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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Fixed point results of generalized almost G- contractions in metric spaces endowed with graphs

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ABSTRACT. The main aim of this paper is to introduce a class of generalized contractions in the sense of Berinde. Some examples and fixed point theorems for such introduced mappings in the setting of metric spaces endowed with a graph are discussed. Our results extend and include many existing results in the literature.

1. INTRODUCTION

Fixed point theory of multivalued mappings plays an important role in science and applied science. It has applications in control theory, convex optimization, differiential inclusions and economics.

For a metric space (X, d), we let $CB(X)$ and $Comp(X)$ to be the set of all nonempty closed bounded subsets of X and the set of all nonempty compact subsets of X, respectively. A point $x \in X$ is a *fixed point* of a multivalued mapping $T : X \to 2^X$ if $x \in Tx$. For any $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(a, B) : a \in A\},$$
$$\delta(B, A) = \sup\{d(b, A) : b \in B\},$$
$$d(a, C) = \inf\{\|a - x\| : x \in C\}.$$

Note that H is called the *Pompeiu-Hausdorff metric* induced by metric d [10]. The first well-known theorem for multivalued contraction mappings was given by Nadler in 1969 [23].

Theorem 1.1. ([23]) Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

The Nadler's fixed point theorem for multivalued contractive mappings has been extended in many directions (see [7], [9], [10], [16], [27]).

Definition 1.1. ([22]) A function $\varphi : [0, \infty) \to [0, 1)$ is said to be $\mathcal{MT}$*-function* if

$$\lim_{r \to t^+} \sup \varphi(r) < 1 \quad \text{for each } t \in (0, \infty)$$

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In 1989, Mizoguchi and Takahashi proved the following fixed point theorem for multivalued mappings.

Theorem 1.2. ([22]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Suppose that there exists a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \text{ for all } x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

In 2007, Berinde and Berinde [9] introduced a class of multivalued mappings which is more general than that of $\mathcal{MT}$-contractions.

Definition 1.2. ([9]) Let (X, d) be a metric space and $T : X \to CB(X)$ a multivalued mapping. T is said to be a *generalized multivalued* (α, L)*-weak contraction* if there exist $L \geq 0$ and a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx), \text{ for all } x, y \in X.$$

They also showed that in a complete metric space, every generalized multivalued (α, L)-weak contraction has a fixed point. Note that the $(\alpha$,$L)$ contractive mapping is larger than those of Banach contractions, and it is not necessary to by a continuous mapping. For more works on this type of mappings, one may see ([6], [17], [19], [24], [25], [29]) and literatures therein, for example.

Most recently, Du and Hung [17] established a generalization of Mizoguchi-Takashi's fixed point theorem.

Theorem 1.3. ([17]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping on X. Suppose that there exists an $\mathcal{MT}$-function φ such that

$$\min\{H(Tx, Ty), d(y, Ty)\} \leq \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then there exists $z \in X$ such that $z \in Tz$.

Next, we recall the concept of fixed point theorems in metric spaces endowed with graphs. Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of a graph and $E(G)$ be a set of its edges. Assume that G has no parallel edges. We denote G^{-1} by the directed graph obtained from reversing the direction of edges of G, that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

In 2008, Jachymski [21] combined the concept of fixed point theory and graph theory to study fixed point theory in a metric space endowed with a directed graph. He introduced a concept of G-contraction and generalized Banach contraction principle in a metric space endowed with a directed graph.

Definition 1.3. ([21]) Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$. A mapping $f : X \to X$ is said to be *G-contractive* if f preserves edges of G, i.e.,

$$\forall\, x, y \in X,\ (x, y) \in E(G) \implies (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in [0, 1)$ such that, for any $x, y \in X$,

$$(x, y) \in E(G) \implies d(f(x), f(y)) \leq \alpha d(x, y).$$

Property A ([21]) For any sequence $(x_n)_{n\in\mathbb{N}}$ in X. If $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{n_k})_{n\in\mathbb{N}}$ with $(x_{n_k}, x) \in E(G)$ for $n \in \mathbb{N}$.

Using this concept, in [21], he proved the following theorem by

Theorem 1.4. ([21]) Let (X,d) be complete metric space. Suppose that a triple (X,d,G) have the property A. Let $f : X \to X$ be a G-contractive mapping and $X_f = \{x \in X : (x, f(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_f \neq \emptyset$.

Jachymski's fixed point theorem has been generalized and extended in several directions, see for example ([1], [4], [7], [8], [12], [16], [20] [27], [28]).

Inspired by the works of Berinde [9], Du and Hung [17] and Jachymski [21], we introduced the concept of multivalued generalized G-almost contractions in metric spaces and establish some fixed point theorems for this type mappings in metric spaces endowed with a directed graph. We give some examples to illustrate our main results.

2. Main results

We start with defining a new type of multivalued mappings.

Definition 2.4. Let (X,d) be a metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and $T : X \to CB(X)$. T is said to be a *generalized almost G-contraction* if

(1) there exist an MT-function $\alpha : [0,\infty) \to [0,1)$ and $L \geq 0$ with
$$\min\{H(Tx,Ty), d(y,Ty)\} \leq \alpha(d(x,y))d(x,y) + LD(y,Tx)$$
for all $x, y \in X$ such that $(x,y) \in E(G)$,

(2) if $u \in Tx$ and $v \in Ty$ are such that $d(u,v) \leq d(x,y)$ then $(u,v) \in E(G)$.

Remark 2.1.

(1) The class of generalized multivalued(α, L)-weak contraction is a special class of generalized almost G-contraction, when $E(G) = X \times X$.

(2) The class of generalized contractions in Theorem 1.3 is a special class of generalized almost G-contraction, when $E(G) = X \times X$ and $L = 0$.

The following theorem is the main result in the framework of complete metric spaces endowed with graphs.

Theorem 2.5. *Let (X,d) be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and X has Property A. Let $T : X \to CB(X)$ is a generalized G-almost contraction. Suppose that there exists $x_0 \in X$ such that $(x_0, y) \in E(G)$, for some $y \in Tx_0$. Then there exists $v \in X$ such that $v \in Tv$.*

Proof. Define the function $\mu : [0,\infty) \to [0,1)$ by
$$\mu(t) = \frac{1+\varphi(t)}{2} \quad \text{for all } t \in [0,\infty).$$
Therefore $0 \leq \varphi(t) < \mu(t) < 1$ for all $t \in [0,\infty)$. Let $x_0 \in X$ be such that $(x_0, x_1) \in E(G)$ where $x_1 \in Tx_0$. This implies that
$$\min\{H(Tx_0,Tx_1), d(x_1,Tx_1)\} \leq \alpha(d(x_0,x_1))d(x_0,x_1) + LD(x_1,Tx_0). \tag{2.1}$$
Since
$$d(x_1,Tx_1) \leq \sup_{u \in Tx_0} d(u,Tx_1) \leq H(Tx_0,Tx_1),$$
we obtain
$$\min\{H(Tx_0,Tx_1), d(x_1,Tx_1)\} = d(x_1,Tx_1). \tag{2.2}$$
So, by (2.1) and (2.2), we get
$$d(x_1,Tx_1) < \mu(d(x_0,x_1))d(x_0,x_1). \tag{2.3}$$

By (2.3), there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < \mu(d(x_0, x_1))d(x_0, x_1) < d(x_0, x_1)$$

Hence $(x_1, x_2) \in E(G)$. If $x_2 = x_1$, then $x_1 \in Tx_1$ which means that x_1 is a fixed point of T and the desired conclusion is proved. Assume that $x_2 \neq x_1$. Since T is a generalized G-almost contraction, we get

$$\begin{aligned} d(x_2, Tx_2) &= \min\{H(Tx_1, Tx_2), d(x_2, Tx_2)\} \\ &< \mu(d(x_1, x_2))d(x_1, x_2) + Ld(x_2, Tx_1) \\ &= \mu(d(x_1, x_2))d(x_1, x_2). \end{aligned}$$

So there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < \mu(d(x_1, x_2))d(x_1, x_2) < d(x_1, x_2).$$

Hence $(x_2, x_3) \in E(G)$. By induction, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that for each $n \in \mathbb{N}$

(i) $x_n \in Tx_{n-1}$ with $x_n \neq x_{n-1}$;
(ii) $d(x_n, x_{n+1}) < \mu(d(x_{n-1}, x_n))d(x_{n-1}, x_n) < d(x_{n-1}, x_n)$;
(iii) $(x_n, x_{n+1}) \in E(G)$.

By (ii), the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}\cup\{0\}}$ is strictly decreasing in $[0, \infty)$. Since φ is an $\mathcal{MT}$-function, by Definition 1.1, we have

$$0 \leq \sup_{n\in\mathbb{N}\cup\{0\}} \varphi(d(x_n, x_{n+1})) < 1$$

and hence deduces

$$0 < \sup_{n\in\mathbb{N}\cup\{0\}} \mu(d(x_n, x_{n+1})) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}\cup\{0\}} \varphi(d(x_n, x_{n+1}))\right] < 1$$

We denote $\gamma := \sup_{n\in\mathbb{N}\cup\{0\}} \mu(d(x_n, x_{n+1}))$. Hence $\gamma \in [0, 1)$. For each $n \in \mathbb{N} \cup \{0\}$, by (ii), we obtain

$$d(x_n, x_{n+1}) < \mu(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \leq \gamma d(x_{n-1}, x_n) \tag{2.4}$$

It follows from (2.4) that

$$d(x_n, x_{n+1}) < \gamma d(x_{n-1}, x_n) < \cdots < \gamma^n d(x_0, x_1). \tag{2.5}$$

We denote $\xi_n = \frac{\gamma^n}{1-\gamma} d(x_0, x_1)$. For $m, n \in \mathbb{N}$ with $m > n$, by (2.5), we get

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < \xi_n.$$

Since $0 < \gamma < 1$, $\lim_{n\to\infty} \xi_n = 0$, which implies that

$$\lim_{n\to\infty} \sup\{d(x_m, x_n) : m > n\} = 0.$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Since X has Property (A), $(x_n, v) \in E(G)$ for all $n \in \mathbb{N}$. So, we have

$$\min\{H(Tx_n, Tv), d(v, Tv)\} \leq \varphi(d(x_n, v))d(x_n, v) + Ld(v, Tx_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

Suppose that

$$\mathcal{A} = \{n \in \mathbb{N} : \min\{H(Tx_n, Tv), d(v, Tv)\} = H(Tx_n, Tv)\}.$$

We conclude that there are two possibilities.

Case 1. Assume that $\sharp(\mathcal{A}) = \infty$, where $\sharp(\mathcal{A})$ denotes the cardinal number of $\mathcal{A}$. Thus there exists $\{n_j\} \subset \mathcal{A}$ such that

$$\min\{H(Tx_{n_k}, Tv), d(v, Tv)\} = H(Tx_{n_k}, Tv) \quad \text{for all } k \in \mathbb{N}. \tag{2.7}$$

Since $(x_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned} d(v, Tv) &\leq d(v, x_{n_k+1}) + d(x_{n_k+1}, Tv) \\ &\leq d(v, x_{n_k+1}) + H(Tx_{n_k}, Tv) \\ &= d(v, x_{n_k+1}) + \min\{H(Tx_{n_k}, Tv), d(v, Tv)\} \\ &\leq d(v, x_{n_k+1}) + \varphi(d(x_{n_k}, v))d(x_{n_k}, v) + Ld(v, Tx_{n_k}) \\ &< d(v, x_{n_k+1}) + d(x_{n_k}, v) + Ld(v, x_{n_k+1}) \end{aligned}$$

Since $x_{n_k} \to v$ as $k \to \infty$, it follows that $d(v, Tv) = 0$. By the closedness of Tv, we conclude that $v \in Tv$.

Case 2. Suppose that $\sharp(\mathcal{A}) < \infty$. Then there exists a sequence $\{n_k\}$ of natural numbers such that

$$\min\{H(x_{n_k}, Tv), d(v, Tv)\} = d(v, Tv) \quad \text{for all } k \in \mathbb{N}. \tag{2.8}$$

Since $(x_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned} d(v, Tv) &= \min\{H(x_{n_k}, Tv), d(v, Tv)\} \\ &\leq \varphi(d(x_{n_k}, v))d(x_{n_k}, v) + Ld(v, Tx_{n_k}) \\ &< d(x_{n_k}, v) + Ld(v, x_{n_k+1}). \end{aligned}$$

Since $x_n \to v$ as $k \to \infty$, it follows that $d(v, Tv) = 0$. By the closedness of Tv, we obtain $v \in Tv$. The proof is completed.

□

Remark 2.2. Theorem 2.5 improves the the following results.

(a) If we take $E(G) = X \times X$ and $L = 0$ in Theorem 2.5, then we obtain the result of Mizoguchi-Takahashi [22].
(b) If we take $E(G) = X \times X$ in Theorem 2.5, then we obtain the result of Berinde [9].
(c) If we take $E(G) = X \times X$ in Theorem 2.5, then we obtain Theorem 1.5
(d) If we take $CB(X) = \{\{x\} : x \in X\}$, $\varphi(t) = k$ where $0 \leq k < 1$ and $L = 0$ in Theorem 2.5, then we obtain the result of Jachymski [21].

Next, we give an example which can illustrate Theorem 2.5 but Mizoguchi-Takahashi's fixed point theorm is not applicable.

Example 2.1. Let l^∞ be the Banach space consisting of all bounded real sequence with supremum norm d_∞. Let $\{\tau_n\}$ be a sequence defined by $\tau_n = \frac{1}{n}$ for each $n \in \mathbb{N}$ and $\{e_n\}$ be the canonical basis of l^∞. Put $v_n = \tau_n e_n$ for $n \in \mathbb{N}$ and $X = \{v_n\}_{n \in \mathbb{N}}$. Then (X, d_∞) be a complete metric space and $d_\infty(v_n, v_m) = \frac{1}{n}$ if $m > n$. Let $G = (V(G), E(G))$ be such that $V(G) = X$ and

$$E(G) = \{(v_n, v_m) \in X \times X : m \geq n\}.$$

Notice that X has Property A. Let $T : X \to CB(X)$ be a mapping defined by

$$Tv_n = \begin{cases} \{v_1, v_2\} & \text{, if } n \in \{1, 2\}, \\ X \setminus \{v_1, v_2, \ldots, v_n, v_{n+1}\} & \text{, if } n \geq 3. \end{cases}$$

Define $\varphi : [0,\infty) \to [0,1)$ by

$$\varphi(t) = \begin{cases} \dfrac{\tau_{n+2}}{\tau_n} & \text{, if } t = \tau_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{, otherwise.} \end{cases}$$

We see that $\limsup_{s\to t^+} \varphi(s) = 0 < 1$ for all $t \in [0,\infty)$, so φ is an $\mathcal{MT}$-function. We now show that T is generalized almost G- contraction. Let $x, y \in X$ such that $(x,y) \in E(G)$, we consider the following necessary four cases.

Cases 1: Let $x = v_1, y = v_2$. Then $Tv_1 = Tv_2 = \{v_1, v_2\}$. Moreover, we get

$$\min\{H(Tv_1, Tv_2), d(v_2, Tv_2)\} = 0 < \tau_3 = \varphi(d(v_1, v_2))d(v_1, v_2).$$

Cases 2: Let $x = v_1, y = v_m$ for each $m \geq 3$. Then $Tv_1 = \{v_1, v_2\}$ and $Tv_m = \{v_{m+2}, v_{m+3}, \ldots\}$. Moreover, we get

$$\min\{H(Tv_1, Tv_m), d(v_m, Tv_m)\} = \tau_m \leq \varphi(d(v_1, v_m))d(v_1, v_m) + 2d(v_m, Tv_1).$$

Cases 3: Let $x = v_2, y = v_m$ for each $m \geq 3$. Then $Tv_2 = \{v_1, v_2\}$ and $Tv_m = \{v_{m+2}, v_{m+3}, \ldots\}$. Moreover, we get

$$\min\{H(Tv_2, Tv_m), d(v_m, Tv_m)\} = \tau_m \leq \varphi(d(v_2, v_m))d(v_2, v_m) + 2d(v_m, Tv_2).$$

Cases 4: Let $x = v_n, = v_m$ for each $n \geq 3$ and $m > n$. Then $Tv_n = \{v_{n+2}, v_{n+3}, \ldots\}$ and $Tv_m = \{v_{m+2}, v_{m+3}, \ldots\}$. Moreover, we get

$$\min\{H(Tv_n, Tv_m), d(v_m, Tv_m)\} = \tau_{n+2} = \varphi(d(v_n, v_m))d(v_n, v_m) + 2d(v_m, Tv_n).$$

Hence, from the above cases, we can conclude that T is generalized G-almost contraction or $(\varphi, 2)$-G-contraction. Choosing $v_1 \in X$, we see that $(v_1, v_2) \in E(G)$ where $v_2 \in Tv_1 = \{v_1, v_2\}$. Therefore, all conditions of Theorem 2.5 are satisfied and we see that $F(T) = \{1, 2\}$. Notice that

$$H(Tv_n, Tv_m) = \tau_1 > \tau_3 = \varphi(d(v_1, v_m))d(v_1, v_m) \quad \text{for all } m \geq 3,$$

which means that Mizoguchi-Takahashi's fixed point theorem is not applicable here.

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