CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
34 (2018), No. 3, 379 - 390 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Fractional evolution equations with nonlocal conditions in partially ordered Banach space

HEMANT KUMAR NASHINE[1,2], HE YANG[3] and RAVI P. AGARWAL[1,4]

ABSTRACT. In the present work, we discuss the existence of mild solutions for the initial value problem of fractional evolution equation of the form

$$\text{(A)} \qquad \begin{cases} {}^C D_t^\sigma x(t) + Ax(t) = f(t, x(t)), & t \in J := [0, b], \\ x(0) = x_0 \in X, \end{cases}$$

where ${}^C D_t^\sigma$ denotes the Caputo fractional derivative of order $\sigma \in (0, 1)$, $-A : D(A) \subset X \to X$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ of uniformly bounded linear operator in X, $b > 0$ is a constant, f is a given functions. For this, we use the concept of measure of noncompactness in partially ordered Banach spaces whose positive cone K is normal, and establish some basic fixed point results under the said concepts. In addition, we relaxed the conditions of boundedness, closedness and convexity of the set at the expense that the operator is monotone and bounded. We also supply some new coupled fixed point results via MNC. To justify the result, we prove an illustrative example that rational of the abstract results for fractional parabolic equations.

1. INTRODUCTION AND PRELIMINARIES

It is well-known that the fixed point theorems are very important for proving the existence of solutions for some nonlinear differential and integral equations, see [1, 27, 28] and the references therein. The mixed arguments from different branches of mathematics are used in the research of fixed point theory. The first hybrid fixed point theorems in partially ordered sets is obtained by Ran and Reurings [29], where they extended the Banach contraction principle to partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [23, 24] extended the results in [26] to the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra, analysis and geometry and applied the abstract results to study the unique solution for a first order ordinary differential equation with periodic boundary conditions. Further improvements of the above mentioned results in partially ordered linear spaces can be found in [22] and the references therein.

In 1930, Kuratowski [19] opened up a new direction of research with the introduction of measure of noncompactness. The measure of noncompactness [19] combine with some algebraic arguments are useful for studying the mathematical formulations, particularly for solving the existence of solutions of some nonlinear problems under certain conditions. The Kuratowskii and Hausdorff measure of noncompactness [5, 6, 7, 17] in a metric space are well-known in the literature. However, as far as we know that the applications of measure of noncompactness in partially ordered normed linear spaces are seldom [10, 11, 12, 25].

Received: 21.09.2017. In revised form: 18.06.2018. Accepted: 15.07.2018
2010 *Mathematics Subject Classification.* 35F25, 45N05, 47H10.
Key words and phrases. *partially ordered Banach space, measure of noncompactness, fixed point, coupled fixed point, fractional integro-differential evolution equation.*
Corresponding author: Ravi P. Agarwal; Ravi.Agarwal@tamuk.edu

Throughout this paper, we assume that $(\mathcal{E}, \|\cdot\|)$ is an infinite dimensional Banach space. Let $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N} = \{1, 2, 3, \cdots\}$. If D is a subset of $\mathcal{E}$, we denote by $conv(D)$ and $\overline{conv}(D)$ the closure and the convex closure of D, respectively. Moreover, we denote by $\mathfrak{M}_{\mathcal{E}}$ the family of nonempty bounded subset of $\mathcal{E}$ and by $\mathfrak{N}_{\mathcal{E}}$ the family of all relatively compact subset of $\mathcal{E}$. We use the following definition of measure of noncompactness(MNC, for short) given in [7].

Definition 1.1. A mapping $\beta : \mathfrak{M}_{\mathcal{E}} \to \mathbb{R}^+$ is said to be a MNC in $\mathcal{E}$ if it satisfies the following conditions:

(1) The family $Ker\beta = \{D \in \mathfrak{M}_{\mathcal{E}} : \beta(D) = 0\}$ is nonempty and $Ker\beta \subset \mathfrak{N}_{\mathcal{E}}$;
(2) $\beta(C) \leq \beta(D)$ for any nonempty subsets $C, D \in \mathfrak{M}_{\mathcal{E}}$ with $C \subset D$;
(3) $\beta(conv(D)) = \beta(D)$ for any nonempty subset $D \in \mathfrak{M}_{\mathcal{E}}$;
(4) $\beta(\overline{conv}(D)) = \beta(D)$ for any nonempty subset $D \in \mathfrak{M}_{\mathcal{E}}$;
(5) $\beta(\lambda C + (1-\lambda)D) \leq \lambda\beta(C) + (1-\lambda)\beta(D)$ for any nonempty subsets $C, D \in \mathfrak{M}_{\mathcal{E}}$ and $\lambda \in [0, 1]$;
(6) If $\{D_n\}$ is a sequence of closed sets from $\mathfrak{M}_{\mathcal{E}}$ such that $D_{n+1} \subset D_n, n \geq 1$ and if $\lim_{n\to\infty} \beta(D_n) = 0$, the intersection set $D_\infty = \bigcap_{n=1}^{\infty} D_n$ is nonempty.

It follows from Definition 1.1 (6) that D_∞ is a member of the family $Ker\beta$. Since $\beta(D_\infty) \leq \beta(D_n)$ for any n, we can deduce that $\beta(D_\infty) = 0$. This implies that $D_\infty \in Ker\beta$.

Definition 1.2. [15] Let Ψ be a set of functions $\chi : \mathbb{R}^+ \to [0, 1)$ satisfying

$$\chi(t_n) \to 1 \Rightarrow t_n \to 0.$$

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and Banach's fixed point theorem. The following fixed point theorem of Darbo type proved by Banaś and Goebel in [7].

Lemma 1.1. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{Q} : D \to D$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\beta(\mathcal{Q}(S)) \leq k\beta(S),$$

for any nonempty subset $S \subset D$. Then $\mathcal{Q}$ has at least one fixed point in D.

Recently, Lemma 1.1 has been extended by Aghajani et al. in [4]. They obtained the following two fixed point theorems.

Lemma 1.2. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $\chi \in \Psi$, $\mathcal{Q} : D \to D$ be a continuous mapping satisfying*

$$\beta(\mathcal{Q}(S)) \leq \chi(\beta(S))\beta(S),$$

for any nonempty subset $S \subset D$, where β is MNC. Then $\mathcal{Q}$ has at least one fixed point in D.

Lemma 1.3. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{Q} : D \to D$ be a continuous mapping satisfying*

$$\beta(\mathcal{Q}(S)) \leq \phi(\beta(S)),$$

for any nonempty subset $S \subset D$, where β is MNC, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and upper semi-continuous function such that $\phi(t) < t$ for all $t > 0$. Then $\mathcal{Q}$ has at least one fixed point in D.

Clearly, if we take $\chi(t) \equiv k \in [0, 1)$ for any $t \in \mathbb{R}^+$ in Lemma 1.2, or take $\phi(t) = kt$ for any $t \in \mathbb{R}^+$ and $k \in [0, 1)$ in Lemma 1.3, the Lemma 1.2 and Lemma 1.3 degenerate into Lemma 1.1.

Very recently, Dhage et al. [13] defined new control function Φ and proved following result and generalized Theorem 1.4 due to Aghajani et. al. [3].

Definition 1.3. [13] Let Υ be a set of continuous functions $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t_n) \to 0 \Rightarrow t_n \to 0.$$

For shorten the text, we denote $\phi \in \Phi$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that and ϕ is lower semicontinuous on $\mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$.

Lemma 1.4. *Let D be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{Q} : D \to D$ be a continuous operator such that*

$$\beta(\mathcal{Q}S) \leq \beta(S) - \phi(\zeta(\beta(S)))$$

for every nonempty subset S of D and each $\zeta \in \Upsilon$, where β is MNC and $\phi \in \Phi$. Then $\mathcal{Q}$ has at least one fixed point in D.

In the present paper, we will extend the results in Lemma 1.4 into partially ordered Banach spaces. By doing this, we also improve and generalize the work mentioned in [10, 11, 12, 25]. For this, we first define a notion of measure of noncompactness in partially ordered Banach spaces. We use this notion to prove some fixed point theorems for $\beta - \phi-$contraction condition (2.1) in partially ordered Banach spaces whose positive cone K is normal, and then to prove some coupled fixed point theorems in partially ordered Banach spaces. To achieve this result,we relaxed the conditions of boundedness, closedness and convexity of the set at the expense that the operator is monotone and bounded. Further, we apply the obtained fixed point theorems to prove the existence of mild solutions for fractional integro-differential evolution equations with nonlocal conditions. At the end, an example is given to illustrate the rationality of the abstract results for fractional parabolic equations.

2. Fixed point theorems

Let $\mathcal{E}$ be a Banach space with the norm $\|\cdot\|$ whose positive cone is defined by $K = \{x \in \mathcal{E} : x \geq 0\}$. Then $(\mathcal{E}, \|\cdot\|)$ is now a partially ordered Banach space with the order relation $\sqsubseteq$ induced by cone K.

Now, we establish the fixed point theorems via MNC in partially ordered Banach spaces.

Theorem 2.1. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space, whose positive cone K is normal. Suppose that $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$ is a continuous, nondecreasing and bounded mapping satisfying the following contraction:*

$$\beta(\mathcal{Q}(\mathcal{C})) \leq \beta(\mathcal{C}) - \phi(\zeta(\beta(\mathcal{C}))) \tag{2.1}$$

for all bounded subset $\mathcal{C}$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$, $\phi \in \Phi$ and $\zeta \in \Upsilon$.

If there exists an element $u_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{Q}u_0$, then $\mathcal{Q}$ has a fixed point u^* and the sequence $\{\mathcal{Q}^n x_0\}$ of successive iterations converges monotonically to u^*.

Proof. Starting from the given $u_0 \in \mathcal{E}$, we define a sequence $\{u_n\}$ of points in $\mathcal{E}$ by

$$u_{n+1} = \mathcal{Q}u_n, n \in \mathbb{N}^* := \mathbb{N} \cup \{0\}. \tag{2.2}$$

Since $\mathcal{Q}$ is nondecreasing and $u_0 \sqsubseteq \mathcal{Q}u_0$, we have

$$u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \ldots \sqsubseteq u_n \sqsubseteq \ldots \tag{2.3}$$

Denote $\mathcal{C}_n = \overline{conv}\{u_n, u_{n+1}, \ldots\}$ for $n \in \mathbb{N}^*$. By (2.2) and (2.3), each $\mathcal{C}_n$ is a bounded and closed subset in $\mathcal{E}$ and

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \ldots \supset \mathcal{C}_n \supset \ldots. \tag{2.4}$$

Following (2.1), we obtain

$$\beta(\mathcal{C}_{n+1}) = \beta(\overline{conv}\mathcal{Q}\mathcal{C}_n) = \beta(\mathcal{Q}\mathcal{C}_n) \leq \beta(\mathcal{C}_n) - \phi(\zeta(\beta(\mathcal{C}_n))). \tag{2.5}$$

By the construction of $\mathcal{C}_n$, it is clear that $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ and so by the Definition 1.1, the sequence $\{\beta(\mathcal{C}_n)\}$ is nonincreasing and nonnegative. Thus, there exists $r \geq 0$ such that $\beta(\mathcal{C}_n) \to r$ when $n \to \infty$. Continuity of β implies that $\zeta(\beta(\mathcal{C}_n)) \to \zeta(r)$ as $n \to \infty$. Now, in view of (2.5) we obtain

$$\limsup_{n\to\infty} \beta(\mathcal{C}_{n+1}) \leq \limsup_{n\to\infty} \beta(\mathcal{C}_n) - \liminf_{n\to\infty} \phi(\zeta(\beta(\mathcal{C}_n))).$$

This yields $r \leq r - \liminf_{n\to\infty} \phi(\zeta(\beta(\mathcal{C}_n)))$. Since ϕ is nondecreasing, we obtain

$$\phi(\zeta(r)) \leq \liminf_{n\to\infty} \phi(\zeta(\beta(\mathcal{C}_n))) = 0.$$

By the virtue of ϕ, we deduce that

$$\zeta(\beta(\mathcal{C}_n)) \to 0 \text{ as } n \to \infty.$$

Since $\zeta \in \Upsilon$, we get $r = 0$, and hence

$$\beta(\mathcal{C}_n) \to 0, \text{ as } n \to \infty. \tag{2.6}$$

Since $\mathcal{C}_n \subset \mathcal{C}_{n-1}$, we have

$$\overline{\mathcal{C}}_\infty = \bigcap_{n=1}^{\infty} \mathcal{C}_n \neq \emptyset \text{ and } \mathcal{C}_\infty \in Ker\beta.$$

Hence, for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\beta(\mathcal{C}_n) < \epsilon \;\forall\, n \geq n_0.$$

This concluded that $\overline{\mathcal{C}}_{n_0}$ and consequently $\mathcal{C}_0$ is a compact chain in $\mathcal{E}$. Hence, $\{u_n\}$ has a convergent subsequence. Applying the monotone property of $\mathcal{Q}$ and the normality of cone K, the whole sequence $\{u_n\} = \{\mathcal{Q}^n u_0\}$ converges monotonically to a point, say $u^* \in \mathcal{C}_0$. Finally, from the continuity of $\mathcal{Q}$, we get

$$\mathcal{Q}u^* = \mathcal{Q}(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} u_{n+1} = u^*.$$

□

Remark 2.1. Taking various concrete functions $\phi \in \Phi$ and $\zeta \in \Upsilon$ in the condition (2.1) of Theorem 2.1, we can get various classes of μ-set contractive conditions. We state just a few examples which include results existing in the literature:

(A1) Taking $\zeta(t) = t$ $(t > 0)$, we have condition

$$\beta(\mathcal{Q}(\mathcal{C})) \leq \beta(\mathcal{C}) - \phi(\beta(\mathcal{C}))$$

(A2) Taking $\phi(t) = (1-k)t$ $(t > 0)$, $k \in [0, 1)$, we have condition

$$\beta(\mathcal{Q}(\mathcal{C})) \leq k\zeta(\beta(\mathcal{C})).$$

(A3) Taking $\zeta(t) = t$ $(t > 0)$ and $\phi(t) = (1-k)t$ $(t > 0)$, $k \in [0, 1)$, we have Darbo's μ-set contraction condition (see Lemma 1.1)

$$\beta(\mathcal{Q}(\mathcal{C})) \leq k\beta(\mathcal{C}).$$

Proposition 2.1. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space, whose positive cone K is normal. Suppose that $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$ is a continuous, nondecreasing and bounded mapping satisfying the following contraction:*

$$diam(\mathcal{Q}(\mathcal{C})) \leq diam(\mathcal{C}) - \phi(\zeta(diam(\mathcal{C}))) \tag{2.7}$$

for all bounded subset $\mathcal{C}$ in $\mathcal{E}$, where $\phi \in \Phi$, $\zeta \in \Upsilon$, and $diam(\mathcal{C})$ denote the diameter of $\mathcal{C}$.

If there exists an element $u_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{Q}u_0$, then $\mathcal{Q}$ has a fixed point u^* and the sequence $\{\mathcal{Q}^n x_0\}$ of successive iterations converges monotonically to u^*.

Proof. Following Theorem 2.1 and argument of Proposition 3.2 [14], $\mathcal{Q}$ has a fixed point in $\mathcal{C}$.
To prove uniqueness, we suppose that there exist two distinct fixed points $\zeta, \xi \in \mathcal{C}$, then we may define the set $\Lambda := \{\zeta, \xi\}$. In this case $diam(\Lambda) = diam(\mathcal{Q}(\Lambda)) = \|\xi - \zeta\| > 0$. Then using (2.7), we get

$$diam(\mathcal{Q}(\Lambda)) \leq diam(\Lambda) - \phi(\zeta(diam(\Lambda))),$$

and $\phi(\zeta(diam(\Lambda))) \leq 0$, a contradiction to the virtue of ϕ and hence $\xi = \zeta$. □

3. Coupled fixed point theorems

In this section, we prove some coupled fixed point theorems. We begin our discussion by recalling some definitions and notions.

Definition 3.4. [16] An element $(u^*, v^*) \in \mathcal{E}^2$ is called a coupled fixed point of a mapping $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ if $\mathcal{G}(u^*, v^*) = u^*$ and $\mathcal{G}(v^*, u^*) = v^*$.

Definition 3.5. Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space and let $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ be a mapping. A mapping $\mathcal{G}$ is said to have the monotone property if $\mathcal{G}(u, v)$ is monotone nondecreasing in both variables u and v, that is, for any $u, v \in \mathcal{E}$,

$$u_1, u_2 \in \mathcal{E},\ u_1 \sqsubseteq u_2 \Rightarrow \mathcal{G}(u_1, v) \sqsubseteq \mathcal{G}(u_2, v)$$

and

$$v_1, v_2 \in \mathcal{E},\ v_1 \sqsubseteq v_2 \Rightarrow \mathcal{G}(u, v_1) \sqsubseteq \mathcal{G}(u, v_2).$$

Lemma 3.5. [5] *Suppose that $\beta_1, \beta_2, \ldots, \beta_n$ are measures of noncompactness in Banach spaces $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$, respectively. Moreover assume that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, 3, \ldots, n$. Then*

$$\beta(\mathcal{C}) = F(\beta_1(\mathcal{C}_1), \beta_2(\mathcal{C}_2), \ldots, \beta_n(\mathcal{C}_n)),$$

defines a measure of noncompactness in $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \times \ldots \times \mathcal{E}_n$ where $\mathcal{C}_i$ denotes the natural projection of $\mathcal{C}$ into $\mathcal{E}_i$, for $i = 1, 2, 3, \ldots, n$.

Theorem 3.2. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) \leq \frac{1}{2}[\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2) - \phi(\zeta(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))] \tag{3.8}$$

for all bounded subsets $\mathcal{C}_1, \mathcal{C}_2$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$, $\phi \in \Phi$ and $\zeta \in \Upsilon$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*, v^*).

Proof. Consider the mapping $\widetilde{\mathcal{G}} : \mathcal{E}^2 \to \mathcal{E}^2$ defined by the formula

$$\widetilde{\mathcal{G}}(u, v) = (\mathcal{G}(u, v), \mathcal{G}(v, u)).$$

Since $\mathcal{G}$ is a continuous and bounded mapping, having monotone property, it follows that $\widetilde{\mathcal{G}}$ is also a continuous and bounded mapping, having monotone property.

Following Lemma 3.5, for $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, we define a new measure of noncompactness in the space $\mathcal{E}^2$ as

$$\widetilde{\beta}(\mathcal{C}) = \beta(\mathcal{C}_1) + \beta(\mathcal{C}_2),$$

where $\mathcal{C}_i$, $i = 1, 2$ denote the natural projections of $\mathcal{C}$. Now let $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \subset \mathcal{E}^2$ be a nonempty bounded subset. Due to (3.8) we conclude that

$$\begin{aligned}
\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) &\leq \widetilde{\beta}(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2) \times \mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\
&= \beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) + \beta(\mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\
&\leq \frac{1}{2}[\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2) - \phi(\zeta(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))] \\
&\qquad + \frac{1}{2}[\beta(\mathcal{C}_2) + \beta(\mathcal{C}_1) - \phi(\zeta(\beta(\mathcal{C}_2) + \beta(\mathcal{C}_1)))] \\
&= \beta(\mathcal{C}_1) + \beta(\mathcal{C}_2) - \phi(\zeta(\beta(\mathcal{C}_1) + \beta(\mathcal{C}_2))) \\
&= \widetilde{\beta}(\mathcal{C}) - \phi(\zeta(\widetilde{\beta}(\mathcal{C}))).
\end{aligned}$$

That is,

$$\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) \leq \widetilde{\beta}(\mathcal{C}) - \phi(\zeta(\widetilde{\beta}(\mathcal{C}))).$$

Next, we show that there is a $\widetilde{u}_0 \in \mathcal{C}$ such that $\widetilde{u}_0 \sqsubseteq \widetilde{\mathcal{G}}(\widetilde{u}_0)$. Indeed, since there exist two elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, set $\widetilde{u}_0 = (u_0, v_0)$. Then by the definition of $\widetilde{\mathcal{G}}$, we have

$$\begin{aligned}
\widetilde{u}_0 = (u_0, v_0) &\sqsubseteq (\mathcal{G}(u_0, v_0), \mathcal{G}(v_0, u_0)) = \widetilde{\mathcal{G}}(u_0, v_0) \\
&= \widetilde{\mathcal{G}}(\widetilde{u}_0).
\end{aligned}$$

Thus, following from Theorem 2.1, we conclude that $\widetilde{\mathcal{G}}$ has a fixed point, and hence $\mathcal{G}$ has a coupled fixed point. □

Theorem 3.3. *Let $(\mathcal{E}, \|\cdot\|, \sqsubseteq)$ be a partially ordered Banach space whose positive cone K is normal. Suppose that $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ is a continuous and bounded mapping, having monotone property and satisfying*

$$\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)) \leq \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\} - \phi(\zeta(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\})) \tag{3.9}$$

for all bounded subsets $\mathcal{C}_1, \mathcal{C}_2$ in $\mathcal{E}$, where β denotes the MNC in $\mathcal{E}$, $\phi \in \Phi$ and $\zeta \in \Upsilon$.

If there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, then $\mathcal{G}$ has at least a coupled fixed point (u^*, v^*).

Proof. Consider the mapping $\widetilde{\mathcal{G}} : \mathcal{E}^2 \to \mathcal{E}^2$ defined by the formula

$$\widetilde{\mathcal{G}}(u, v) = (\mathcal{G}(u, v), \mathcal{G}(v, u)).$$

Then $\widetilde{\mathcal{G}}$ is a continuous and bounded mapping, having monotone property.

For any $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, we define a new MNC in the space $\mathcal{E}^2$ as

$$\widetilde{\beta}(\mathcal{C}) = \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\}$$

where $\mathcal{C}_i$, $i = 1, 2$ denote the natural projections of $\mathcal{C}$. Now let $\mathcal{C} \subset \mathcal{E}^2$ with $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ be a nonempty bounded subset. We can conclude

$$\begin{aligned} \widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) &\leq \widetilde{\beta}(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2) \times \mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1)) \\ &= \max\{\beta(\mathcal{G}(\mathcal{C}_1 \times \mathcal{C}_2)), \beta(\mathcal{G}(\mathcal{C}_2 \times \mathcal{C}_1))\} \\ &\leq \max\left\{ \begin{array}{c} \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\} - \phi(\zeta(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\})), \\ \max\{\beta(\mathcal{C}_2), \beta(\mathcal{C}_1) - \phi(\zeta(\max\{\beta(\mathcal{C}_2), \beta(\mathcal{C}_1)\})) \end{array} \right\} \\ &= \max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\} - \phi(\zeta(\max\{\beta(\mathcal{C}_1), \beta(\mathcal{C}_2)\})) \\ &= \widetilde{\beta}(\mathcal{C}) - \phi(\zeta(\widetilde{\beta}(\mathcal{C}))). \end{aligned}$$

That is,

$$\widetilde{\beta}((\widetilde{\mathcal{G}}(\mathcal{C})) \leq \chi(\widetilde{\beta}(\mathcal{C}))\widetilde{\beta}(\mathcal{C}).$$

Next, we show that there is a $\widetilde{u}_0 \in \mathcal{C}$ such that $\widetilde{u}_0 \sqsubseteq \widetilde{\mathcal{G}}(\widetilde{u}_0)$. Since there exist elements $u_0, v_0 \in \mathcal{E}$ such that $u_0 \sqsubseteq \mathcal{G}(u_0, v)$ for any $v \in \mathcal{E}$ and $v_0 \sqsubseteq \mathcal{G}(v_0, u)$ for any $u \in \mathcal{E}$, set $\widetilde{u}_0 = (u_0, v_0)$. Then by the definition of $\widetilde{\mathcal{G}}$, we have

$$\begin{aligned} \widetilde{u}_0 = (u_0, v_0) &\sqsubseteq (\mathcal{G}(u_0, v_0), \mathcal{G}(v_0, u_0)) = \widetilde{\mathcal{G}}(u_0, v_0) \\ &= \widetilde{\mathcal{G}}(\widetilde{u}_0). \end{aligned}$$

Following Theorem 2.1, $\widetilde{\mathcal{G}}$ has a fixed point, and hence $\mathcal{G}$ has a coupled fixed point. Thus we conclude the result. □

Remark 3.2. In view of the Remark 2.1 (A1)-(A3), some new coupled fixed point results can be derived from Theorem 3.2 and Theorem 3.3.

4. Existence of mild solutions for fractional evolution equations

Let $(X, \|\cdot\|, \leq)$ be a partially ordered Banach space, whose positive cone K is normal. Consider the existence of mild solutions for the following fractional evolution equation with initial value condition

$$\left\{ \begin{array}{l} {}^C D_t^q x(t) + Ax(t) = f(t, x(t)), \quad t \in J := [0, a], \\ x(0) = x_0 \in X, \end{array} \right. \tag{4.10}$$

where ${}^C D_t^q$ denotes the Caputo fractional derivative of order $q \in \left(\frac{1}{2}, 1\right]$, $-A : D(A) \subset X \to X$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ of uniformly bounded linear operator in X, $a > 0$ is a constant, f is a given function.

Definition 4.6. A C_0-semigroup $T(t)(t \geq 0)$ is called a positive C_0-semigroup if $T(t)x \geq 0$ for all $x \geq 0$.

Define two operator families $\{\mathcal{U}_q(t)\}_{t\geq 0}$ and $\{\mathcal{V}_q(t)\}_{t\geq 0}$ as

$$\mathcal{U}_q(t)x = \int_0^\infty \zeta_q(\tau) S(t^q \tau) x d\tau,$$

$$\mathcal{V}_q(t)x = q \int_0^\infty \tau \zeta_q(\tau) S(t^q \tau) x d\tau, \quad 0 < q < 1,$$

where

$$\zeta_q(\tau) = \frac{1}{q}\tau^{-1-\frac{1}{q}}\varrho_q(\tau^{-\frac{1}{q}}),$$

$$\varrho_q(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\tau^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \tau \in (0,\infty).$$

The following lemma can be found in [21, 30].

Lemma 4.6. (i) *For any $x \in X$ and fixed $t \geq 0$, one has*

$$\|\mathcal{U}_q(t)x\| \leq M\|x\|, \qquad \|\mathcal{V}_q(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

(ii) *If $T(t)(t \geq 0)$ is an equi-continuous semigroup, $\mathcal{U}_q(t)$ and $\mathcal{V}_q(t)$ are equi-continuous in X for $t > 0$.*

(iii) *If $T(t)(t \geq 0)$ is a positive C_0-semigroup, $\mathcal{U}_q(t)$ and $\mathcal{V}_q(t)$ are positive operators for all $t \geq 0$.*

Definition 4.7. A function $x \in C(J, X)$ is called a mild solution of the initial value problem of fractional integro-differential evolution equation (4.10) if it satisfies the following integral equation

$$x(t) = \mathcal{U}_q(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{V}_q(t-s)f(s, x(s))ds,\ \ t \in J.$$

To prove our main results, we list the following assumptions:

$(H1)$ $T(t)(t \geq 0)$ is a positive and equi-continuous C_0-semigroup, and there is a constant $M > 0$ such that

$$\|T(t)\| \leq M \qquad \forall t \geq 0.$$

$(H2)$ $f : J \times X \to X$ is continuous and satisfies the following conditions:

(i) For any $t \in J$ and $x_1, x_2 \in X$ with $x_1 \leq x_2$, one has

$$f(t, x_1) \leq f(t, x_2).$$

(ii) There exist functions $\rho_i \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, where $i = 1, 2$ and $\sigma \in [0, 2q-1)$ such that

$$\|f(t, x)\| \leq \rho_1(t)\|x\| + \rho_2(t),$$

for all $t \in J$ and $x \in X$.

(iii) There is a function $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$\beta(f(t, D)) \leq \gamma(t)\ln(1 + \beta(D)),$$

for any $t \in J$ and any nonempty bounded subsets $D \subset X$.

$(H3)$ There exists an element $u_0 \in C(J, X)$ satisfying

$$\begin{cases} {}^C D_t^q u_0(t) + Au_0(t) \leq f(t, u_0(t)), & t \in J, \\ u_0(0) \leq x_0. \end{cases}$$

$(H4)$ $\frac{Mb}{\Gamma(q)}a^{2q-1-\sigma}\|\rho_1\|_{L^{\frac{1}{\sigma}}} \leq 1$, where $b = (\frac{1-\sigma}{2q-1-\sigma})^{1-\sigma}$.

Theorem 4.4. *Assume that the conditions $(H1) - (H4)$ hold true. Then the initial value problem of fractional evolution equation (4.10) has at least one mild solution provided that*

$$\frac{4Ma^q}{\Gamma(q+1)}\|\gamma\|_{L^1} \leq 1. \tag{4.11}$$

Proof. Define an operator $\mathcal{Q}: C(J,X) \to C(J,X)$ by the formula:

$$(\mathcal{Q}x)(t) = \mathcal{U}_q(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{V}_q(t-s)f(s,x(s))ds, \;\; t \in J. \tag{4.12}$$

It follows from the continuity of f that $\mathcal{Q}: C(J,X) \to C(J,X)$ is continuous. From the assumption $(H2)(i)$ and the Lemma 1.2 (iii), it is easy to see that $\mathcal{Q}: C(J,X) \to C(J,X)$ is nondecreasing. The remaining proof will be given in four steps.

Step 1. We prove that there is $r > 0$ such that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x \in C(J,X) : \|x\| \le r\}$.

If this were not the case, there would exist $t \in J$ and $x_r \in B_r$ such that $r < \|(\mathcal{Q}x_r)(t)\|$ for each $r > 0$. Combining with $(H2)(ii)$, Lemma 1.2 and Hölder inequality, we have

$$\begin{aligned} r < \|(\mathcal{Q}x_r)(t)\| &\le M\|x_0\| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\rho_1(s)\|x_r(s)\| + \rho_2(s)]ds \\ &\le M\|x_0\| + \frac{Mr}{\Gamma(q)}\int_0^t (t-s)^{q-1}\rho_1(s)ds + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\rho_2(s)ds \\ &\le M\|x_0\| + \frac{Mbr}{\Gamma(q)}a^{2q-1-\sigma}\|\rho_1\|_{L^{\frac{1}{\sigma}}} + \frac{Mb}{\Gamma(q)}a^{2q-1-\sigma}\|\rho_2\|_{L^{\frac{1}{\sigma}}}. \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$ in above inequality, we obtain

$$1 \le \frac{Mb}{\Gamma(q)}a^{2q-1-\sigma}\|\rho_1\|_{L^{\frac{1}{\sigma}}},$$

which is a contradiction of assumption $(H4)$. Therefore, $\mathcal{Q}B_r \subset B_r$ for some $r > 0$.

Step 2. We prove that $\mathcal{Q}(B_r)$ is equi-continuous.

Using $(H1)$, it is a similar proof as in the proof of Theorem 1 of [21]. So, we omit the details here.

Step 3. We prove that $\beta(\mathcal{Q}(B_r)) \le \beta(\mathcal{Q}(B_r)) - \phi(\zeta(\beta(\mathcal{Q}(B_r))))$.

Since $\mathcal{Q}(B_r)$ is bounded and equi-continuous, there exists a countable subset $B_r^n = \{x_n\} \subset B_r$ such that $\mathcal{Q}(B_r^n)$ is bounded and equi-continuous and

$$\beta(\mathcal{Q}(B_r)) \le 2\beta(\mathcal{Q}(B_r^n)). \tag{4.13}$$

For any $x \in B_r$ and $t \in J$, by $(H2)(iii)$ and (4.12), we have

$$\begin{aligned} \beta((\mathcal{Q}B_r^n)(t)) &\le \beta(\{\int_0^t (t-s)^{q-1}\mathcal{V}_q(t-s)f(s,x_n(s))ds\}) \\ &\le \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\gamma(s)\ln(1+\beta(B_r^n(s)))ds \\ &\le \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\gamma(s)\ln(1+\beta(B_r))ds \\ &\le \frac{2Ma^q}{\Gamma(q+1)}\|\gamma\|_{L^1}\ln(1+\beta(B_r)) \\ &\le \frac{1}{2}\ln(1+\beta(B_r)). \end{aligned}$$

Since $\mathcal{Q}B_r^n$ is bounded and equi-continuous, we can obtain

$$\beta(\mathcal{Q}B_r) \le 2\beta(\mathcal{Q}B_r^n) = 2\max_{t\in J}\beta((\mathcal{Q}B_r^n)(t)) \le \ln(1+\beta(B_r)).$$

Now we consider the functions $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$\zeta(t) = t, \ \ \phi(t) = t - \ln(1+t) \ \forall\, t \in \mathbb{R}^+.$$

Then $\phi \in \Phi$, $\zeta \in \Upsilon$ and

$$\beta(\mathcal{Q}B_r) \le \beta(B_r) - [\beta(B_r) - \ln(1+\beta(B_r))] = \beta(B_r) - \phi(\zeta(\beta(B_r))).$$

Step 4. We prove that there is $u_0 \in B_r$ satisfying $u_0 \le \mathcal{Q}u_0$.

Let $h(t) =^C D_t^q u_0(t) + Au_0(t)$ and $u_0(0) = u_0(0)$. Then by Definition 4.7, we can obtain

$$u_0(t) = \mathcal{U}_q(t)u_0(0) + \int_0^t (t-s)^{q-1}\mathcal{V}_q(t-s)h(s)ds, \quad t \in J.$$

Combining this fact with $(H4)$ and (4.12), we have

$$u_0(t) \le \mathcal{U}_q(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{V}_q(t-s)f(t, u_0(t), \int_0^t k(t,s,u_0(s))ds)ds = (\mathcal{Q}u_0)(t)$$

for all $t \in J$.

Hence, all the conditions of Theorem 2.1 are satisfied. By Theorem 2.1, the initial value problem of fractional integro-differential evolution equation (4.10) has at least one mild solution. □

5. An example

In this section, we apply Theorem 4.4 to consider the existence of solutions for the following fractional integro-differential equation

$$\begin{cases} \dfrac{\partial^{\frac{3}{4}}}{\partial t^{\frac{3}{4}}} u - \Delta u = g(z,t,u), & (z,t) \in \Omega \times J, \\ u|_{\partial\Omega} = 0, & \\ x(z,0) = x_0(z), & z \in \Omega, \end{cases} \tag{5.14}$$

where $J = [0,1]$ $\Omega \in \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, Δ is the Laplace operator, $g : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition:

(F) (i) For any $(z,t) \in \Omega \times J$ and $u_1, u_2 \in \mathbb{R}$ with $u_1 \le u_2$, one has

$$g(z,t,u_1) \le g(z,t,u_2).$$

(ii) There exist functions $\rho_i \in L^{\frac{1}{\sigma}}(J, \mathbb{R}^+)$, where $i = 1, 2$ and $\sigma \in [0, \frac{1}{2})$ such that

$$|g(z,t,u)| \le \rho_1(t)|u| + \rho_2(t),$$

for all$(z,t) \in \Omega \times J$ and $u \in \mathbb{R}$.

(iii) There is a function $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$\beta(g(z,t,D)) \le \gamma(t)\ln(1+\beta(D)),$$

for any $(z,t) \in \Omega \times J$ and any nonempty bounded subsets $D \subset \mathbb{R}$.

Let $X = L^p(\Omega)(0 \le p < +\infty)$. Then X is the partially ordered Banach space, whose positive cone $K := \{u \in L^p(\Omega) : u(x) \ge 0, a.e. x \in \Omega\}$ is normal with constant N. Let

$$D(A) = \{u \in W^{2,p}(\Omega) : u|_{\partial\Omega} = 0\},$$

$$Au = -\Delta u.$$

Then $-A : D(A) \subset X \to X$ generates an analytic semigroup $T(t)(t \ge 0)$ in X. By the maximum principle of ellipse equation, $T(t)(t \ge 0)$ is a positive C_0-semigroup. Its growth index is $\nu_0 = -\lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of Laplace operator $-\Delta$ with boundary condition $u|_{\partial\Omega} = 0$. Therefore, there is a constant $M > 0$ such that

$$\|T(t)\| \le M \qquad \forall t \ge 0.$$

Let $u(t)(z) = u(z,t)$ and $f(t,u(t))(z) = g(z,t,u(z,t))$. Then the fractional differential equation (5.14) can be rewritten into (4.10). It is clear that $f(t,u)$ is a continuous and nondecreasing function with respect to $u \in X$ and the assumption $(H2)$ holds. If there exists a function $\widetilde{v} \in C(J,X)$ such that

$$\begin{cases} \frac{\partial^{\frac{3}{4}}}{\partial t^{\frac{3}{4}}}\widetilde{v} - \Delta\widetilde{v} \leq g(z,t,\widetilde{v}), \quad (z,t) \in \Omega \times J, \\ \widetilde{v}|_{\partial\Omega} = 0, \\ \widetilde{v}(z,0) \leq \widetilde{v}_0(z), \quad z \in \Omega, \end{cases}$$

then $\widetilde{v}$ satisfies the assumption $(H3)$. Therefore, by Theorem 4.4, the fractional differential equation (5.14) has at least one solution provided that

$$\frac{M\|\gamma\|_{L^1}}{\Gamma(\frac{3}{4})} < \frac{3}{16}$$

and

$$\frac{Mb}{\Gamma(\frac{3}{4})}\|\rho_1\|_{L^{\frac{1}{\sigma}}} < 1,$$

where

$$b = \left(\frac{1-\sigma}{\frac{1}{2}-\sigma}\right)^{1-\sigma}.$$

Acknowledgements. The first author is thankful to the United State-India Education Foundation, New Delhi, India and IIE/CIES, Washington, DC, USA for Fulbright-Nehru PDF Award (No. 2052/FNPDR/2015) and second author is thankful to the NSF(No. 11701457).

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[1]DEPARTMENT OF MATHEMATICS
TEXAS A & M UNIVERSITY - KINGSVILLE
78363-8202 TEXAS, USA
E-mail address: drhknashine@gmail.com
E-mail address: Ravi.Agarwal@tamuk.edu

[2]DEPARTMENT OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
VELLORE INSTITUTE OF TECHNOLOGY
VELLORE-432014 TN, INDIA
E-mail address: hemant.nashine@vit.ac.in

[3]COLLEGE OF MATHEMATICS AND STATISTICS
NORTHWEST NORMAL UNIVERSITY, LANZHOU 730070 P. R. CHINA
E-mail address: yanghe256@163.com

[4]DISTINGUISHED UNIVERSITY PROFESSOR OF MATHEMATICS,
FLORIDA INSTITUTE OF TECHNOLOGY,
MELBOURNE, FL 32901, USA
E-mail address: Ravi.Agarwal@tamuk.edu