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*Dedicated to Professor Yeol Je Cho on the occasion of his retirement*

# Hybrid Bregman projection methods for fixed point and equilibrium problems

ZI-MING WANG[1], AIRONG WEI[1] and POOM KUMAM[2]

ABSTRACT. The purpose of this article is to investigate a projection algorithm for solving a fixed point problem of a closed multi-valued Bregman quasi-strict pseudocontraction and an equilibrium problem of a bifunction. Strong convergence of the projection algorithm is obtained without any compact assumption in a reflexive Banach space. As applications, monotone variational inequality problems are considered. Finally, a numerical simulation example is presented for demonstrating the feasibility and convergence of the algorithm proposed in main result.

## 1. INTRODUCTION

Let $E$ be a Banach space and $C$ be a nonempty closed convex subset of $E$. $N(C)$ and $CB(C)$ stand for the family of nonempty subsets and nonempty closed bounded subsets of $C$, respectively. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(C)$ defined as $H(A,B) = \max\{\sup_{y\in B} d(y,A), \sup_{x\in A} d(x,B)\}, \forall A, B \in CB(C)$, where $d(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from point $a$ to subset $B$. Let $T : C \to CB(C)$ be a multi-valued mapping. $F(T) := \{p \in C : p = T(p)\}$ represents the fixed point set of $T$.

Lots of problems can be studied via fixed point techniques of multi-valued mappings, such as optimal control, signal processing, image reconstruction, which makes construction of iterative algorithms for approximating fixed points of multi-valued mappings become one of the main concerns of fixed point theory [5, 6, 7, 8, 10]. On the other hand, the "so-called" equilibrium problem with respect to a bifunction $g : C \times C \to \mathbb{R}$ is described as follows: find $\tilde{x}$ such that $g(\tilde{x}, y) \geq 0$, for all $y \in C$. The set of solutions of the equilibrium problem is denoted as $EP(g)$. To solve the equilibrium problem, the following assumptions hold: (A1) $g(x,x) = 0$ for all $x \in C$; (A2) $g$ is monotone, i.e., $g(x,y) + g(y,x) \leq 0$ for all $x,\ y \in C$; (A3) for all $x,\ y,\ z \in C$, $\limsup_{t\downarrow 0} g(tz + (1-t)x, y) \leq g(x,y)$; (A4) for all $x \in C$, $g(x,\cdot)$ is convex and lower semi-continuous.

In this paper, our main goal is to address the convergence of iterative algorithms for approximating a common element in the fixed points set of a multi-valued Bregman quasi-strict pseudo-contraction and the solutions set of an equilibrium problem in a reflexive Banach space. The results presented in this paper improve some corresponding results announced in [9, 14, 15, 16].

## 2. PRELIMINARIES

In this section, we collect some preliminaries which are used in the following section. Unless mentioned otherwise, all throughout the paper, $E$ is a real reflexive Banach space

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Corresponding author: A. Wei; weiairong@sdu.edu.cn

with the norm $\|\cdot\|$ and $E^*$ is the dual space of $E$, $C$ is a nonempty, closed, and convex subset of $E$. $f : E \to (-\infty, +\infty]$ is a proper, convex and lower semi-continuous function. Denote the domain of $f$ by dom$f$, i.e., dom$f := \{x \in E : f(x) < +\infty\}$. $\mathbb{N}$ and $\mathbb{R}$ are denoted as the sets of positive integers and real numbers, respectively. Let any $x \in$ int dom$f$ and $y \in E$, the right-hand derivative of $f$ at $x$ in the direction of $y$ is defined by

$$f^\circ(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

**Definition 2.1.** The function $f$ is said to be: (i) Gâteaux differentiable at $x$ if the limit $f^\circ(x, y)$ exists for any $y$; (ii) Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom$f$; (iii) Fréchet differentiable at $x$ if the limit $f^\circ(x, y)$ is attained uniformly in $\|y\| = 1$; (iv) uniformly Fréchet differentiable on a subset $C$ of $E$ if the limit $f^\circ(x, y)$ is attained uniformly for $x \in C$ and $\|y\| = 1$.

**Remark 2.1.** (i) If $f$ is Gâteaux differentiable at $x$, then $f^\circ(x, y)$ coincides with $\triangledown f(x)$, the value of the gradient $\triangledown f$ of $f$ at $x$; (ii) if a continuous convex function $f \to \mathbb{R}$ is Gâteaux differentiable, $\nabla f$ is norm-to-weak* continuous; (iii) if $f$ is Fréchet differentiable, $\nabla f$ is norm-to-norm continuous.

Let $x \in$ int dom$f$, the subdifferential of $f$ at $x$ is the convex set defined by $\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \forall\, y \in E\}$. The Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$, where $x^* \in E^*$.

**Definition 2.2.** The function $f$ is called: (i) essentially smooth if $\partial f$ is both locally bounded and single-valued on its domain; (ii) essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and $f$ is strictly convex on every convex subset of dom$\partial f$; (iii) Legendre, if it is both essentially smooth and essentially strictly convex.

**Remark 2.2.** Let $E$ be a reflexive Banach space, the following conclusions hold: (i) $f$ is essentially smooth if and only if $f^*$ is essentially strictly convex; (ii)$(\partial f)^{-1} = \partial f^*$; (iii) $f$ is Legendre if and only if $f^*$ is Legendre; (iv)if $f$ is Legendre, then $\nabla f$ is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$=dom$\nabla f^*$=int dom$f^*$ and ran$\nabla f^*$=dom$\nabla f$=int dom$f$, see [1].

**Definition 2.3.** Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The Bregman distance with respect to $f$ [4] is the function $D_f$ : dom$f\times$int dom$f \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - \langle \triangledown f(x), y - x\rangle - f(x).$$

Recall that the bifunction $V_f : E \times E^* \to [0, \infty)$ associated with $f$ is defined by $V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \forall\, x \in E,\ x^* \in E^*$. Then $V_f$ is nonnegative and satisfies $V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \forall x \in E, x^* \in E^*$. Although $D_f(\cdot, \cdot)$ does not normally satisfy the symmetry and the triangle inequality, it has the following important property, called "three point identity": for any $x \in$ dom$f$ and $y,\ z \in$ int dom$f$, $\langle \triangledown f(z) - \triangledown f(y), x - y\rangle = D_f(x, y) + D_f(y, z) - D_f(x, z)$.

**Definition 2.4.** If $f : E \to (-\infty, +\infty]$ is convex and Gâteaux differentiable, $C \subset$ dom$f$ is a nonempty, closed, and convex set. The Bregman projection [11] $x \in$ int dom$f$ onto $C$ is the unique vector $P^f_C(x) \in C$ satisfying $D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}$.

In fact, the Bregman projection $P^f_C(x)$, which is more general than the generalized projection $\Pi_C(x)$ defined by $\Pi_C(x) = \arg\min_{y\in C} \phi(y, x)$ from $E$ onto $C$, reduces to the generalized projection by taking $f(x) = \|x\|^2$ for all $x \in E$.

Let $f : E \to (-\infty, +\infty]$ be Gâteaux differentiable. The modulus of total convexity of $f$ at $x \in$ dom$f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by $\nu_f(x, t) := \inf\{D_f(y, x) : y \in \mathrm{dom}\, f, \|y - x\| = t\}$. The modulus of total convexity of the function $f$ on the set $B$ is the function $\nu_f$ : int dom$f \times [0, +\infty) \to [0, +\infty]$ defined by $\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \mathrm{dom}\, f\}$.

**Definition 2.5.** A function $f$ is said to be: (i) totally convex at $x$ if $\nu_f(x,t) > 0$, whenever $t > 0$; (ii) totally convex if it is totally convex at any point $x \in \operatorname{int} \operatorname{dom} f$; (iii) totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset $B$ of $E$ and $t > 0$.

**Definition 2.6.** A function $f$ is said to be: (i) strongly coercive if $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$; (ii) sequentially consistent [3], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that the first one is bounded, $\lim_{n\to\infty} D_f(y_n,x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Definition 2.7.** A multi-valued mapping $T : C \to CB(C)$ is said to be multi-valued Bregman quasi-strictly pseudo-contractive with respect to $f$ if $F(T) \neq \emptyset$ and

$$D_f(p,u) \le D_f(p,x) + kD_f(x,u), \forall u \in Tx,\ x \in C,\ p \in F(T).$$

In the following, we list some lemmas which are important in our proof.

**Lemma 2.1.** [3] *Suppose that $f$ is Gâteaux differentiable and totally convex on int dom$f$. For a nonempty, closed and convex set $C \subset$ int dom$f$, $x \in$ int dom$f$ and $\hat{x} \in C$, then the following conditions are equivalent: (i) the vector $\hat{x}$ is the Bregman projection of $x$ onto $C$ with respect to $f$, i.e., $z = P_C^f(x)$; (ii) the vector $\hat{x}$ is the unique solution for $\langle \nabla f(x) - \nabla f(z), z - y\rangle \ge 0, \quad \forall\, y \in C$; (iii) The vector $\hat{x}$ is the unique solution for $D_f(y,z) + D_f(z,x) \le D_f(y,x), \quad \forall\, y \in C$.*

**Lemma 2.2.** [1] *Suppose $x \in E$ and $y \in$ int dom$f$. If $f$ is essentially strictly convex, then $D_f(x,y) = 0 \Leftrightarrow x = y$.*

**Lemma 2.3.** [2] *The function $f$ is totally convex on bounded sets if and only if the function $f$ is sequentially consistent.*

**Lemma 2.4.** [12] *Suppose that $f : E \to \mathbb{R}$ is Gâteaux differentiable and totally convex. If $x_0 \in E$ and the sequence $\{D_f(x_n,x_0)\}$ is bounded, the sequence $\{x_n\}$ is bounded too.*

**Lemma 2.5.** [11] *Suppose that the convex function $f : E \to \mathbb{R}$ is bounded on bounded subsets of $E$. Then the following assertions are equivalent: (a) $f$ is strongly coercive and uniformly convex on bounded subsets of $E$; (b) $f^*$ is Fréchet differentiable and $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of dom$f^*$=$E^*$.*

**Lemma 2.6.** [13] *Suppose that the convex, continuous and strongly coercive function $f : E \to \mathbb{R}$ is bounded on bounded subsets and uniformly convex on bounded subsets of $E$, $g : C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)-(A4), $Res_r^g : E \to C$ is the resolvent operator defined by $Res_r^g(x) = \left\{ z \in C : g(z,y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall\, y \in C \right\}$, where $r > 0$, $x \in E$. Then the following statements hold: (a) $Res_r^g$ is single-valued; (b) $F(Res_r^g) = EP(g)$; (c) $EP(g)$ is closed and convex; (d) $D_f(p, Res_r^g x) + D_f(Res_r^g x, x) \le D_f(p,x), \forall\, p \in EP(g), \forall\, x \in E$.*

## 3. Main results

In this section, we state and prove our main theorem.

**Theorem 3.1.** *Suppose that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$, $g$ is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $T : C \to CB(C)$ is a closed mapping defined as Definition 2.7 such that $F(T) \cap EP(g) \neq \emptyset$. For an arbitrary element $x_0 \in C$, let $C_0 = C$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(z_n)], \quad z_n \in Tx_n, \\ g(u_n,y) + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle \ge 0, \quad \forall\, y \in C, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.1}$$

*where* $C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, y_n) \leq D_f(z, x_n) + \frac{\kappa}{1-\kappa}\langle x_n - z, \nabla f(x_n) - \nabla f(z_n)\rangle \right\}$, $\kappa \in [0, 1)$, $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\widehat{p} = P^f_{F(T)\cap EP(g)}(x_0)$, *where* $P^f_{F(T)\cap EP(g)}$ *is the Bregman projection of* $E$ *onto* $F(T) \cap EP(g)$.

*Proof.* Due to the construction of $C_n$, one sees that $C_n$ is closed for all $n \in \mathbb{N} \cup \{0\}$. Since $D_f(z, u_n) \leq D_f(z, y_n)$ and $D_f(z, y_n) \leq D_f(z, x_n) + \frac{k}{1-k}\langle x_n - z, \nabla f(x_n) - \nabla f(z_n)\rangle$ are equal to $\langle z, \nabla f(y_n) - \nabla f(u_n)\rangle \leq f(u_n) - f(y_n) + \langle y_n, \nabla f(y_n)\rangle - \langle u_n, \nabla f(u_n)\rangle$, and $\left\langle z, \frac{1}{1-k}\nabla f(x_n) - \nabla f(y_n) - \frac{k}{1-k}\nabla f(z_n)\right\rangle \leq f(y_n) - f(x_n) + \left\langle x_n, \frac{1}{1-k}\nabla f(x_n)\right\rangle - \left\langle x_n, \frac{k}{1-k}\nabla f(z_n)\right\rangle$
$- \langle y_n, \nabla f(y_n)\rangle$ respectively, thus $C_n$ is convex and closed for $n \in \mathbb{N} \cup \{0\}$.

Next, we show that $F(T)\cap EP(g) \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$. It is clear that $F(T)\cap EP(g) \subset C_0 = C$. Suppose that $F(T) \cap EP(g) \subset C_m$ for some $m \in \mathbb{N}$. For any $w \in F(T) \cap EP(g) \subset C_m$, since $u_m = \operatorname{Res}^g_{r_m} y_m$, one has from Lemma 2.6 (d) that

$$\begin{aligned}
D_f(w, u_m) &\leq D_f(w, y_m) \leq \alpha_m V(w, \nabla f(x_m)) + (1 - \alpha_m)V(w, \nabla f(z_m)) \\
&= \alpha_m D_f(w, x_m) + (1 - \alpha_m)D_f(w, z_m) \\
&\leq \alpha_m D_f(w, x_m) + (1 - \alpha_m)[D_f(w, x_m) + kD_f(x_m, z_m)] \\
&\leq D_f(w, x_m) + \frac{k}{1-k}\langle x_m - w, \nabla f(x_m) - \nabla f(z_m)\rangle.
\end{aligned}$$

This implies that $w \in C_{m+1}$. Therefore, one has $F(T) \cap EP(g) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we are in a position to show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. In fact, since $x_n = P^f_{C_n}(x_0)$, from Lemma 2.1 (ii), one has $\langle y - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, y \in C_n$, and since $F(T) \cap EP(g) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$, we arrive at

$$\langle w - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, w \in F(T) \cap EP(g). \tag{3.2}$$

From Lemma 2.1 (iii), one has $D_f(x_n, x_0) = D_f(P^f_{C_n}(x_0), x_0) \leq D_f(w, x_0) - D_f(w, P^f_{C_n}(x_0)) \leq D_f(w, x_0)$, for each $w \in F(T)\cap EP(g)$ and for each $n \in \mathbb{N}\cup\{0\}$. Therefore, $\{D_f(x_n, x_0)\}$ is bounded. From Lemma 2.4, one has $\{x_n\}$ is also bounded. Since $x_n = P^f_{C_n}(x_0)$ and $x_{n+1} = P^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, one has $D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{D_f(x_n, x_0)\}$ is a nondecreasing sequence. Therefore $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Since $\{x_n\}$ is bounded and $E$ is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \widehat{p} \in C = C_1$. Since $C_n$ is closed and convex and $C_{n+1} \subset C_n$, this implies that $C_n$ is weakly closed and $\widehat{p} \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $\widehat{p} \in C_{n_i}$ for all $n_i \in \mathbb{N} \cup \{0\}$. In view of $x_{n_i} = P^f_{C_{n_i}}(x_0)$, one has from the definition of Bregman projection that $D_f(x_{n_i}, x_0) \leq D_f(\widehat{p}, x_0), \quad \forall\, n_i \in \mathbb{N} \cup \{0\}$. Since $f$ is a lower semi-continuous function on convex set $C$, it is weakly lower semi-continuous on $C$. Hence we have

$$\begin{aligned}
\liminf_{i\to\infty} D_f(x_{n_i}, x_0) &= \liminf_{i\to\infty}\{f(x_{n_i}) - f(x_0) - \langle \nabla f(x_0), x_{n_i} - x_0\rangle\} \\
&\geq f(\widehat{p}) - f(x_0) - \langle \nabla f(x_0), \widehat{p} - x_0\rangle = D_f(\widehat{p}, x_0).
\end{aligned}$$

Thus, one has $D_f(\widehat{p}, x_0) \leq \liminf_{i\to\infty} D_f(x_{n_i}, x_0) \leq \limsup_{i\to\infty} D_f(x_{n_i}, x_0) \leq D_f(\widehat{p}, x_0)$, which implies that $\lim_{i\to\infty} D_f(x_{n_i}, x_0) = D_f(\widehat{p}, x_0)$. Employing 2.1 (iii), one obtains that $D_f(\widehat{p}, x_{n_i}) \leq D_f(\widehat{p}, x_0) - D_f(x_{n_i}, x_0)$. When $n_i \to \infty$ in the above inequality, one obtains $\lim_{n_i\to\infty} D_f(\widehat{p}, x_{n_i}) = 0$, which implies from Lemma 2.2 that $\lim_{n_i\to\infty} x_{n_i} = \widehat{p}$. Besides, noticing that $\{D_f(x_n, x_0)\}$ is convergent, hence, one gets $\lim_{n\to\infty} D_f(x_n, x_0) = D_f(\widehat{p}, x_0)$.

And since $x_n = P^f_{C_n} x_0$, from Lemma 2.1 (iii), one has $D_f(\widehat{p}, x_n) \leq D_f(\widehat{p}, x_0) - D_f(x_n, x_0)$. Similarly, one also obtains $\lim_{n_i\to\infty} x_n = \widehat{p}$. Since $x_{n+1} \in C_{n+1}$, from the construction of $C_{n+1}$, one has $D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n) + \dfrac{k}{1-k}\langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(z_n)\rangle$. Noticing that $\lim_{n_i\to\infty} x_n = \widehat{p}$, one has $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$, and $\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0$. In view of Lemma 2.3 and Definition 2.6, one has $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$, furthermore, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since $\nabla f$ is uniformly continuous on each bounded subset of $E$, one has $\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0$. Due to $y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(z_n)]$, one has $\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_n)\| = \lim_{n\to\infty} \dfrac{1}{1-\alpha_n}\|\nabla f(x_n) - \nabla f(y_n)\| = 0$. From Lemma 2.5, one has $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Therefore $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = \widehat{p}$. In view of $z_n \in Tx_n$, and from the closedness of $T$, it follows $\widehat{p} \in T\widehat{p}$, that is, $\widehat{p} \in F(T)$.

Next, we prove $\widehat{p} \in EP(g)$. Obviously, $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Hence $\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(y_n)\| = 0$. By the assumption $\liminf_{n\to\infty} r_n > 0$, one has $\lim_{n\to\infty} \frac{\|\nabla f(u_n) - \nabla f(y_n)\|}{r_n} = 0$, which together with $u_n = T^g_{r_n} y_n$ implies that $g(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle \geq 0, \forall\, y \in C$. From (A2), we deduce that $\|y - u_n\| \frac{\|\nabla f(u_n) - \nabla f(y_n)\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle \geq -g(u_n, y) \geq g(y, u_n), \forall y \in C$. Letting $n \to \infty$ in the above inequality, one has from (A4) that $g(y, \widehat{p}) \leq 0$, $\forall y \in C$. For $t \in (0,1)$ and $y \in C$, let $y_t = ty + (1-t)\widehat{p}$. Then $y_t \in C$, which yields that $g(y_t, \widehat{p}) \leq 0$. Therefore, from (A1) and (A4) one has $0 = g(y_t, y_t) \leq tg(y_t, y) + (1-t)g(y_t, p) \leq tg(y_t, y)$. Dividing by $t$, one has $g(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from (A3), one has $g(\widehat{p}, y) \geq 0$, $\forall y \in C$. Hence $\widehat{p} \in EP(g)$.

Finally, we take $n \to \infty$ in (3.2) and obtain that $\langle w - \widehat{p}, \nabla f(x_0) - \nabla f(\widehat{p})\rangle \leq 0, \forall\, w \in F(T) \cap EP(g)$. In view of Lemma 2.10 (i) and (ii), one has $\widehat{p} = P^f_{F(T)\cap EP(g)}(x_0)$. □

## 4. Applications and examples

Let $A : C \subseteq E \to E^*$ be a nonlinear mapping. The variational inequality problem for a nonlinear mapping $A$ and its domain $C$ is to find $\bar{x} \in C$ such that $\langle A\bar{x}, y - \bar{x}\rangle \geq 0$, $\forall\, y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. Recall that a mapping $A : C \to E^*$ is called monotone if $\langle Ax - Ay, x - y\rangle \geq 0$, $\forall\, x,\ y \in C$.

Assume that $A$ is a continuous and monotone mapping. For $r > 0$, define the resolvent operator $\mathrm{Res}^f_r : E \to C$ as follows: for all $x \in E$, $\mathrm{Res}^A_r := \{z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall y \in C\}$. Similar to Lemma 2.15, the following conclusions hold: (1) $\mathrm{Res}^A_r$ is single-valued; (2) $F(\mathrm{Res}^A_r) = VI(C, A)$; (3) $D_f(p, \mathrm{Res}^A_r\, x) + D_f(\mathrm{Res}^A_r\, x, x) \leq D_f(p, x)$, for $p \in F(\mathrm{Res}^A_r)$; (4) $VI(C, A)$ is closed and convex.

**Theorem 4.2.** *Suppose that $f$ and $g$ are defined as Theorem 3.1, $A : C \to E^*$ is a continuous monotone mapping such that $VI(C, A) \cap EP(g) \neq \emptyset$. Reset $Tx_n = \mathrm{Res}^A_r\, x_n$ in the algorithm* (3.1), *let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the algorithm* (3.1). *Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P^f_{VI(C,A)\cap EP(g)}(x_0)$, where $P^f_{VI(C,A)\cap EP(g)}$ is the Bregman projection of $E$ onto $VI(C, A) \cap EP(g)$.*

Finally, a numerical experiment will be carried out to demonstrate the efficiency of the algorithm (3.1). Based on Example 5.1 of Wang and Wei [16] and Example 1 of Saewan, Cho, Kumam [13], the following example could be obtained easily.

**Example 4.1.** Let $E = \mathbb{R}$, $C = [-\pi, \pi]$, $f(x) = x^2$, $Tx = \sin\left(\dfrac{1}{2}x\right)$, $g(z, y) = y^2 + zy - 2z^2$. Then $T$ is a closed Bregman quasi-strict pseudo-contraction with $EP(g) \bigcap F(T) = \{0\}$.

Based on the assumption of Example 4.1, replace $Tx_n$ and $g(u_n, y)$ by $Tx_n = \sin\left(\frac{1}{2}x_n\right)$, $g(u_n, y) = y^2 + u_n y - 2u_n^2$ in the algorithm (3.1). For the initial conditions $x_0 = -0.8, 1$, $r_n \equiv 1$, $\alpha_n = \frac{1}{n}$, the picture (a) in Fig.1 shows that the sequence $\{x_n\}$ converge to the same value for the different initial points. For the initial conditions $r_n = 10^{-4},\ 1,\ n^2$, $x_0 = 1$, $\alpha_n = \frac{1}{n}$, the picture (b) in Fig.1 shows that the different values of parameter sequence $\{r_n\}$ do not significantly influence on the rates of convergence. Therefore, in a real world application, the parameter sequence $\{r_n\}$ of algorithm (3.1) can be regarded as the constant 1.

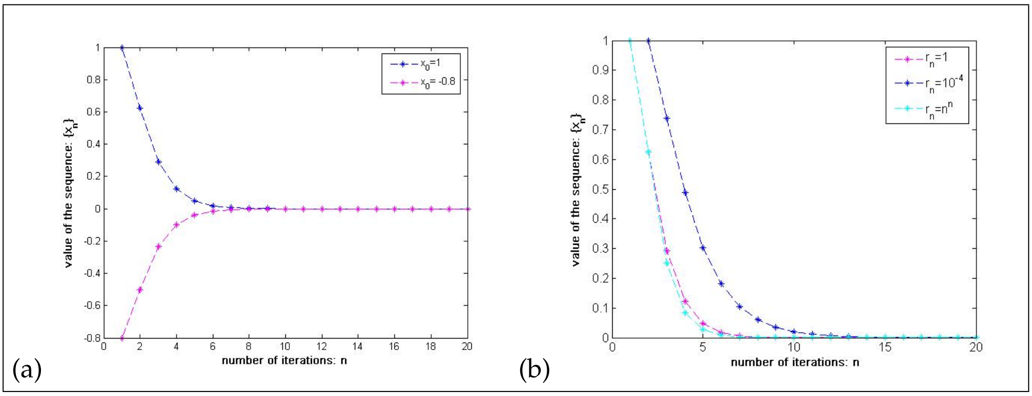


**Fig. 1.** the convergence process of the sequence $\{x_n\}$ with different initial conditions.

## 5. CONCLUSIONS

In the paper, we have investigated a fixed point problem of a closed multi-valued Bregman quasi-strict pseudocontraction and an equilibrium problem via hybrid Bregman projection methods, and obtained a strong convergence result. Furthermore, a kind of variational inequality problem has been solved as an application and a numerical example has been given to illustrate the effectiveness of the proposed algorithm.

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[1]School of Control Science and Engineering
Shandong University
Jinan 250061 China
*E-mail address*: `wangziming@ymail.com`
*E-mail address*: `weiairong@sdu.edu.cn`

[2]KMUTTFixed Point Research Laboratory, Department of Mathematics
Room SCL 802 Fixed Point Laboratory
KMUTT-Fixed Point Theory and Applications Research Group
Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140 Thailand
*E-mail address*: `poom.kum@kmutt.ac.th`