of measure of noncompactness, some generalizations of Darbo fixed point theorem and to extend some recent results of Arab[4] are proved. Finally in Section 4, using the obtained results in Section 3, we investigate the problem of existence of solutions for the nonlinear integral equation (1.1).

We use the following definition of the measure of noncompactness given in [13].

Definition 1.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1^0) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,
(2^0) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
(3^0) $\mu(\bar{X}) = \mu(X)$,
(4^0) $\mu(ConvX) = \mu(X)$,
(5^0) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,
(6^0) If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n (n = 1, 2, ...)$ and if $\lim_{n \longrightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 1.1. (Schauder [1]) *Let C be a nonempty, bounded, closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

In the following we state a fixed-point theorem of Darbo type proved by Banaś and Goebel [13].

Theorem 1.2. *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $F : C \to C$ be a continuous mapping. Assume that there exist a constant $k \in [0,1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset of C. Then F has a fixed-point in C.*

2. New fixed point theorems on Banach spaces

The main result of the present paper is the following fixed point theorem which is a generalization of Darbo fixed point theorem (cf. Theorem1.2) and extend Theorem 4 of [4].

In the sequel, we fixed the set of functions by $F, \psi, \phi : [0, +\infty) \to [0, +\infty)$ such that

(a) F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;
(b) $\phi(t) < \psi(t)$ for each $t > 0$, $\phi(0) = \psi(0) = 0$;
(c) $\phi(t)$ and $\psi(t)$ are continuous functions;
(d) ψ is increasing.

Define $\mathbb{F} = \{F : F$ satisfies (a)$\}$, $\Psi = \{(\psi, \phi) : \psi$ and ϕ satisfy (b),(c) and (d)$\}$.

In the sequel, we denote by Θ the class of functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following conditions:

(θ_1) $\theta(t,s) < s - t$, for all $t, s > 0$;
(θ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \longrightarrow +\infty} t_n = l > 0$ and $\lim_{n \longrightarrow +\infty} s_n = s > 0$, then

$$\limsup_{n \longrightarrow +\infty} \theta(t_n, s_n) < s - l.$$

Now, we are in a position to state and prove our main result.

Theorem 2.3. *Let C be a nonempty subset of a Banach space E, $T : C \to C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions. Suppose that if for any $0 < a < b < \infty$ there exists $0 < \gamma(a,b) < 1$*

such that for all $X \subseteq C$*,*

$$a \leq F(\mu(X) + \varphi(\mu(X))) \leq b \tag{2.2}$$

$$\implies \theta[\psi(F(\mu(T(X)) + \varphi(\mu(TX)))), \gamma(a,b)\, \phi(F(\mu(X) + \varphi(\mu(X))))] \geq 0,$$

where μ *is an arbitrary measure of noncompactness,* $\theta \in \Theta$*,* $F \in \mathbb{F}$ *and* $(\psi, \phi) \in \Psi$*. Then* T *has at least one fixed point in* C*.*

Proof. Let $C_0 = C$, we construct a sequence $\{C_n\}$ such that $C_{n+1} = Conv(TC_n)$, for $n \geq 0$. $TC_0 = TC \subseteq C = C_0, C_1 = Conv(TC_0) \subseteq C = C_0$, therefore by continuing this process, we have

$$C_0 \supseteq C_1 \supseteq ... \supseteq C_n \supseteq C_{n+1} \supseteq ... .$$

If there exists a positive integer $N \in \mathbb{N}$ such that $\mu(C_N) + \varphi(\mu(C_N)) = 0$, i.e, $\mu(C_N) = 0$, then C_N is relatively compact. On the other hand, we have $T(C_N) \subseteq Conv(TC_N) = C_{N+1} \subseteq C_N$. Then Theorem 1.1 implies that T has a fixed point. So we assume that $0 < \mu(C_n) + \varphi(\mu(C_n))$ for all $n \in \mathbb{N}$ and by property of function F, we have

$$0 < F(\mu(C_n) + \varphi(\mu(C_n))), \ \forall n \geq 1. \tag{2.3}$$

Suppose that

$$F(\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))) < F(\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1}))), \tag{2.4}$$

for some $n_0 \in \mathbb{N}$. In addition, by (2.3) and (2.4), we have

$$\begin{aligned} 0 < a := F(\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))) &\leq F(\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))) \\ &< F(\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1}))) := b. \end{aligned}$$

By using (2.2) and (θ_1) with $X = C_{n_0}$, there exists $0 < \gamma(a,b) < 1$ such that

$$\begin{aligned} 0 &\leq \theta[\psi(F(\mu(TC_{n_0}) + \varphi(\mu(TC_{n_0})))), \gamma(a,b)\phi(F(\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))))] \\ &< \gamma(a,b)\psi(F(\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})))) - \psi(F(\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})))), \end{aligned}$$

which implies that $\gamma(a,b) > 1$, a contradiction. This implies that

$$\mu(C_{n+1}) + \varphi(\mu(C_{n+1})) \leq \mu(C_n) + \varphi(\mu(C_n)),$$

for all $n \in \mathbb{N}$, that is, the sequence $\{\mu(C_n) + \varphi(\mu(C_n))\}$ is non-increasing and nonnegative, we infer that

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = r. \tag{2.5}$$

Now, we show that $r = 0$. Suppose, to the contrary, that $r > 0$. Then

$$0 < a := F(r) \leq F(\mu(C_n) + \varphi(\mu(C_n))) \leq F(\mu(C_0) + \varphi(\mu(C_0))) =: b, \ for\ all\ n \geq 0.$$

By using (2.2) with $X = C_n$, there exists $0 < \gamma(a,b) < 1$ such that

$$\begin{aligned} 0 &\leq \theta[\psi(F(\mu(TC_n) + \varphi(\mu(TC_n)))), \gamma(a,b)\phi(F(\mu(C_n) + \varphi(\mu(C_n))))] \\ &= \theta[\psi(F(\mu(C_{n+1}) + \varphi(\mu(C_{n+1})))), \gamma(a,b)\phi(F(\mu(C_n) + \varphi(\mu(C_n))))]. \end{aligned}$$

The above inequality and the condition (θ_2), with $t_n = \psi(F(\mu(C_{n+1}) + \varphi(\mu(C_{n+1}))))$ and $s_n = \gamma(a,b)\phi(F(\mu(C_n) + \varphi(\mu(C_n))))$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \theta[\psi(F(\mu(C_{n+1}) + \varphi(\mu(C_{n+1})))), \gamma(a,b)\phi(F(\mu(C_n) + \varphi(\mu(C_n))))] \\ &< \gamma(a,b)\phi(F(r)) - \psi(F(r)) < \gamma(a,b)\psi(F(r)) - \psi(F(r)) < 0, \end{aligned}$$

which is a contradiction. Then we conclude that $r = 0$ and from (2.5), since $\varphi \geq 0$, we get

$$\lim_{n\to\infty} \mu(C_n) = 0 \ \ and \ \lim_{n\to\infty} \varphi(\mu(C_n)) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, ...$, it follows from (6^0) that $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore Theorem 1.1 completes the proof. □

We show the unifying power of simulation functions by applying Theorem 2.3 to deduce different kinds of contractive conditions in the existing literature.

Theorem 2.4. *Let C be a nonempty subset of a Banach space E, $T : C \to C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions. Suppose that if for any $0 < a < b < \infty$ there exists $0 < \lambda(a, b) < 1$ such that for all $X \subseteq C$,*

$$\begin{aligned} & a \leq F(\mu(X) + \varphi(\mu(X))) \leq b \\ \Longrightarrow \quad & \psi(F(\mu(T(X)) + \varphi(\mu(TX)))) \leq \lambda(a, b)\, \phi(F(\mu(X) + \varphi(\mu(X)))), \end{aligned}$$

where μ is an arbitrary measure of noncompactness, $F \in \mathbb{F}$ and $(\psi, \phi) \in \Psi$. Then T has at least one fixed point in C.

Proof. The result follows from Theorem 2.3, by taking as function $\theta(t, s) = k\, s - t$, for all $t, s \geq 0$ and $\lambda(a, b) = k\, \gamma(a, b)$. □

Now, the following fixed point theorem follows immediately from Theorem 2.4 is a generalization of [4].

Corollary 2.1. *Let C be a nonempty subset of a Banach space E, $T : C \to C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions. Suppose that if for any $0 < a < b < \infty$ there exists $0 < \lambda(a, b) < 1$ such that for all $X \subseteq C$,*

$$a \leq F(\mu(X) + \varphi(\mu(X))) \leq b \Longrightarrow F(\mu(T(X)) + \varphi(\mu(TX)) \leq \lambda(a, b)\, F(\mu(X) + \varphi(\mu(X))),$$

where μ is an arbitrary measure of noncompactness and $F \in \mathbb{F}$. Then T has at least one fixed point in C.

Proof. The result follows from Theorem 2.4, by taking as function $\theta(t, s) = k_1\, s - t$, for all $t, s \geq 0$, $0 < k_1 < 1$, $0 < k_2 < 1$, $\psi(t) = t$, $\phi(t) = k_2 t$ and $\lambda(a, b) = k_1 k_2\, \gamma(a, b)$. □

3. Application

Let $C(I) = C[0, 1]$ be the Banach space of all continuous functions on $I = [0, 1]$ equipped with the standard norm

$$||x|| = max\{|x(t)| : t \geq 0\}.$$

Next, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in this Section. This measure was introduced and studied in [11]. Let X be a fixed nonempty and bounded subset of $C(I)$. For $x \in X$ and $\epsilon \geq 0$, denote by $\omega(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, 1]$, i.e.

$$\omega(x, \epsilon) := sup\{|x(t) - x(s)| : t, s \in [0, 1], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega(X, \epsilon) := sup\{\omega(x, \epsilon) : x \in X\},\ \omega_0(X) := \lim_{\epsilon \to 0} \omega(X, \epsilon).$$

Define

$$i(x) := sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \leq s\},$$

and

$$i(X) := sup\{i(x) : x \in X\}.$$

Observe that all functions belonging to X are nondecreasing on I if and only if $i(X) = 0$. Now, let us define the function μ on the family $\mathfrak{M}_C(I)$ by the formula

$$\mu(X) := \omega_0(X) + i(X).$$

It can be shown [11] that the function μ is a measure of noncompactness in the space $C(I)$. Now, equation (1.1) will be investigated under the assumptions:

(a_1) $a : I \to \mathbb{R}_+$ is a continuous, nondecreasing and nonnegative function on I.

(a_2) $h : I \times \mathbb{R} \to \mathbb{R}$ is continuous function in t, x such that $h(I \times \mathbb{R}_+) \subseteq \mathbb{R}_+$ and there exists a continuous and nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and for each $t > 0$, $\varphi(t) < t$ such that

$$|h(t,x) - h(t,y)| \le \lambda\,\varphi(|x-y|) \tag{3.6}$$

for all $t \in I$ and all $x, y \in \mathbb{R}$ where $0 < \lambda < 1$. Additionally we assume that φ is superadditive i.e., $\varphi(t) + \varphi(s) \le \varphi(t+s)$ for all $t, s \in \mathbb{R}_+$.

(a_3) The superposition operator H generated by the function $h(t,x)$ satisfies for any nonnegative function x the condition

$$i(Hx) \le \lambda\,\varphi(i(x))$$

where φ is the same function as in (a_2).

(a_4) $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing function for each variable separately, such that $F(I \times \mathbb{R}_+ \times \mathbb{R}_+) \subseteq \mathbb{R}_+$ and satisfies the following condition

$$|F(t,x,y) - F(t,u,v)| \le |x-u| + |y-v|, \tag{3.7}$$

for all $t \in I$ and all $x, y, u, v \in \mathbb{R}$.

(a_5) $f_1, f_2 : I \times I \to \mathbb{R}$ are continuous and the functions $f_1(t,s)$ and $f_2(t,s)$ are nondecreasing for each variable t and s, separately.

(a_6) $k_1 : Imf_1 \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Imf_1.

(a_7) $k_2 : Imf_2 \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Imf_2.

(a_8) The inequality

$$M_1 + \lambda\,\varphi(r) + M_2 + \frac{||k_1||r}{\Gamma(\alpha+1)} + \frac{||k_2||r}{\Gamma(\beta+1)} + M_3 \le r, \tag{3.8}$$

has a positive solution r_0, where $M_1 = max\{|a(t)| : t \in I\}$, $M_2 = max\{|h(t,0)| : t \in I\}$ and $M_3 = max\{|F(t,0,0)| : t \in I\}$.

Theorem 3.5. *Under assumptions $(a_1) - (a_8)$, the equation (1.1) has at least one solution $x = x(t)$ which belongs to the space $C(I)$.*

Proof. Consider the operators G_1, G_2 and T defined on the space $C(I)$ by the formulas:

$$\begin{aligned}
(G_1x)(t) &= \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(f_1(t,s))x(s)\,ds,\\
(G_2x)(t) &= \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)}\, ns^{n-1}k_2(f_2(t,s))x(s)\,ds,\\
(Tx)(t) &= a(t) + h(t,x(t)) + F(t,(G_1x)(t),(G_2x)(t)).
\end{aligned}$$

Solving Eq.(1.1) is equivalent to finding a fixed point of the operator T defined on the space $C(I)$. By considering the conditions of theorem we infer that Tx is continuous on I

for any function $x \in C(I)$, i.e., T transforms the space $C(I)$ into itself. Moreover, in virtue of $(a_1) - (a_7)$ for each $t \in I$, we have

$$|(G_1x)(t)| \leq \frac{||k_2||||x||}{\Gamma(\beta+1)}, |(G_2x)(t)| \leq \frac{||k_2||||x||}{\Gamma(\beta+1)}. \tag{3.9}$$

So, we obtain

$$\begin{aligned}|(Tx)(t)| \leq & M_1 + [\lambda\,\varphi(||x||) + M_2] + |(G_1x)(t)| + |(G_2x)(t)| + |F(t,0,0,0)| \\ \leq & M_1 + \lambda\,\varphi(||x||) + M_2 + \frac{||k_1||||x||}{\Gamma(\alpha+1)} + \frac{||k_2||||x||}{\Gamma(\beta+1)} + M_3.\end{aligned}$$

Hence,

$$||Tx|| \leq M_1 + \lambda\,\varphi(||x||) + M_2 + \frac{||k_1||||x||}{\Gamma(\alpha+1)} + \frac{||k_2||||x||}{\Gamma(\beta+1)} + M_3.$$

Thus, if $||x|| \leq r_0$ we obtain from assumption (a_8) the estimate

$$||Tx|| \leq M_1 + \lambda\,\varphi(r_0) + M_2 + \frac{||k_1||r_0}{\Gamma(\alpha+1)} + \frac{||k_2||r_0}{\Gamma(\beta+1)} + M_3 \leq r_0.$$

Consequently, the operator T maps the ball $B_{r_0} \subset C(I)$ into itself. Next, we prove that the operator T is continuous on B_{r_0}. To do this, let $\{x_n\}$ be a sequence in B_{r_0} such that $x_n \to x$. We have to show that $Tx_n \to Tx$. In fact, for each $t \in I$, we have

$$|(G_1x_n)(t) - (G_1x)(t)| \leq \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f_1(t,s))||x_n(s) - x(s)|ds,$$

thus

$$||G_1x_n - G_1x|| \leq \frac{||k_1||}{\Gamma(\alpha+1)}||x_n - x||.$$

Similarly, we have

$$||G_2x_n - G_2x|| \leq \frac{||k_2||}{\Gamma(\beta+1)}||x_n - x||.$$

As,

$$|(Tx_n)(t) - (Tx)(t)| \leq \lambda\,\varphi(||x_n - x||) + ||G_1x_n - G_1x|| + ||G_2x_n - G_2x||.$$

It follows that

$$||Tx_n - Tx| \leq \lambda\,\varphi(||x_n - x||) + \frac{||k_1||}{\Gamma(\alpha+1)}||x_n - x|| + \frac{||k_2||}{\Gamma(\beta+1)}||x_n - x||.$$

This proves that T is continuous on B_{r_0}. The operator T on the subset $B^+_{r_0}$ of the ball B_{r_0} defined in the following way: $B^+_{r_0} = \{x \in B_{r_0} : x(t) \geq 0,\ for\ t \in I\}$. Obviously, the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. In view of our assumptions (a_2) and (a_4), if $x(t) \geq 0$ then $(Tx)(t) \geq 0$ for all $t \in I$. Thus T transforms the set $B^+_{r_0}$ into itself. Moreover, T is continuous on $B^+_{r_0}$. Let X be a nonempty subset of $B^+_{r_0}$. Fix $\epsilon > 0$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \epsilon$. Without loss of generality assume that $t_2 \geq t_1$. Then we get

$$\begin{aligned}|(G_1x)(t_2) - (G_1x)(t_1)| \leq & \int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f_1(t_2,s)) - k(f(t_1,s))||x(s)|ds \\ & + \int_{t_1}^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f_1(t_1,s))||x(s)|ds \\ & + \int_0^{t_1} \frac{|(t_2^m - s^m)^{\alpha-1} - (t_1^m - s^m)^{\alpha-1}|}{\Gamma(\alpha)} ms^{m-1}|k_1(f_1(t_1,s))||x(s)|ds.\end{aligned}$$

Therefore, if we denote

$$\omega_{k_i of_i}(\epsilon,.) = sup\{|k_i(f_i(t,s)) - k_i(f_i(t',s))| : t,t',s \in I \; and \; |t-t'| \leq \epsilon, \; i=1,2\},$$

then

$$|(G_1x)(t_2) - (G_1x)(t_1)| \leq \frac{||x||\omega_{k_1of_1}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2||x||||k_1||}{\Gamma(\alpha+1)}(t_2^m - t_1^m)^\alpha.$$

Similarly, one can show that

$$|(G_2x)(t_2) - (G_2x)(t_1)| \leq \frac{||x||\omega_{k_2of_2}(\epsilon,.)}{\Gamma(\beta+1)} + \frac{2||x||||k_2||}{\Gamma(\beta+1)}(t_2^n - t_1^n)^\beta.$$

So, by applying the mean value theorem on $[t_1,t_2]$, we get

$$|t_2^m - t_1^m|^\alpha \leq m^\alpha |t_2 - t_1|^\alpha, \; |t_2^n - t_1^n|^\alpha \leq n^\alpha |t_2 - t_1|^\alpha.$$

$$|(Tx)(t_2) - (Tx)(t_1)| \leq \omega(a,\epsilon) + \gamma_{r_0}(h,\epsilon) + \lambda\,\varphi(\omega(x,\epsilon)) +$$

$$+\frac{||x||\omega_{k_1of_1}(\epsilon,.)}{\Gamma(\alpha+1)} + +\frac{2||x||||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha + \frac{||x||\omega_{k_2of_2}(\epsilon,.)}{\Gamma(\beta+1)} + \frac{2||x||||k_2||}{\Gamma(\beta+1)}(n\epsilon)^\beta,$$

where we denoted

$$\gamma_{r_0}(h,\epsilon) = \sup\{|h(t,x) - h(t',x)| : t,t' \in I, x \in [0,r_0], |t-t'| \leq \epsilon\}.$$

Hence,

$$\omega(Tx,\epsilon) \leq \omega(a,\epsilon) + \gamma_{r_0}(h,\epsilon) + \lambda\,\varphi(\omega(x,\epsilon)) +$$
$$+ \frac{r_0\omega_{k_1of_1}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2r_0||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha + \frac{r_0\omega_{k_2of_2}(\epsilon,.)}{\Gamma(\beta+1)} + \frac{2r_0||k_2||}{\Gamma(\beta+1)}(n\epsilon)^\beta.$$

Thus, taking the supremum on X, we obtain

$$\omega(TX,\epsilon) \leq \omega(a,\epsilon) + \gamma_{r_0}(h,\epsilon) + \lambda\,\varphi(\omega(X,\epsilon)) +$$
$$+ \frac{r_0\omega_{k_1of_1}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2r_0||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^\alpha + \frac{r_0\omega_{k_2of_2}(\epsilon,.)}{\Gamma(\beta+1)} + \frac{2r_0||k_2||}{\Gamma(\beta+1)}(n\epsilon)^\beta.$$

From the uniform continuity of the functions k_1of and k_2og on the set $I \times I$ and h on the set $I \times [0,r_0]$ and the continuity of the function a on I, we have that $\omega_{k_1of_1}(\epsilon,.) \to 0$, $\omega_{k_2of_2}(\epsilon,.) \to 0$, $\gamma_{r_0}(h,\epsilon) \to 0$ and $\omega(a,\epsilon) \to 0$ as $\epsilon \to 0$. So, let $\epsilon \to 0$ to obtain

$$\omega_0(TX) \leq \lambda\,\varphi(\omega_0(X)). \tag{3.10}$$

Let $x \in X$ and $t_1,t_2 \in I$ with $t_1 < t_2$. Then

$$(G_1x)(t_2) - (G_1x)(t_1) \geq 0, \; (G_2x)(t_2) - (G_2x)(t_1) \geq 0$$

and so $|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)] \leq i(Hx)$. The above estimate implies that

$$i(Tx) \leq \lambda\,\varphi(i(x)) \Longrightarrow i(TX) \leq \lambda\,\varphi(i(X)). \tag{3.11}$$

From (3.10) and (3.11) and the definition of the measure of noncompactness μ, we obtain

$$\mu(TX) = \omega_0(TX) + i(TX) \leq \lambda\,\varphi(\omega_0(X) + i(X)) \leq \lambda\varphi(\mu(X)).$$

Now, taking into account the above inequality and the fact that $\lambda < 1$ and applying Corollary 2.1, we complete the proof. □

Remark 3.1. If $F(t,x,y)=x$, $h(t,x)\equiv 0$, $k_1(t)=t$ and $m=1$, then Eq. (1.1) becomes the well-known linear Abel integral equation of the second kind

$$x(t)=a(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(t,s)x(s)\,ds.$$

Abel integral equations have applications in many fields of physics and experimental sciences. For example, problems in mechanics, spectroscopy, scattering theory, elasticity theory and plasma physics often lead to such equations [16].

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