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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Coupled coincidence and coupled common fixed point theorems on a fuzzy metric space with a graph

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ABSTRACT. Inspired by the work of Dakjum et al. [Eshi, D., Das, P. K. and Debnath, P., *Coupled coincidence and coupled common fixed point theorems on a metric space with a graph*, Fixed Point Theory Appl., **37** (2016), 1–14], we introduce a new class of $G-f-$contraction mappings in complete fuzzy metric spaces endowed with a directed graph and prove some existence results for coupled coincidence and coupled common fixed point theorems of this type of contraction mappings in complete fuzzy metric spaces endowed with a directed graph.

1. INTRODUCTION

Zadeh [18] introduced the concept of fuzzy set theory. There are many viewpoints of the notion of the metric space in fuzzy topology. We can divide them into following two groups. First group relate these results to the fuzzy metrics on a set X is treated as a map d: X x $X \to \mathbb{R}^+$, where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. In such an approach numerical distances are set up between fuzzy objects. Second group focussed on those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. Erceg [4], Kaleva and Seikkala [12] and Kramosil and Michalek [13] discussed in length about fuzzy metric spaces. Grabiec's [8] proved a fixed point theorem in fuzzy metric space. Subramanyam [16] generalized Grabiec's result for a pair of commuting maps in the setting of Jungck 1976 [11]. George and Veermani [6] modified the concept of fuzzy metric spaces and defined a Hausdorff topology on this fuzzy metric space which has some applications in quantum particle physics. The concept of coupled fixed point was introduce by Bhaskar and Lakshmikantham [7] which has been further generalized and extended by many authors. Sedghi et al. [14] initiated the study of coupled fixed point in the setup of fuzzy metric spaces which become a topic of interest for many mathematician working in this area. (for example see [3, 15, 2, 17]).
In this paper, we study and establish the existence theorems for $G-f$-contraction mapping in the new set up of complete fuzzy metric spaces endowed with a directed graph.

Let $(M, X, *)$ be a fuzzy metric space and Δ be the diagonal of the Cartesian product $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subset E(G)$, where $E(G)$ is the set of edges of the graph G. Assume also that G has no parallel edges and, thus one can identified G with the pair $(V(G),\ E(G))$. Also, denote by

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G^{-1} the graph obtained from G by reversing the direction of the edges. Thus,

$$E(G^{-1}) = \{(x,\ y) \in X \times X : (y,\ x) \in E(G)\}.$$

2. Preliminaries

In this section, we present some basic definitions and concepts related to the main results of this article. In the sequel, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers.

Definition 2.1. [13] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for every $a \in [0,1]$;
(4) $a * b \le c * d$ if $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 2.2. [9] A t - norm $*$ is said to be of H - type if the family of functions $\{*^m(a)\}_{m=1}^{\infty}$ is equicontinuous at $a = 1$, where $*^2(a) = a*a, \quad *^{m+1}(a) = a*(*^m(a)), \quad m = 1,2,...,a \in [0,1]$. The t - norm $*_M$ = min is an example of t - norm of H - type, but there are other t - norms $*$ of H - type (see[9]). Obviously, $*$ is a H - type t norm if and only if for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $*^m(a) > 1 - \lambda$ for all $m \in N$, when $a > 1 - \delta$.

Definition 2.3. [6] A triplet $(X, M, *)$ is said to be a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ and $t,s > 0$:

(M_1) $M(x,y,t) > 0$,
(M_2) $M(x,y,t) = 1$ if and only if $x = y$,
(M_3) $M(x,y,t) = M(y,x,t)$,
(M_4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
(M_5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

In view of (M_1) and (M_2), it is worth pointing out that $0 < M(x,y,t) < 1$ for all $t > 0$, provided $x \neq y$. In view of Definition (2.3), George and Veermani [6] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric space induces a fuzzy metric space. In fact, we can fuzzify metric spaces into fuzzy metric spaces in a natural way as is shown by the following example. In other words, every metric induces a fuzzy metric.

Example 2.1. Let (X,d) be a metric space and define $a * b = ab$ for all $a,b \in [0,1]$. Also define $M(x,y,t) = \dfrac{t}{t + d(x,y)}$ for all $x,y \in X$ and $t > 0$. Then $(X,M,*)$ is a fuzzy metric space, called standard fuzzy metric space induced by (X,d).

Definition 2.4. [6] Let $(X,M,*)$ be a fuzzy metric space, then

(i) a sequence $\{x_n\}$ in X is said to be convergent and converges to say x if $\lim_{n\to\infty} M(x_n,x,t) = 1$, for all $t > 0$;
(ii) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $M(x_n,x_m,t) > 1 - \epsilon$, for all $t > 0$ and $n,m \ge n_0$;
(iii) a fuzzy metric space $(X,M,*)$ is said to be complete if and only if every Cauchy sequence in X is a convergent sequence.

Remark 2.1. (see [8]).

(a) For all $x,y \in X$, $M(x,y,.)$ is non - decreasing.

(b) It is easy to prove that if $x_n \to x, y_n \to y, t_n \to t$, then $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$.

(c) In a fuzzy metric space $(X, M, *)$, whenever $M(x, y, t) > 1 - r$ for $x, y \in X, t > 0, 0 < r < 1$, we can find a $t_0,\ 0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$.

(d) For any $r_1 > r_2$, we can find r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find r_5 such that $r_3 * r_5 \geq r_4$,, where $r_1, r_2, r_3, r_4, r_5 \in (0, 1)$.

Lemma 2.1. [8] *M is a continuous function on $X \times X \times (0, \infty)$.*

Definition 2.5. [14] Let $(X, M, *)$ be a fuzzy metric space. M is said to satisfy the n - property on $X \times X \times (0, \infty)$ if $\lim_{n\to\infty}[M(x, y, k^n t)]^{n^p} = 1$, whenever $x, y \in X, k > 1$ and $p > 0$.

Lemma 2.2. [10] *Let $(X, M, *)$ be a fuzzy metric space and M satisfies the n - property; then $lim_{t\to+\infty} M(x, y, t) = 1$, for all $x, y \in X$.*

Our aim in this paper is to prove a coupled coincidence point theorem for two mappings in a complete fuzzy metric space endowed with graph.

Definition 2.6. [7] Let X be a nonempty set. An element $(x,\ y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if $T(x,\ y) = x$ and $T(y,\ x) = y$.

Definition 2.7. [7] An element $(x,\ y) \in X \times X$ is called a coupled coincidence point of the mappings $T : X \times X \to X$ and $f : X \to X$ if $T(x,\ y) = fx$ and $T(y,\ x) = fy$.

Let us denote the set of all coupled coincidence points of T and f as $CCoin(Tf)$.

Definition 2.8. [7] An element $(x,\ y) \in X \times X$ is called a coupled common fixed point of the mappings $T : X \times X \to X$ and $f : X \to X$ if $T(x,\ y) = fx = x$ and $T(y,\ x) = fy = y$.

Definition 2.9. [7] Let X be a nonempty set. The mappings $T : X \times X \to X$ and $f : X \to X$ are called commutative if $f(T(x,\ y)) = T(fx,\ fy)$, for all $x,\ y \in X$.

Definition 2.10. [1, 7] A function $f : X \to X$ is called G-continuous if

(i) for all $x,\ x^* \in X$ and for any sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers with $\{f^{n_i}x\} = \{x_{n_i}\} \to x^*$, and $(f^{n_i}x, f^{n_{i+1}}x) = (x_{n_i},\ x_{n_{i+1}}) \in E(G)$, implies $f(f^{n_i}x) = f(x_{n_i}) \to fx^*$ as $i \to \infty$,

(ii) for all $y,\ y^* \in X$ and for any sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers with $\{f^{n_i}y\} = \{y_{n_i}\} \to y^*$ and $(f^{n_i}y, f^{n_{i+1}}y) = (y_{n_i},\ y_{n_{i+1}}) \in E(G^{-1})$, implies $f(f^{n_i}y) = f(y_{n_i}) \to fy^*$ as $i \to \infty$.

Definition 2.11. [1] A function $T : X \times X \to X$ is G-continuous if for all $(x,\ y),\ (x^*,\ y^*) \in X \times X$ and for any sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers with $\{x_{n_i}\} \to x^*,\ \{y_{n_i}\} \to y^*$ as $i \to \infty$ and $(T^{n_i}(x, y), T^{n_{i+1}}(x, y)) = (x_{n_i},\ x_{n_{i+1}}) \in E(G),\ (T^{n_i}(y, x), T^{n_{i+1}}(y, x)) = (y_{n_i},\ y_{n_{i+1}}) \in E(G^{-1})$ implies $T(T^{n_i}(x, y), T^{n_i}(y, x)) = T(x_{n_i},\ y_{n_i}) \to T(x^*,\ y^*)$, $T(T^{n_i}(y, x), T^{n_i}(x, y)) = T(y_{n_i},\ x_{n_i}) \to T(y^*,\ x^*)$ as $i \to \infty$.

3. Main results

In this section, we discuss certain definitions and lemmas which will be necessary for establishing the results of the next section.

Suppose that $(X, M, *)$ is a fuzzy metric space endowed with a directed graph G. Let us consider the mappings $T : X \times X \to X$ and $f : X \to X$. Define the set $(X \times X)_{Tf}$ as $(X \times X)_{Tf} = \{(x,\ y) \in X \times X : (fx,\ T(x,\ y)) \in E(G))$ and $(fy,\ T(y,\ x)) \in E(G^{-1})\}$. We also call the sequences $\{x_n\}$ and $\{y_n\}$ in X as Picard sequences if $T(x_n, y_n) = fx_{n+1}$ and $T(y_n, x_n) = fy_{n+1}$ for all $n = 0, 1, 2, ...$

Definition 3.12. The mapping $T : X \times X \to X$ is called a $G - f$-contraction if

(i) for all $x,\ y,\ u,\ v \in X$, T is $f-$edge preserving, i.e., $(fx,\ fu) \in E(G)$ and $(fy,\ fv) \in E(G^{-1})$ then $(T(x,\ y),\ T(u,\ v)) \in E(G)$ and $(T(y,\ x),\ T(v,\ u)) \in E(G^{-1})$;

(ii) for all $x,\ y,\ u,\ v \in X$ such that $(fx,\ fu) \in E(G)$ and $(fy,\ fv) \in E(G^{-1})$, $M(T(x,\ y),\ T(u,\ v), \alpha t) \geq [M(fx,\ fu, t) * M(fy,\ fv, t))]$, where $\alpha \in (0, 1)$ is called the contraction constant of T.

Lemma 3.3. *Let $T : X \times X \to X$ be a f-edge preserving and $T(X \times X) \subseteq f(X)$, then for every Picard sequences $\{x_n\}, \{y_n\}$ in X the following holds:*

(i) *if $(x_0, y_0) \in (X \times X)_{Tf}$, then $(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \in E(G)$ and $(T(y_n, x_n), T(y_{n+1}, x_{n+1})) \in E(G^{-1})$;*

(ii) *if $(x_0, y_0) \in (X \times X)_{Tf}$, then $(x_{n+1},\ y_{n+1}) \in (X \times X)_{Tf}$*

Proof. Let x_0, y_0 be two arbitrary points in X. Since $T(X \times X) \subseteq f(x)$, we can choose x_n, y_n in X such that $f(x_1) = T(x_0, y_0)$ and $f(y_1) = T(y_0, x_0)$. Continuity this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$f(x_{n+1}) = T(x_n, y_n) \text{ and } f(y_{n+1}) = T(y_n, x_n) \text{ for all } n = 0, 1, 2... \tag{3.1}$$

(i) Let $(x_0, y_0) \in (X \times X)_{Tf}$ i.e. $(fx_0, T(x_0, y_0)) \in E(G)$ and $(fy_0, T(y_0, x_0)) \in E(G^{-1})$ which imply $(fx_0, fx_1) \in E(G)$ and $(fy_0, fy_1) \in E(G^{-1})$ Now, by the f-edge preserving property of T, we get that $(T(x_0, y_0), T(x_1, y_1)) \in E(G)$ and $(T(y_0, x_0), T(y_1, x_1)) \in E(G^{-1})$. Continuity in this way, we set $(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \in E(G)$ and $(T(y_n, x_n), T(y_{n+1}, x_{n+1})) \in E(G^{-1})\ \forall\ n = 0, 1, 2.....$

(ii) Let $(x_0, y_0) \in (X \times X)_{Tf}$, then from ($i$) we have $(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \in E(G)$ and $(T(y_n, x_n), T(y_{n+1}, x_{n+1})) \in E(G^{-1})$. Thus, by the definition of $(X \times X)_{Tf}$, we have $(x_{n+1},\ y_{n+1}) \in (X \times X)_{Tf}$. □

Lemma 3.4. *Suppose $(X, M, *)$ is a complete fuzzy metric space endowed with a directed graph G with $*$ is a continuous t - norms of H- type. Let $T : X \times X \to X$ be a $G - f-$contraction with contraction constant $\alpha \in (0,\ 1)$ and $T(X \times X) \subseteq f(X)$. Also let $\{x_n\},\ \{y_n\}$ be Picard sequences in X. Then, for each $(x_0, y_0) \in (X \times X)_{Tf}$ there exist $x^*,\ y^* \in X$ such that $fx_n \to x^*$ and $fy_n \to y^*$, as $n \to \infty$.*

Proof. Let $(x_0, y_0) \in (X \times X)_{Tf}$. Then by Lemma 3.3, we have $(T(x_n, y_n), T(x_{n+1}, y_{n+1})) \in E(G)$ and $(T(y_n, x_n), T(y_{n+1}, x_{n+1})) \in E(G^{-1})\ \forall\ n = 0, 1, 2, ...$ i.e. $(fx_n, fx_{n+1}) \in E(G)$ and $(fy_n, fy_{n+1}) \in E(G^{-1})\ \forall\ n = 0, 1, 2, ...$

Since $*$ is a t - norm of H - type, then for any $\lambda > 0$, there exists a $\theta > 0$ such that

$$\underbrace{(1-\theta) * (1-\theta) * ... * (1-\theta)}_{k-times} \geq 1 - \lambda, \text{ for all } k \in N. \tag{3.2}$$

Since $M(x, y, .)$ is continuous and $lim_{t \to +\infty} M(x, y, t) = 1$ for all $x, y \in X$ there exists $t_0 > 0$ such that

$$\begin{cases} M(fx_0, fx_1, t_0) \geq 1 - \theta, \\ M(fy_0, fy_1, t_0) \geq 1 - \theta. \end{cases} \tag{3.3}$$

On the other hand, for any $t > 0$, there exists $n_0 \in \mathbb{N}$ and $t_0 < t$ such that

$$t > \sum_{i=n_0}^{\infty} \alpha^i(t_0), \tag{3.4}$$

As F is a $G-f$-contraction, so we have $M(fx_1, fx_2, \alpha(t_0)) = M(T(x_0, y_0), T(x_1, y_1), \alpha(t_0))$

$$\geq M(fx_0, fx_1, t_0)*M(fy_0, fy_1, t_0).$$

and $M(fy_1, fy_2, \alpha(t_0)) = M(T(y_0, x_0), T(y_1, x_1), \alpha(t_0)) \geq M(fy_0, fy_1, t_0)*M(fx_0, fx_1, t_0)$.
Similarly, we can also get

$$\begin{aligned} M(fx_2, fx_3, \alpha^2(t_0)) &= M(T(x_1, y_1), T(x_2, y_2), \alpha^2(t_0)) \\ &\geq M(fx_1, fx_2, \alpha(t_0)) * M(fy_1, fy_2, \alpha(t_0)) \\ &= M(T(x_0, y_0), T(x_1, y_1), \alpha(t_0)) \\ &\quad *M(T(y_0, x_0), T(y_1, x_1), \alpha(t_0)) \\ &\geq M(fx_0, fx_1, t_0) * M(fy_0, fy_1, t_0) \\ &\quad *M(fy_0, fy_1, t_0) * M(fx_0, fx_1, t_0) \\ &= [M(fx_0, fx_1, t_0)]^2 * [M(fy_0, fy_1, t_0)]^2. \end{aligned}$$

$$\begin{aligned} M(fy_2, fy_3, \alpha^2(t_0)) &= M(T(y_1, x_1), T(y_2, x_2), \alpha^2(t_0)) \\ &\geq M(fy_1, fy_2, \alpha(t_0)) * M(fx_1, fx_2, \alpha(t_0)) \\ &= M(T(y_0, x_0), T(y_1, x_1), \alpha(t_0)) \\ &\quad *M(T(x_0, y_0), T(x_1, y_1), \alpha(t_0)) \\ &\geq M(fy_0, fy_1, t_0) * M(fx_0, fx_1, t_0) \\ &\quad *M(fx_0, fx_1, t_0) * M(fy_0, fy_1, t_0) \\ &= [M(fy_0, fy_1, t_0)]^2 * [M(fx_0, fx_1, t_0)]^2. \end{aligned}$$

Continuing in the same way, we get

$$\begin{aligned} M(fx_n, fx_{n+1}, \alpha^n(t_0)) &\geq [M(fx_0, fx_1, t_0)]^{2^{n-1}} * [M(fy_0, fy_1, t_0)]^{2^{n-1}}, \\ M(fy_n, fy_{n+1}, \alpha^n(t_0)) &\geq [M(fy_0, fy_1, t_0)]^{2^{n-1}} * [M(fx_0, fx_1, t_0)]^{2^{n-1}}, \end{aligned}$$

So, from (3.3) and (3.4), for $m > n \geq n_0$, we have

$$\begin{aligned} M(fx_n, fx_m, t) &\geq M(fx_n, fx_m, \sum_{i=n_0}^{\infty} \alpha^i(t_0)) \geq M(fx_n, fx_m, \sum_{i=n}^{m-1} \alpha^i(t_0)) \\ &\geq M(fx_n, fx_{n+1}, \alpha^n(t_0)) * M(fx_{n+1}, fx_{n+2}, \alpha^{n+1}(t_0)) \\ &\quad * \cdots * M(fx_{m-1}, fx_m, \alpha^{m-1}(t_0)) \\ &\geq [M(fy_0, fy_1, t_0)]^{2^{n-1}} * [M(fx_0, fx_1, t_0)]^{2^{n-1}} \\ &\quad *[M(fy_0, fy_1, t_0)]^{2^n} * [M(fx_0, fx_1, t_0)]^{2^n} \\ &\quad * \cdots * [M(fy_0, fy_1, t_0)]^{2^{m-2}} * [M(fx_0, fx_1, t_0)]^{2^{m-2}} \\ &= [(M(fy_0, fy_1, t_0))]^{2^{(m-n)(m+n-3)}} * [(M(fx_0, fx_1, t_0))^{2^{(m-n)(m+n-3)}} \\ &\geq (1-\theta) * (1-\theta) * \ldots * (1-\theta) \geq 1-\lambda \end{aligned}$$

which implies that

$$M(fx_n, fx_m, t) > 1 - \lambda, \tag{3.5}$$

for all $m, n \in N$ with $m > n \geq n_0$ and $t > 0$. Hence, $\{fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Similarly, we can get that $\{fy_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Also, $(X, M, *)$ is complete. So, there exist, (say) $x^*,\ y^* \in X$ such that $\lim_{n\to\infty} fx_n = x^*$ and $\lim_{n\to\infty} fy_n = y^*$. □

We now discuss our main results.

Theorem 3.1. *Suppose that $(X, M, *)$ is a complete fuzzy metric space endowed with a directed graph G with $*$ is a continuous t - norms of H- type. Let $T : X \times X \to X$ be a $G - f$-contraction with contraction constant $\alpha \in (0,1)$ and $T(X \times X) \subseteq f(X)$. Let f be G-continuous and commutes with T. Also, T is G - continuous. Then $CCoin(Tf) \neq \phi$ iff $(X \times X)_{Tf} \neq \phi$.*

Proof. Suppose that $CCoin(Tf) \neq \phi$. Then there exists some $(x^*,\ y^*) \in CCoin(Tf)$, i.e., $fx^* = T(x^*,\ y^*)$ and $fy^* = T(y^*,\ x^*)$. So, $(fx^*,\ T(x^*,\ y^*)) = (fx^*,\ fx^*) \in \Delta \subset E(G)$ and $(fy^*,\ T(y^*,\ x^*) = (fy^*,\ fy^*) \in \Delta \subset E(G^{-1})$ which means $(x^*,\ y^*) \in (X \times X)_{Tf}$ i. e. $(X \times X)_{Tf} \neq \phi$. Next, let us assume that $(X \times X)_{Tf} \neq \phi$. Then there exists $(x_0,\ y_0) \in (X \times X)_{Tf}$, i.e., $(fx_0, T(x_0, y_0)) \in E(G)$ and $(fy_0,\ T(y_0, x_0)) \in E(G^{-1})$. Then, by Lemma (3.3), we have a sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers such that $(T(x_{n_i},\ y_{n_i}),\ T(x_{n_{i+1}},\ y_{n_{i+1}})) \in E(G)$ and $(T(y_{n_i},\ x_{n_i}),\ T(y_{n_{i+1}},\ x_{n_{i+1}})) \in E(G^{-1})$. Using (3.1), we have

$$(fx_{n_{i+1}},\ fx_{n_{i+2}}) \in E(G) \text{ and } (fy_{n_{i+1}},\ fy_{n_{i+2}}) \in E(G^{-1}). \tag{3.6}$$

Also, from Lemma (3.4), we have

$$\lim_{n\to\infty} fx_{n_i} = x^* \text{ and } \lim_{n\to\infty} fy_{n_i} = y^* \tag{3.7}$$

But f is G-continuous, so we get that

$$\lim_{n\to\infty} f(fx_{n_i}) = fx^* \text{ and } \lim_{n\to\infty} f(fy_{n_i}) = fy^* \tag{3.8}$$

Using (3.4) and the commutativity of T and f gives us

$$f(fx_{n_{i+1}}) = f(T(x_{n_i},\ y_{n_i}) \text{ and } f(fy_{n_{i+1}}) = f(T(y_{n_i},\ x_{n_i}))$$

$$\text{i.e. } f(fx_{n_{i+1}}) = (T(fx_{n_i},\ fy_{n_i}) \text{ and } f(fy_{n_{i+1}}) = (T(fy_{n_i},\ fx_{n_i})) \tag{3.9}$$

Finally, we show that $fx^* = T(x^*,\ y^*)$ and $fy^* = T(y^*,\ x^*)$. Let T be G - continuous. Then from (3.6), (3.7) and (3.9), we have $\lim_{n\to\infty} f(fx_{n_{i+1}}) = \lim_{n\to\infty} T(fx_{n_i},\ fy_{n_i})$ i.e. $fx^* = T(x^*,\ y^*))$ and $\lim_{n\to\infty} f(fy_{n_{i+1}}) = T(fy_{n_i},\ fx_{n_i})$ i.e. $fy^* = T(y^*, x^*)$. Thus, $(x^*,\ y^*)$ is coupled coincidence point of the mapping T and f, i.e., $CCoin(Tf) \neq \phi$. □

Theorem 3.2. *Suppose that the hypotheses of Theorem (3.1) hold. Besides, let for every $(x, y), (x^*, y^*) \in X \times X$ there exist $(u,\ v) \in X \times X$ such that $(T(x,\ y),\ T(u,\ v)) \in E(G)$, $(T(y,\ x),\ T(v,\ u)) \in E(G^{-1})$ and $(T(x^*,\ y^*),\ T(u,\ v)) \in E(G)$, $(T(y^*,\ x^*),\ T(v,\ u)) \in E(G^{-1})$. Then T and f have unique coupled common fixed point.*

Proof. Let $(x,\ y)$ and (x^*, y^*) be coupled coincidence points of T and f , i.e.,

$$fx = T(x,\ y) \text{ and } fy = T(y,\ x) \tag{3.10}$$

and

$$fx^* = T(x^*, y^*) \text{ and } fy^* = T(y^*,\ x^*) \tag{3.11}$$

By hypothesis, we have

$$(T(x,\ y),\ T(u,\ v)) \in E(G), \text{ and } (T(y,\ x),\ T(v,\ u)) \in E(G^{-1}) \tag{3.12}$$

$$(T(x^*,\ y^*),\ T(u,\ v)) \in E(G), \text{ and } (T(y^*,\ x^*),\ T(v,\ u)) \in E(G^{-1}) \tag{3.13}$$

Set $T(u_n,\ v_n) = fu_{n+1},\ u_0 = u$, and $T(v_n,\ u_n) = fv_{n+1},\ v_0 = v$. Then, using (3.10) and (3.11); (3.12) and (3.13) we have $(fx,\ fu_1) \in E(G),\ (fy,\ fv_1) \in E(G^{-1})$ and$(fx^*,\ fu_1) \in E(G),\ (fy^*,\ fv_1) \in E(G^{-1})$. But T is f-edge preserving, so $(T(x,\ y),\ T(u_1,\ v_1)) \in E(G),\ (T(y,\ x),\ T(v_1,\ u_1)) \in E(G^{-1})$ and $(T(x^*,\ y^*),\ T(u_1,\ v_1)) \in E(G),\ (T(y^*,\ x^*), T(v_1,\ u_1) \in E(G^{-1})$i.e. $(fx,\ fu_2) \in E(G),\ (fy,\ fv_2) \in E(G^{-1})$ and $(fx^*,\ fu_2) \in E(G),\ (fy^*,\ fv_2) \in E(G^{-1})$ Using the f-edge preserving property of T repeatedly, we obtain $(fx,\ fu_n) \in E(G)$, $(fy,\ fv_n) \in E(G^{-1})$ and $(fx^*,\ fu_n) \in E(G)$, $(fy^*,\ fv_n) \in E(G^{-1})$ Since $*$ is a t - norm of H - type, then for any $\lambda > 0$, there exists a $\theta > 0$ such that

$$\underbrace{(1-\theta) * (1-\theta) * ... * (1-\theta)}_{k-times} \geq 1-\lambda, \text{ for all } k \in N. \tag{3.14}$$

Since $M(x, y, .)$ is continuous and $lim_{t\to+\infty}M(x, y, t) = 1$ for all $x, y \in X$ there exists $t_0 > 0$ such that $M(fx, fx^*, t_0) \geq 1-\theta, M(fy, fy^*, t_0) \geq 1-\theta$. On the other hand, for any $t > 0$, there exists $n_0 \in \mathbb{N}$ and $t_0 < t$ such that $t > \sum_{i=n_0}^{\infty} \alpha^i(t_0)$, and for $t_0 > 0$, we have

$$M(fx,\ fx^*, t_0) \geq M\left(fx,\ fu_{n+1}, \frac{t_0}{2}\right) * M\left(fu_{n+1},\ fx^*, \frac{t_0}{2}\right)$$

$$= M\left(T(x,\ y),\ T(u_n,\ v_n), \frac{t_0}{2}\right) * M\left(T(u_n,\ v_n),\ T\left(x^*,\ y^*, \frac{t_0}{2}\right)\right)$$

$$\geq M\left(fx,\ fu_n, \frac{t_0}{2^2}\right) * M\left(fy,\ fv_n, \frac{t_0}{2^2}\right) * M\left(fu_n,\ fx^*, \frac{t_0}{2^2}\right) * M\left(fv_n,\ fy^*, \frac{t_0}{2^2}\right)$$

$$= M\left(T(x,\ y),\ T(u_{n-1},\ v_{n-1}), \frac{t_0}{2^2}\right) * M\left(T(y,\ x),\ T(v_{n-1},\ u_{n-1}), \frac{t_0}{2^2}\right)$$

$$* M\left(T(u_{n-1},\ v_{n-1}), T(x^*,\ y^*), \frac{t_0}{2^2}\right) * M\left(T(v_{n-1},\ u_{n-1}),\ T(y^*,\ x^*), \frac{t_0}{2^2}\right)$$

$$\geq \left[M\left(fx,\ fu_{n-2}, \frac{t_0}{2^3}\right)\right]^2 * \left[M\left(fy,\ fv_{n-2}, \frac{t_0}{2^3}\right)\right]^2 * M\left[\left(fu_{n-2},\ fx^*, \frac{t_0}{2^3}\right)\right]^2$$

$$* \left[M\left(fv_{n-2},\ fy^*, \frac{t_0}{2^3}\right)\right]^2$$

$$\vdots$$

$$\geq \left[M\left(fx,\ fu_0, \frac{t_0}{2^{n+1}}\right)\right]^{2^n} * \left[M\left(fy,\ fv_0, \frac{t_0}{2^{n+1}}\right)\right]^{2^n} * \left[M\left(fu_0,\ fx^*, \frac{t_0}{2^{n+1}}\right)\right]^{2^n}$$

$$* \left[M\left(fv_0,\ fy^*, \frac{t_0}{2^{n+1}}\right)\right]^{2^n}$$

i.e $\geq (1-\theta) * (1-\theta) * ... * (1-\theta) \geq 1-\lambda$

which implies that

$$fx = fx^* \text{ as } n \to \infty. \tag{3.15}$$

Similarly,

$$fy = fy^*. \tag{3.16}$$

Let $fx = fx^* = m$ and $fy = fy^* = n$. Then, using commutativity of T and f along with (3.10) gives $f(fx) = f(T(x, y)) = T(fx, fy)$ and $f(fy) = f(T(y, x)) = T(fy, fx)$ i.e. $fm = T(m, n)$ and $fn = T(n, m)$. Thus, $(m,\ n)$ is a coupled coincidence point. So, repeating the earlier argument for $(x,\ y)$ and $(m,\ n), fx = fm$ and $fy = fn$ i.e. $m = fm$ and $n = fn$. Thus, $m = fm = T(m,\ n)$ and $n = fn = T(n,\ m)$. So, $(m,\ n)$ is coupled common fixed point of T and f. Finally, we prove that the coupled common fixed point of T and f is unique. Let us suppose that (p, q) is another coupled common fixed point of T and f. Then

$$p = gp = T(p, q) \text{ and } q = gq = T(q, p). \tag{3.17}$$

But, from (3.15) and (3.16), we have $fp = fm = m$ and $fq = fn = n$. So, from (3.17), we have $p = m$ and $q = n$. Hence the coupled common fixed point of T and f is unique. □

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