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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Improvements on Bogin-type fixed point theorems in complete metric spaces

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ABSTRACT. We improve Bogin-type fixed point theorems in complete metric spaces. We also compare these theorems with a Ćirić-type fixed point theorem and others.

1. INTRODUCTION

In 1976, Bogin proved the following fixed point theorem.

Theorem 1.1. (Theorem 1 in [1]) *Let T be a mapping on a complete metric space (X, d). Assume that there exist $r \in [0, 1)$ and $s, t \in (0, 1/2)$ satisfying $r + 2\,s + 2\,t = 1$ and*

$$d(Tx, Ty) \le r\,d(x, y) + s\,d(x, Ty) + s\,d(Tx, y) + t\,d(x, Tx) + t\,d(y, Ty)$$

for all $x, y \in X$. Then T has a unique fixed point.

The author thinks that Theorem 1.1 is splendid because the value of $r + 2\,s + 2\,t$ is equal to 1, that is, the assumption is not "$r + 2\,s + 2\,t < 1$" in Theorem 1.1.
In [12], the following theorem was proved. See also [4, 6, 7, 9, 10] and references therein.

Theorem 1.2. (Theorem 5.4 in [12]) *Let T be a mapping on a complete metric space (X, d). Assume that there exist $r \in [0, 1)$, $s, t \in (0, 1/2)$ and $\eta \in H\big((1-s)/t\big)$ satisfying $r + 2\,s + 2\,t = 1$ and*

$$\eta\big(d(Tx, Ty)\big) \le r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big) + t\,\eta\big(d(x, Tx)\big) + t\,\eta\big(d(y, Ty)\big) \tag{1.1}$$

for all $x, y \in X$, where for $\upsilon \in (1, \infty)$, the set $H(\upsilon)$ is defined as follows: $\eta \in H(\upsilon)$ iff η is a function from $[0, \infty)$ into itself satisfying the following:

- *(H1) For any sequence $\{a_n\}$ in $[0, \infty)$, $\lim_n \eta(a_n) = 0$ iff $\lim_n a_n = 0$.*
- *(H2:υ) For any sequence $\{a_n\}$ in $[0, \infty)$ which converges to some $\alpha \in (0, \infty)$,*

$$\eta(\alpha) < \upsilon \limsup_{n\to\infty} \eta(a_n)$$

holds.

Assume also the following for some $u \in X$:

- *(H3) $\big\{\eta\big(d(T^m u, T^n u)\big) : m, n \in \mathbb{N} \cup \{0\}\big\}$ is bounded.*

Then T has a unique fixed point.

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We note that in Theorem 1.1, we can prove the boundedness of $\{T^n u\}$ for any $u \in X$. So it is unclear whether Theorem 1.2 is a generalization of Theorem 1.1. That is, some mathematicians consider that and some do not. Therefore it is meaningful to give a sufficient condition for (H3). In [11], it was proved that the following is a sufficient condition for (H3):

(H4) For any $\beta > 0$ and $\varepsilon > 0$, there exists $M > 0$ such that $\eta(a) < (1+\varepsilon)\,\eta(a+b)$ holds for any $a > 0$ and $b \in [-\beta, +\beta]$ satisfying $\eta(a) > M$ and $a + b > 0$.

On the other hand, very recently, fixed point theorems for Kannan [5], Chatterjea [2] and Ćirić [3] types of contractions was proved in [8] (Theorems 2.3–2.5 below). For example, in Theorem 2.5, we assume the following:

(H5) There exists $\varepsilon > 0$ such that for any $\beta > 0$, there exists $M > 0$ such that

$$\eta(a) < (1+\varepsilon)\,\eta(a+b)$$

holds for any $a > 0$ and $b \in [-\beta, +\beta]$ satisfying $\eta(a) > M$ and $a + b > 0$.

It is obvious that (H5) is weaker than (H4). So it is a natural question of whether (H5) is a sufficient condition for (H3). In this paper, we give an affirmative answer to this question (see Corollary 3.1). Also, in order to compare our theorems with the former results (Theorems 2.3–2.5), we prove fixed point theorems under the inequality (3.7) below which is a little weaker than (1.1).

2. Known results

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In this section, we give some results in [8, 12]. Examples 2.1 and 2.2 show that (H4) is much stronger than (H5).

Example 2.1. (Example 2.5 in [8]) Let η be a function from $[0,\infty)$ into itself. Assume that there exist $q \in (0,\infty)$ and $s, t \in (0,\infty)$ satisfying $s\,a^q \le \eta(a) \le t\,a^q$ for any $a \in [0,\infty)$. Then η satisfies (H1) and (H5).

Example 2.2. (Example 4.2 in [8]) Fix $q \in (0,\infty)$ and define a continuous function f from $[0,\infty)$ into $[1,2]$ by $f(a) = 2 - \min\{|a - 2\,n| : n \in \mathbb{N} \cup \{0\}\}$. Define a continuous function η from $[0,\infty)$ into itself by $\eta(a) = f(a)\,a^q$. Then the following hold:

(i) $\eta \in H(v)$ holds for any $v \in (1,\infty)$ and η satisfies (H5).
(ii) η does not satisfy (H4).

Lemma 2.1. (Lemma 2.2 in [12]) *Let $\eta \in H(v)$ for some $v \in (1,\infty)$. Then the following hold*:

(i) $\eta(\alpha) = 0$ iff $\alpha = 0$.
(ii) If $\{a_n\}$ is a sequence in $[0,\infty)$ which converges to some $\alpha \in [0,\infty)$ and

$$v \limsup_{n\to\infty} \eta(a_n) \le \eta(\alpha)$$

holds, then $\alpha = 0$ holds.

Lemma 2.2. (Lemma 5.3 in [12]) *Let T be a mapping on a metric space (X,d). Assume that there exist $r \in [0,1)$, $s, t \in (0,1/2)$ and a function η from $[0,\infty)$ into itself satisfying* (H1), *$r + 2\,s + 2\,t = 1$ and* (1.1) *for all $x, y \in X$. Assume also that* (H3) *holds for some $u \in X$. Then $\{T^n u\}$ is a Cauchy sequence.*

Remark 2.1. We note that we do not use (H2:$(1-s)/t$) in the proof of Lemma 5.3 in [12].

In the following three theorems, we let T be a mapping on a complete metric space (X,d) and let η be a function from $[0,\infty)$ into itself satisfying (H1).

Theorem 2.3. (Theorem 3.3 in [8]) *Assume that there exists* $\alpha \in (0, 1/2)$ *satisfying*

$$\eta\big(d(Tx, Ty)\big) \le \alpha\, \eta\big(d(x, Tx)\big) + \alpha\, \eta\big(d(y, Ty)\big)$$

for all $x, y \in X$. *Assume also* (H2:$1/\alpha$). *Then the following holds.*

(A) T *has a unique fixed point* z. *Moreover* $\{T^n x\}$ *converges to* z *for any* $x \in X$.

Theorem 2.4. (Corollary 3.5 in [8]) *Assume that there exists* $\alpha \in [0, 1/2)$ *satisfying*

$$\eta\big(d(Tx, Ty)\big) \le \alpha\, \eta\big(d(x, Ty)\big) + \alpha\, \eta\big(d(Tx, y)\big)$$

for all $x, y \in X$. *Assume also* (H5). *Then* (A) *holds.*

Theorem 2.5. (Theorem 3.2 in [8]) *Assume that there exists* $\rho \in (0, 1)$ *satisfying*

$$\begin{aligned}\eta\big(d(Tx, Ty)\big) \le \rho \max\big\{&\eta\big(d(x, y)\big),\ \eta\big(d(x, Ty)\big),\ \eta\big(d(Tx, y)\big),\\ &\eta\big(d(x, Tx)\big),\ \eta\big(d(y, Ty)\big)\big\}\end{aligned}$$

for all $x, y \in X$. *Assume also* (H2:$1/\rho$) *and* (H5). *Then* (A) *holds.*

3. Fixed point theorems

In this section, we prove fixed point theorems.

Theorem 3.6. *Let* T *be a mapping on a complete metric space* (X, d). *Assume that there exist* $r, s, \sigma, t, \tau \in \mathbb{R}$ *and a function* η *from* $[0, \infty)$ *into itself satisfying* (H1),

$$r + s + \sigma + t + \tau = 1, \tag{3.2}$$

$$r \ge 0, \tag{3.3}$$

$$1 - s > 0, \tag{3.4}$$

$$s + \sigma > 0, \tag{3.5}$$

$$t + \tau > 0 \tag{3.6}$$

and

$$\begin{aligned}\eta\big(d(Tx, Ty)\big) \le{}& r\, \eta\big(d(x, y)\big) + s\, \eta\big(d(x, Ty)\big) + \sigma\, \eta\big(d(Tx, y)\big)\\ &+ t\, \eta\big(d(x, Tx)\big) + \tau\, \eta\big(d(y, Ty)\big)\end{aligned} \tag{3.7}$$

for all $x, y \in X$. *Assume also that there exists* $u \in X$ *satisfying* (H3). *In the case where* $\tau > 0$, *assume* $\eta \in H\big((1-s)/\tau\big)$. *Then* T *has a unique fixed point* z. *Moreover* $\{T^n u\}$ *converges to* z.

Proof. By (3.7), we have

$$\begin{aligned}\eta\big(d(Tx, Ty)\big) \le{}& r\, \eta\big(d(x, y)\big) + \frac{s + \sigma}{2}\Big(\eta\big(d(x, Ty)\big) + \eta\big(d(Tx, y)\big)\Big)\\ &+ \frac{t + \tau}{2}\Big(\eta\big(d(x, Tx)\big) + \eta\big(d(y, Ty)\big)\Big)\end{aligned}$$

for all $x, y \in X$; see the proof of Proposition 2.6 in [12]. By Lemma 2.2, $\{T^n u\}$ is a Cauchy sequence in X. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. Since

$$\begin{aligned}\eta\big(d(T^{n+1}u, Tz)\big) \le{}& r\, \eta\big(d(T^n u, z)\big) + s\, \eta\big(d(T^n u, Tz)\big) + \sigma\, \eta\big(d(T^{n+1}u, z)\big)\\ &+ t\, \eta\big(d(T^n u, T^{n+1}u)\big) + \tau\, \eta\big(d(z, Tz)\big),\end{aligned}$$

we have

$$(1 - s) \liminf_{n \to \infty} \eta\big(d(T^n u, Tz)\big) \le \tau\, \eta\big(d(z, Tz)\big)$$

by (H1). We consider the following two cases:

- $\tau \le 0$
- $\tau > 0$

In the first case, $\liminf_n \eta\big(d(T^n u, Tz)\big) = 0$ holds. By (H1), $\{T^n u\}$ converges to Tz. Hence $Tz = z$ holds. In the second case, $d(Tz, z) = 0$ holds by Lemma 2.1 (ii). So, $Tz = z$ holds. Therefore in both cases, z is a fixed point of T. Let us prove that the fixed point z is unique. Let $w \in X$ be a fixed point of T. Then we have

$$\begin{aligned} \eta\big(d(z,w)\big) &= \eta\big(d(Tz,Tw)\big) \\ &\le r\,\eta\big(d(z,w)\big) + s\,\eta\big(d(z,Tw)\big) + \sigma\,\eta\big(d(Tz,w)\big) \\ &\qquad + t\,\eta\big(d(z,Tz)\big) + \tau\,\eta\big(d(w,Tw)\big) \\ &= (r+s+\sigma)\,\eta\big(d(z,w)\big). \end{aligned}$$

Since $r+s+\sigma = 1-(t+\tau) < 1$, we have $\eta\big(d(z,w)\big) = 0$. Hence by (H1), we obtain $z = w$. We have shown that the fixed point is unique. □

Theorem 3.7. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $r, s, \sigma, t, \tau \in [0,\infty)$ and a function η from $[0,\infty)$ into itself satisfying* (H1), (3.2), (3.5), (3.6) *and* (3.7) *for all $x, y \in X$. In the case where $\tau > 0$, assume $\eta \in H\big((1-s)/\tau\big)$. Assume also* (H5). *Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Proof. Since we assume $s, \sigma, t, \tau \ge 0$, we note that (3.2)–(3.6) hold. By Theorem 3.6, we only have to show that (H3) holds for any $x \in X$. Fix $x \in X$. For $m, n \in \mathbb{N} \cup \{0\}$ with $m \le n$, we set $D(m,n) \in [0,\infty)$ by

$$D(m,n) = \max\big\{\eta\big(d(T^i x, T^j x)\big) : m \le i \le j \le n\big\}.$$

Arguing by contradiction, we will show

$$D(m,n) = \max\big\{\eta\big(d(T^m x, T^j x)\big) : m \le j \le n\big\}. \tag{3.8}$$

If not, there exist $m, n \in \mathbb{N} \cup \{0\}$ satisfying $m \le n$ and

$$D(m,n) > \max\big\{\eta\big(d(T^m x, T^j x)\big) : m \le j \le n\big\}.$$

Then we can choose $k \in \mathbb{N}$ satisfying $m < k \le n$,

$$D(m,n) = \max\big\{\eta\big(d(T^k x, T^j x)\big) : k \le j \le n\big\}$$

and

$$D(m,n) > \max\big\{\eta\big(d(T^i x, T^j x)\big) : i \le j \le n\big\}$$

for any $i \in \mathbb{N} \cup \{0\}$ with $m \le i < k$. We note $D(m,n) > 0$. We choose $\ell \in \mathbb{N}$ satisfying $k < \ell \le n$,

$$D(m,n) = \eta\big(d(T^k x, T^\ell x)\big)$$

and

$$D(m,n) > \eta\big(d(T^k x, T^i x)\big)$$

for any $i \in \mathbb{N} \cup \{0\}$ with $k \le i < \ell$. Noting (3.5), we have

$$\begin{aligned} D(m,n) &= \eta\big(d(T^k x, T^\ell x)\big) \\ &\le r\,\eta\big(d(T^{k-1}x, T^{\ell-1}x)\big) + s\,\eta\big(d(T^{k-1}x, T^\ell x)\big) + \sigma\,\eta\big(d(T^k x, T^{\ell-1}x)\big) \\ &\qquad + t\,\eta\big(d(T^{k-1}x, T^k x)\big) + \tau\,\eta\big(d(T^{\ell-1}x, T^\ell x)\big) \\ &< r\,\eta\big(d(T^{k-1}x, T^{\ell-1}x)\big) + (s+\sigma)\,D(m,n) \\ &\qquad + t\,\eta\big(d(T^{k-1}x, T^k x)\big) + \tau\,\eta\big(d(T^{\ell-1}x, T^\ell x)\big) \\ &\le D(m,n), \end{aligned}$$

which implies a contradiction. Therefore we have shown (3.8). For $m, n \in \mathbb{N} \cup \{0\}$ with $m \le n$, let us prove

$$D(m,n) \le \upsilon^m\, D(0,n) + D(0,m), \tag{3.9}$$

where $\upsilon := 1 - \max\{t, \tau\} \in (0,1)$. We choose $k \in \mathbb{N}$ satisfying $m \le k \le n$ and $D(m,n) = \eta\big(d(T^m x, T^k x)\big)$. We have

$$\begin{aligned} D(m,n) &= \eta\big(d(T^m x, T^k x)\big) \\ &\le r\,\eta\big(d(T^{m-1}x, T^{k-1}x)\big) \\ &\qquad + s\,\eta\big(d(T^{m-1}x, T^k x)\big) + \sigma\,\eta\big(d(T^m x, T^{k-1}x)\big) \\ &\qquad + t\,\eta\big(d(T^{m-1}x, T^m x)\big) + \tau\,\eta\big(d(T^{k-1}x, T^k x)\big) \\ &\le (1-t)\, D(m-1,n) + t\, D(0,m) \end{aligned}$$

and

$$\begin{aligned} D(m,n) &= \eta\big(d(T^k x, T^m x)\big) \\ &\le r\,\eta\big(d(T^{k-1}x, T^{m-1}x)\big) \\ &\qquad + s\,\eta\big(d(T^{k-1}x, T^m x)\big) + \sigma\,\eta\big(d(T^k x, T^{m-1}x)\big) \\ &\qquad + t\,\eta\big(d(T^{k-1}x, T^k x)\big) + \tau\,\eta\big(d(T^{m-1}x, T^m x)\big) \\ &\le (1-\tau)\, D(m-1,n) + \tau\, D(0,m). \end{aligned}$$

Hence

$$D(m,n) \le \upsilon\, D(m-1,n) + (1-\upsilon)\, D(0,m)$$

holds because either $\upsilon = 1-t$ or $\upsilon = 1-\tau$ holds. Using this, we have

$$\begin{aligned} D(m,n) &\le \upsilon\, D(m-1,n) + (1-\upsilon)\, D(0,m) \\ &\le \upsilon^2\, D(m-2,n) + \upsilon\,(1-\upsilon)\, D(0,m-1) + (1-\upsilon)\, D(0,m) \\ &\le \upsilon^2\, D(m-2,n) + (1-\upsilon^2)\, D(0,m) \\ &\le \upsilon^3\, D(m-3,n) + (1-\upsilon^3)\, D(0,m) \\ &\le \cdots \le \upsilon^m\, D(0,n) + (1-\upsilon^m)\, D(0,m) \\ &\le \upsilon^m\, D(0,n) + D(0,m). \end{aligned}$$

Choose $\varepsilon > 0$ appearing in (H5). We can choose $\kappa \in \mathbb{N}$ satisfying

$$(1+\varepsilon)\, \upsilon^\kappa < 1.$$

Arguing by contradiction, we assume $\lim_n D(0,n) = \infty$. Then we note that $T^n x$ $(n = 0, 1, 2, \cdots)$ are all different. Put $\beta = d(x, T^\kappa x)$. Then we can choose $M > 0$ appearing in (H5). Also, we can choose $\ell \in \mathbb{N}$ satisfying

$$\begin{aligned} &\ell > \kappa, \\ &\eta\big(d(x, T^\ell x)\big) = D(0,\ell), \\ &\eta\big(d(x, T^\ell x)\big) > M, \\ &(1+\varepsilon)\left(\upsilon^\kappa + \frac{D(0,\kappa)}{D(0,\ell)}\right) < 1. \end{aligned}$$

We have

$$\begin{aligned} D(0,\ell) &= \eta\big(d(x, T^\ell x)\big) \\ &< (1+\varepsilon)\, \eta\big(d(T^\kappa x, T^\ell x)\big) \end{aligned}$$

$$
\begin{aligned}
&\le (1+\varepsilon)\, D(\kappa,\ell) \\
&\le (1+\varepsilon)\, \big(\upsilon^{\kappa} D(0,\ell) + D(0,\kappa)\big) \\
&\le (1+\varepsilon)\, \Big(\upsilon^{\kappa} + \frac{D(0,\kappa)}{D(0,\ell)}\Big)\, D(0,\ell) \\
&< D(0,\ell),
\end{aligned}
$$

which implies a contradiction. Therefore $\lim_n D(0,n) < \infty$ holds. Thus, (H3) holds. □

By Theorem 3.7, we obtain the following, which is the main result in this paper.

Corollary 3.1. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $r \in [0,1)$, $s,t \in (0,1/2)$ and $\eta \in H\big((1-s)/t\big)$ satisfying $r + 2\,s + 2\,t = 1$ and* (1.1) *for all $x,y \in X$. Assume also* (H5)*. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

4. Comparisons

In this section, we compare Theorem 3.7 with Theorem 2.3–2.5.

Proposition 4.1. *Assume the assumption of Theorem 2.3. Assume also* (H5)*. Then the assumption of Theorem 3.7 holds.*

Remark 4.2. Since we assume (H5), Theorem 3.7 is not a generalization of Theorem 2.3.

Proof. Put $r = 0$, $s = 0$, $\sigma = 1 - 2\,\alpha$, $t = \alpha$ and $\tau = \alpha$. Then (3.2)–(3.7) hold. We also have $(1-s)/\tau = 1/\alpha$. □

Proposition 4.2. *Assume the assumption of Theorem 2.4. Then the assumption of Theorem 3.7 holds.*

Remark 4.3. Theorem 3.7 is a generalization of Theorem 2.4.

Proof. Put $r = 0$, $s := \alpha/2 + 1/4 > \alpha$, $\sigma := \alpha/2 + 1/4 > \alpha$, $t := 1/2 - \alpha > 0$ and $\tau = 0$. Then (3.2)–(3.7) hold. □

Lemma 4.3. *Assume $\eta \in H(\upsilon)$ for some $\upsilon \in (1,\infty)$. Assume also* (H5)*. Define a function h from $[0,\infty)$ into itself by $h(a) = \eta(a)^q$, where $q \in (0,\infty)$. Then $h \in H(\upsilon^q)$ holds and h satisfies* (H5).

Proof. It is obvious that $h \in H(\upsilon^q)$ holds. Noting $\varepsilon > 1 \Rightarrow (1+\varepsilon)^q - 1 > 0$, we can easily prove that h satisfies (H5). □

Proposition 4.3. *Assume the assumption of Theorem 2.5. Assume also $\eta \in \bigcup[H(\upsilon) : 1 < \upsilon < 1/\rho]$. Then the assumption of Theorem 3.7 holds.*

Remark 4.4. Since $H(1/\rho) \setminus \big(\bigcup[H(\upsilon) : 1 < \upsilon < 1/\rho]\big) \neq \varnothing$ holds, we cannot tell that Theorem 3.7 is a generalization of Theorem 2.5.

Proof. We can choose $\upsilon \in (1, 1/\rho)$ satisfying $\eta \in H(\upsilon)$. Also we can choose $q \in (1,\infty)$ satisfying

$$\upsilon^q < 1/\rho^q - 1 \quad \text{and} \quad \rho^q < 1/5.$$

We define a function h from $[0,\infty)$ into itself by $h(a) = \eta(a)^q$. Then by Lemma 4.3, $h \in H(\upsilon^q) \subset H(1/\rho^q - 1)$ holds. Also h satisfies (H5). For any $x,y \in X$, we have

$$
\begin{aligned}
h\big(d(Tx,Ty)\big) &= \eta\big(d(Tx,Ty)\big)^q \\
&\le \rho^q \max\big\{\eta\big(d(x,y)\big),\ \eta\big(d(x,Ty)\big),\ \eta\big(d(Tx,y)\big), \\
&\qquad \eta\big(d(x,Tx)\big),\ \eta\big(d(y,Ty)\big)\big\}^q
\end{aligned}
$$

$$
\begin{aligned}
&= \rho^q \max\big\{h\big(d(x,y)\big),\ h\big(d(x,Ty)\big),\ h\big(d(Tx,y)\big), \\
&\qquad h\big(d(x,Tx)\big),\ h\big(d(y,Ty)\big)\big\} \\
&\le \rho^q\, h\big(d(x,y)\big) + \rho^q\, h\big(d(x,Ty)\big) + \rho^q\, h\big(d(Tx,y)\big) \\
&\qquad + \rho^q\, h\big(d(x,Tx)\big) + \rho^q\, h\big(d(y,Ty)\big) \\
&\le (1-4\,\rho^q)\, h\big(d(x,y)\big) + \rho^q\, h\big(d(x,Ty)\big) + \rho^q\, h\big(d(Tx,y)\big) \\
&\qquad + \rho^q\, h\big(d(x,Tx)\big) + \rho^q\, h\big(d(y,Ty)\big).
\end{aligned}
$$

Putting $r = 1 - 4\,\rho^q$, $s = \rho^q$, $\sigma = \rho^q$, $t = \rho^q$ and $\tau = \rho^q$, (3.2)–(3.6) hold. Also (3.7) holds with $\eta := h$. We have

$$(1-s)/\tau = (1-\rho^q)/\rho^q = 1/\rho^q - 1$$

and hence $h \in H\big((1-s)/\tau\big)$ holds. Thus, the assumption of Theorem 3.7 holds. □

Remark 4.5. We note that the assumption of Corollary 3.1 also holds because in the proof of Proposition 4.3, $s = \sigma$ and $t = \tau$ hold.

Corollary 4.2. *Assume the assumption of Theorem 2.5. Assume also that η is continuous. Then the assumption of Theorem 3.7 holds.*

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