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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Global Minimization of best proximity points for semi-cyclic Berinde contractions

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ABSTRACT. In this paper, we introduce a semi-cyclic Berinde contraction pair on a metric space which is more general than that of semi-cyclic contraction pair defined by Gabeleh and Abkar [Gabeleh, M. and Abkar, A., *Best proximity points for semi-cyclic contractive pairs in Banach spaces*, Int. Math. Forum, **6** (2011), 2179–2186] and prove an existence result concerning global monomization of best proximity points of this pair. Our main result can be used to obtain a common fixed point theorem of some contractive mappings related to Berinde's contractions without commutative assumption. An example supporting our main result is also given.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays an important role in solving nonlinear equations. It centers in the process of solving nonlinear equation of the form $Tx = x$ where T is a self-mapping defined on a subset of metric spaces or normed spaces. A solution of above equation is known as a fixed point of T. The most famous fixed point theorem in metric space is Banach contraction principle [3] which assests that every contraction mapping in a complete metric space has a unique fixed point.

A mapping $T : X \to X$, where (X, d) is metric space, is called contraction if there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

Banach contraction principle was extended and generalized in many directions, and one of them has been established by Berinde [5] in 2003. He extended a contraction mapping into a weak contraction mapping.

Let (X, d) be metric space. A mapping $T : X \to X$ is called weak contraction mapping if there exists $\alpha \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \alpha d(Tx, Ty) + L(x, Ty), \text{for all } x, y \in X.$$

He proved the existence and convergence theorem of this mapping in a complete metric space. Fixed point theory is indeed a great tool to solve many nonlinear equations. However, in the case that T is non-self mappings, the equation $Tx = x$ may not have a solution, that is T has no fixed point. In this case, we know that $d(x, T) > 0$ for all x in the domain of T. It is natural to ask the following question, can we find a point x such that

$$d(x, Tx) = \min_{y \in dom(T)} d(y, Ty), \text{where } dom(T) \text{ is a domain of } T.$$

To be more specific, let A and B be two nonempty closed subsets of a metric space (X, d) and $T : A \to B$. We know that $d(x, Tx) \geq d(A, B)$, for all $x \in A$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. A point $x \in A$ such that $d(x, Tx) = d(A, B)$ is called a best

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proximity point of T. Best proximity point theory has recently attracted the attention of many authors (see for instance [1,4,8–11,15,16,19].

In 2003, Kirk, Srinivasan and Veeramani [16] introduced a concepts of cyclic mapping and then proved the following theorem.

Theorem 1.1. *Let A and B be nonempty closed subsets of metric space (X,d). Suppoose that $T : A\cup B \to A\cup B$ be a cyclic, i.e., $T(A) \subseteq B,\ T(B) \subseteq A$ and $d(Tx,Ty) \leq \alpha d(x,y)$ for some $\alpha \in (0,1)$, for all $x \in A,\ y \in B$. Then $A \cap B \neq \emptyset$ and T has a unique fixed point in $A \cap B$.*

In 2006, Eldred and Veeramani [9] extened this result to a case that $A \cap B = \emptyset$ and introduced the concept of cyclic contraction.

A self mapping $T : A\cup B \to A\cup B$ is said to be cyclic contraction if $T(A) \subseteq B, T(B) \subseteq A$ and T satisfies

$$d(Tx,Ty) \leq \alpha d(x,y) + (1-\alpha)d(A,B), \quad \text{for some } \alpha \in (0,1) \text{ and for all } x \in A, y \in B.$$

They also proved the existence and convergence results of best proximity point for this kind of mappings in uniformly convex Banach spaces and metric spaces. We can see that this mapping can be reduced to a contraction mapping if $A \cap B \neq \emptyset$, so T has a unique fixed point. This work has been extended in many directions. For instance, in 2013, Cho et al. [7] proved the existence of tripled best proximity point of some mappings in metric space. For more works which extended [9] see [1,4,8,10,11,15,17–19].

In 2011, Gabeleh and Abkar [11] introduced the concept of semi-cyclic contraction pair as following.

Let A, B be nonempty closed subsets of a complete metric space (X,d), and let S,T be self mappings on $A\cup B$. A pair (S,T) is called semi-cyclic contraction pair if the following conditions hold

(1) $S(A) \subseteq B, T(B) \subseteq A$;
(2) $d(Sx,Ty) \leq \alpha d(x,y)+(1-\alpha)d(A,B)$, for some $\alpha \in (0,1)$ and for all $x \in A, y \in B$.

As we can see if $S = T$ then this mapping is basically a cyclic contraction mapping. They establised interesting results for best proximity point theorems of semi-cyclic contraction pair (S,T) in metric spaces and uniformly convex Banach spaces. In this paper, motivated by these works, by using the idea of Berinde weak contraction mappings, we are interested to extend some results of Gabeleh and Abkar and prove some exsistence theorems of best proximity points of the introduced mappings.

2. Main results

In this section, we first introduce a concept of semi-cyclic Berinde contraction.

Definition 2.1. Let A, B be nonempty subsets of a metric space (X,d) and $S,T : A\cup B \to A\cup B$ be mappings such that $S(A) \subseteq B, T(B) \subseteq A$ and there exists $\alpha \in (0,1)$ such that

$$d(Sx,Ty) \leq \alpha d(x,y) + L\min\{d(Sx,y), d(Ty,x)\} + (1-\alpha)dA,B), \tag{2.1}$$

for all $x \in A$, $y \in B$. In this case, a pair (S,T) is called semi-cyclic Berinde contraction.

Let x_0 be an element in A and define seqeunces $\{x_n\}, \{y_n\}$ by

$$\begin{cases} y_n = Sx_n, \\ x_{n+1} = Ty_n, \quad n = 0,1,2,\ldots \end{cases} \tag{2.2}$$

So $\{x_n\}, \{y_n\}$ are sequences in A, B, respectively.

Proposition 2.1. *Let $\{x_n\}$ and $\{y_n\}$ be sequences defined as in (2.2). Then $d(x_n, Sx_n) \to d(A,B)$ as $n \to \infty$*

Proof.

$$\begin{aligned} d(x_n, Sx_n) &= d(Sx_n, Ty_{n-1}) \\ &\leq \alpha d(x_n, y_{n-1}) + L\min\{d(Sx_n, y_{n-1}), d(x_n, Ty_{n-1})\} + (1-\alpha)d(A,B) \\ &= \alpha d(x_n, y_{n-1}) + (1-\alpha)d(A,B). \end{aligned}$$

$$\begin{aligned} d(x_n, y_{n-1}) &= d(Ty_{n-1}, Sx_{n-1}) \\ &\leq \alpha d(x_{n-1}, y_{n-1}) + L\min\{d(Sx_{n-1}, y_{n-1}), d(x_{n-1}, Ty_{n-1})\} + (1-\alpha)d(A,B) \\ &= \alpha d(x_{n-1}, Sx_{n-1}) + (1-\alpha)d(A,B). \end{aligned}$$

By Combining both inequalities, we then have

$$\begin{aligned} d(x_n, Sx_n) &\leq \alpha(\alpha d(x_{n-1}, Sx_{n-1}) + (1-\alpha)d(A,B)) + (1-\alpha)d(A,B) \\ &= \alpha^2 d(x_{n-1}, Sx_{n-1}) + (1-\alpha^2)d(A,B) \\ &\vdots \\ &\leq \alpha^{2n} d(x_0, Sx_0) + (1-\alpha^{2n})d(A,B). \end{aligned}$$

Because $\alpha^{2n} \to 0$ as $n \to \infty$, the above inequality implies that $d(x_n, Sx_n) \to d(A,B)$. □

Theorem 2.2. *Let $\{x_n\}, \{y_n\}$ be sequences defined by (2.2). If $\{x_n\}$ and $\{y_n\}$ have convergent subsequences in A and B, respectively. Then there exists $x \in A, y \in B$ such that $d(x, Sx) = d(y, Ty) = d(A,B)$, i.e., x is a best proximity point of S and y is a best proximity point of T, respectively.*

Proof. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. We know that

$$d(A,B) \leq d(y_{n_k}, Ty), \quad \forall k \geq 1.$$

We also have,

$$\begin{aligned} d(y_{n_k}, Ty) = d(Sx_{n_k}, Ty) &\leq \alpha d(x_{n_k}, y) + L\min\{d(Sx_{n_k}, y), d(x_{n_k}, Ty)\} + (1-\alpha)d(A,B) \\ &\leq \alpha d(x_{n_k}, y_{n_k}) + \alpha d(y_{n_k}, y) + Ld(y_{n_k}, y) + (1-\alpha)d(A,B). \end{aligned}$$

As $k \to \infty$, we have $d(y, Ty) \leq \alpha d(A,B) + (1-\alpha)d(A,B) = d(A,B)$.
Hence $d(y, Ty) = d(A,B)$.
Next, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \to x$. The relation $d(A,B) \leq d(x_{n_k}, Sx)$ holds for all $k \geq 1$. By (2.2), we have

$$\begin{aligned} d(x_{n_k}, Sx) = d(Sx, Ty_{n_k-1}) &\leq \alpha d(x, y_{n_k-1}) + L\min\{d(Sx, y_{n_k-1}), d(x, Ty_{n_k-1})\} + (1-\alpha)d(A,B) \\ &\leq \alpha d(x, x_{n_k-1}) + \alpha d(x_{n_k-1}, y_{n_k-1}) + Ld(x, x_{n_k}) + (1-\alpha)d(A,B). \end{aligned}$$

As $k \to \infty$, we have $d(x, Sx) \leq \alpha d(A,B) + (1-\alpha)d(A,B) = d(A,B)$.
Hence $d(x, Tx) = d(A,B) = d(y, Ty)$. This complete the proof. □

Example 2.1. Let $A = [-1,1] \times \left[\frac{1}{2}, 1\right] \cup [-1,1] \times \{0\} \cup \left\{\left(\frac{5}{4}, 0\right)\right\}$ and $B = [2,4] \times \{0\} \cup \left\{\left(\frac{3}{2}, 0\right)\right\}$. Both A, B are subspaces of $\mathbb{R}^2$ with euclidean norm. Define mappings $S, T : A \cup B \to A \cup B$ by

$$Sx = \begin{cases} (3 - x_1, 0), & \text{if } x = (x_1, x_2) \in [-1,1] \times \left[\frac{1}{2}, 1\right], \\ \left(\frac{3}{2}, 0\right), & \text{otherwise.} \end{cases}$$

$$Ty = \begin{cases} (3-y_1,0), & \text{if } y=(y_1,y_2)\in[2,4]\times\{0\},\\ \left(\dfrac{5}{4},0\right), & \text{otherwise.}\end{cases}$$

Then a pari (S,T) is a semi-cyclic Berinde contraction which is not a semi-cyclic contraction.

Proof. In oder to prove this, we need to consider several cases of x and y.

Case 1: $x \in [-1,1] \times \left[\dfrac{1}{2},1\right]$ and $y \in [2,4]\times\{0\}$.

In this case, we have $d(Sx,Ty) = y_1 - x_1$ and $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2)^2}$ when $x_1 \in [-1,1]$, $x_2 \in \left[\dfrac{1}{2},1\right]$, and $y_1 \in [2,4]$. By using the elementary calculus, it can be shown that

$$\sup\left\{\frac{d(Sx,Ty)}{d(x,y)} = \frac{y_1-x_1}{\sqrt{(x_1-y_1)^2+(x_2)^2}}\right\} : x_1 \in [-1,1], x_2 \in \left[\frac{1}{2},1\right],$$

and $y_1 \in [2,4]\} = \dfrac{10}{\sqrt{101}}$. Hence $d(Sx,Ty) \leq \dfrac{10}{\sqrt{101}}d(x,y)$.

Case 2: $x \in [-1,1] \times \left[\dfrac{1}{2},1\right]$ and $y = \left(\dfrac{3}{2},0\right)$.

In this case, $d(Sx,y) \geq \dfrac{1}{2}$, $d(x,Ty) \geq \dfrac{\sqrt{5}}{4}$ and $d(Sx,Ty) \leq 5$. If we put L = 20, then we have

$$d(Sx,Ty) \leq \frac{10}{\sqrt{101}}d(x,y) + 20\min\{d(Sx,y),d(x,Ty)\} + \left(1-\frac{10}{\sqrt{101}}\right)d(A,B).$$

However, if we choose $x = \left(1,\dfrac{1}{2}\right)$ then $d(Sx,Ty) = \dfrac{3}{4}$ and $d(x,y) = \dfrac{1}{\sqrt{2}}$. As we can see, $\dfrac{3}{4} > \dfrac{1}{\sqrt{2}}$ and $d(A,B) = \dfrac{1}{4}$. So we can conclude that

$$d(Sx,Ty) > \alpha d(x,y) + (1-\alpha)d(A,B), \text{ for any } \alpha \in (0,1)$$

which means that (S,T) is not a semi-cyclic contraction pair.

It's easy to see that the remaining cases satisfy the inequality where $\alpha = \dfrac{10}{\sqrt{101}}$ and $L = 20$. So

$$d(Sx,Ty) \leq \frac{10}{\sqrt{101}}d(x,y) + 20\min\{d(Sx,y),d(x,Ty)\} + \left(1-\frac{10}{\sqrt{101}}\right)d(A,B)$$

for all $x \in A, y \in B$. Hence (S,T) is a semi-cyclic Berinde contraction pair. □

The study of common fixed points of two mappings satisfying certain contraction is very interesting topic which can be applied to solve the solution of system of operator equations. This topic attracted many authors in the last three decades, see for example [2, 6, 12–14]. Many works in this topic assume commutative assumption of some points on the mappings. In the case that $A \cap B \neq \emptyset$, by Theorem 2.2 we obtain a common fixed point theorem of two mappings satisfying some contractive condition related to Berinde's contraction as seen in the following.

Corollary 2.1. *Let A,B be nonempty compact subsets of metric space (X,d), $S, T : A\cup B \to A\cup B$ be mappings such that $S(A) \subseteq B, T(B) \subseteq A$ and*

$$d(Sx, Ty) \leq \alpha d(x, y) + L\min\{d(Sx, y), d(Ty, x)\}, \text{ for all } x \in A, y \in B.$$

Then S and T have a common fixed point in $A \cup B$, if $A \cap B \neq \emptyset$.

It is noted that we need compactness assumption on the set A and B in above corollary. However, we can relax compactness assumption on A and B as seen on the following theorem.

Theorem 2.3. *Let A, B be nonempty closed subsets of a complete and let metric space (X, d), $S, T : A \cup B \to A \cup B$ be mappings such that $S(A) \subseteq B, T(B) \subseteq A$ and*

$$d(Sx, Ty) \leq \alpha d(x, y) + L\min\{d(Sx, y), d(Ty, x)\}, \text{ for all } x \in A, y \in B.$$

Then S and T have a common fixed point in $A \cup B$, if $A \cap B \neq \emptyset$.

Proof. We first define a sequence $\{z_n\}$ in $A \cup B$ by $z_n = \begin{cases} Ty_k & \text{if } n = 2k, \\ Sx_k & \text{if } n = 2k-1, \end{cases}$
where $\{x_k\}, \{y_k\}$ are defined by (2.2) We claim that $\{z_n\}$ is a cauchy sequence in $A \cup B$. To show this we need to consider 2 cases of n. The first case is $n = 2k-1$, we have

$$\begin{aligned} d(z_{n+1}, z_n) &= d(Sx_k, Ty_k) \\ &\leq \alpha d(x_k, y_k) + Ld(Sx_k, y_k) \\ &= \alpha^2 d(x_k, Sx_k) \\ &\vdots \\ &\leq \alpha^{2k} d(x_1, y_1). \end{aligned}$$

In the case that n = 2k , we have

$$\begin{aligned} d(z_{n+1}, z_n) &= d(Sx_k, Ty_k) \\ &= \alpha d(x_{k+1}, y_k) + Ld(x_{k+1}, Ty_k) \\ &= \alpha d(Sx_k, Ty_k) \\ &\vdots \\ &\leq \alpha^{2k+1} d(x_1, y_1). \end{aligned}$$

By combining these two inequalities, we then have for any $m > n > 0$,

$$d(z_m, z_n) \leq \sum_{i=n}^{m-1} d(z_{i+1}, z_i) \leq \sum_{i=n}^{m-1} \alpha^i d(x_1, y_1) = \frac{\alpha^n - \alpha^{m-1}}{1-\alpha} d(x_1, y_1).$$

So $(z_m, z_n) \to 0$ as $m, n \to \infty$. Hence $\{z_n\}$ is a cauchy sequence in $A \cup B$. Let z be a limit point of $\{z_n\}$. Suppose that $z \in A$, we have that z is also in B, since $\{z_{2k-1}\} \subseteq B$. On the other hand, assume that $z \in B$, we can also show that $z \in A$, since $\{z_{2k}\} \subseteq A$. Hence $z \in A \cup B$. Now, let us consider the follwing equation:

$$\begin{aligned} d(z, Tz) &\leq d(z, z_{2k}) + d(z_{2k}, Tz) = d(z, z_{2k}) + d(Sx_k, Tz) \\ &\leq d(z, z_{2k}) + \alpha d(x_k, z) + Ld(Sx_k, z) \\ &= d(z, z_{2k}) + \alpha d(Ty_{k-1}, z) + Ld(Sx_k, z) \to 0 \text{ as } k \to \infty. \end{aligned}$$

So $d(z, Tz) = 0$. Similary, It can also be shown that $d(z, Sz) = 0$.. Hence z is a common fixed point of mappings S and T. □

The following Corollary is a result from [11].

Corollary 2.2. *Let (S,T) be semi-cyclic contraction pair. Consider the iterative sequences defined by (2.2). If both $\{x_n\}$ and $\{y_n\}$ have a convergent subsequences in A and B, respectively, then there exist $x \in A$ and $y \in B$ such that*

$$d(x,Sx) = d(A,B) = d(y,Ty).$$

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References

[1] Abkar, A. and Gabeleh, M., *Best proximity points for asymptotic cyclic contraction mappings*, Nonlinear Anal., **74** (2011), 7261–7268

[2] Abbas, M. and Jungck, G., *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416-420

[3] Banach, S., *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math., **3** (1922), 133–181

[4] Basha, S. S. and Shahzad, N., *Common best proximity point theorems: Global minimization of some real-valued multi-objective functions*, J. Fixed Point Theory Appl., **18** (2011), 587–600

[5] Berinde, V., *Approximating fixed points of weak φ-contractions using the Picard iteration*, Fixed Point Theory, **4** (2003), 131–142

[6] Berinde, V., *Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces*, Taiwanese J. Math., **14** (2010), 1763–1776

[7] Cho et al., *Tripled best proximity point theorem in metric spaces*, Math. Inequal. Appl., **16** (2013), 1197-1216

[8] Derafshpour, M., Rezapour, Sh. and Shahzad, N., *Best Proximity Points of cyclic-contractions in ordered metric spaces*, Topol. Methods Nonlinear Anal., **37** (2011), 193–202

[9] Eldred, A., A. and Veeramani, P., *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006

[10] Felicit, J. M. and Eldred, A. A., *Best proximity points for cyclical contractive mappings*, Appl. Gen. Topol., **16** (2015), 119–126

[11] Gabeleh, M. and Abkar, A., *Best proximity points for semi-cyclic contractive pairs in Banach spaces*, Int. Math. Forum, **6** (2011), 2179–2186

[12] He et al., *Common fixed points for weak commutative mappings on a multiplicative metric space*, Fixed Point Theory Appl., **2014** (2014):48

[13] Jungck, G., *Commuting maps and fixed points*, Amer. Math. Monthly, **83** (1976), 261–263

[14] Jungck, G., *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc., **103** (1988), 977–983

[15] Karapinar, E., *Best proximity points of cyclic mappings*, Appl. Math. Lett., **25** (2012), 1761–1766

[16] Kirk, W. A., Srinivasan, P. S. and Veeramani, P., *Fixed points for mapping cycli contractions*, Fixed Point Theory, **4** (2003), 79–89

[17] Mongkolkeha, C., Cho, Y. J. and Kumam, P., *Best proximity points for generalized proximal C-contraction mappingsin metric spaces with partial orders*, J. Inequal. Appl., **2013** (2013):94

[18] Mongkolkeha, C., Cho, Y. J. and Kumam, P., *Best proximity points for Geraghtys proximal contraction mapping-mappings*, Fixed Point Theory Appl., **2013**, (2013):180

[19] Rezapour, Sh., Derafshpour, M. and Shahzad, N., *Best proximty points of cyclic ϕ-contractions on reflexive Banach spaces*, Fixed Point Theory Appl., **2010** (2010), Art. ID 946178

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