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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Better approximation of functions by genuine Bernstein-Durrmeyer type operators

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ABSTRACT. The main object of this paper is to construct a new genuine Bernstein-Durrmeyer type operators which have better features than the classical one. Some direct estimates for the modified genuine Bernstein-Durrmeyer operator by means of the first and second modulus of continuity are given. An asymptotic formula for the new operator is proved. Finally, some numerical examples with illustrative graphics have been added to validate the theoretical results and also compare the rate of convergence.

1. INTRODUCTION

Bernstein operators are one of the most important sequences of positive linear operators. These operators were introduced by Bernstein [4] and were intensively studied. For more details on this topic we can refer the readers to excellent monographs [12] and [13]. The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ x \in [0,1].$$

It is well known that the fundamental polynomials verify

$$p_{n,k}(x) = (1-x)\, p_{n-1,k}(x) + x\, p_{n-1,k-1}(x), \ \ 0 < k < n. \tag{1.2}$$

In a recent paper, Khosravian-Arab et al. [14] have introduced a sequence of modified Bernstein operators as follows:

$$B_n^{M,1}(f,x) = \sum_{k=0}^{n} p_{n,k}^{M,1}(x)\, f\left(\frac{k}{n}\right), \ \ x \in [0,1], \tag{1.3}$$

$$\begin{aligned} p_{n,k}^{M,1}(x) &= \alpha(x,n)\, p_{n-1,k}(x) + \alpha(1-x,n)\, p_{n-1,k-1}(x), 1 \le k \le n-1, \\ p_{n,0}^{M,1}(x) &= \alpha(x,n)(1-x)^{n-1}, \quad p_{n,n}^{M,1}(x) = \alpha(1-x,n)x^{n-1}, \end{aligned} \tag{1.4}$$

and

$$\alpha(x,n) = \alpha_1(n)\, x + \alpha_0(n), \ n = 0, 1, \ldots,$$

where $\alpha_0(n)$ and $\alpha_1(n)$ are unknown sequences. For $\alpha_1(n) = -1$, $\alpha_0(n) = 1$, obviously, (1.4) reduces to (1.2).

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A Kantorovich variant of the modified Bernstein operators (1.3) was introduced and studied in [11].

2. The modified genuine Bernstein-Durrmeyer operators

The genuine Bernstein-Durrmeyer operators were introduced by Chen [6] and Goodman and Sharma [10] and were studied by a numbers of authors (see [1], [5], [8], [9], [15], [16]). These operators are defined as follows:

$$U_n(f;x)=(1-x)^n f(0)+x^n f(1)+(n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x),\ \ f\in C[0,1].$$

The genuine Bernstein-Durrmeyer operators are limits of the Bernstein-Durrmeyer operators with Jacobi weights (see [2], [3], [17]), namely

$$U_n f=\lim_{\alpha\to -1,\beta\to -1} M_n^{<\alpha,\beta>}f, \text{ where}$$

$$M_n^{<\alpha,\beta>}:C[0,1]\to \Pi_n,\ \ M_n^{<\alpha,\beta>}(f;x)=\sum_{k=0}^{n}p_{n,k}(x)\frac{\int_0^1 w^{(\alpha,\beta)}(t)p_{n,k}(t)f(t)dt}{\int_0^1 w^{(\alpha,\beta)}(t)p_{n,k}(t)dt},$$
$$w^{(\alpha,\beta)}(t)=x^{\beta}(1-x)^{\alpha},\ x\in(0,1),\ \ \alpha,\beta>-1.$$

In this section, we introduce a new variant of the genuine Bernstein-Durrmeyer operators as follows:

$$\begin{aligned}U_n^1(f;x)&=\alpha(x,n)(1-x)^{n-1}f(0)+\alpha(1-x,n)x^{n-1}f(1)\\&+(n-1)\sum_{k=1}^{n-1}\{\alpha(x,n)p_{n-1,k}(x)+\alpha(1-x,n)p_{n-1,k-1}(x)\}\int_0^1 p_{n-2,k-1}(t)f(t)dt.\end{aligned}\tag{2.5}$$

Throughout this section, we assume $U_n^1(e_0)=1$, namely

$$2\alpha_0(n)+\alpha_1(n)=1.\tag{2.6}$$

In the following, we will consider these two cases:

$$\alpha_0(n)\geq 0,\alpha_0(n)+\alpha_1(n)\geq 0,\tag{2.7}$$

$$\alpha_0(n)<0 \text{ or } \alpha_1(n)+\alpha_0(n)<0.\tag{2.8}$$

Remark 2.1. If the unknown sequences $\alpha_i(n), i=1,2$ verify conditions (2.6) and (2.7), it follows that

$$0\leq\alpha_0(n)\leq 1 \text{ and } -1\leq\alpha_1(n)\leq 1.$$

Thus the sequences $\alpha_i(n), i=1,2$ are bounded and the operator (2.5) is positive. If the sequences $\alpha_i(n), i=1,2$ verify conditions (2.6) and (2.8), we obtain

$$\big(\alpha_0(n)<0,\ \alpha_1(n)+\alpha_0(n)>1\big) \text{ or } \big(\alpha_1(n)+\alpha_0(n)<0,\ \alpha_0(n)>1\big),$$

hence the operator (2.5) is not positive.

Denote $m_{n,k}^1(x):=U_n^1(e_k;x)$, $\mu_{n,k}^1(x):=U_n^1\left((t-x)^k;x\right)$, where $e_k(t)=t^k$, $k=0,1,\ldots$.

Lemma 2.1. *The modified genuine Bernstein-Durrmeyer operators (2.5) verify:*

i) $m_{n,0}^1(x)=1$;

ii) $m_{n,1}^1(x)=\dfrac{1}{n}\{xn+(1-2x)(1-\alpha_0(n))\}$;

iii) $m_{n,2}^1(x)=\dfrac{1}{n(n+1)}\left\{x^2n^2+(4\alpha_0(n)x-2\alpha_0(n)-5x+4)xn+2(1-x)(1-2x)(1-\alpha_0(n))\right\}$.

Lemma 2.2. *The following statements hold:*

i) $\mu_{n,1}^{1}(x) = \dfrac{1}{n}(1-2x)(1-\alpha_0(n))$;

ii) $\mu_{n,2}^{1}(x) = \dfrac{2}{n(n+1)}\left\{x(1-x)n + (2x-1)^2(1-\alpha_0(n))\right\}$;

iii) $\mu_{n,4}^{1}(x) = \dfrac{12}{n(n+1)(n+2)(n+3)}\left\{x^2(1-x)^2n^2 - x(1-x)n\right.$
$\left.\cdot\left[4\alpha_0(n)(1-2x)^2 + 23x(1-x) - 6\right] + 2(1-2x)^4\left[1-\alpha_0(n)\right]\right\}$.

In the following, we will give a direct estimate for the modified genuine Bernstein-Durrmeyer operator U_n^1 by means of the first modulus of continuity $\omega(f,\delta)$.

Theorem 2.1. *Let f be a bounded function for $x \in [0,1]$. If $\alpha_1(n)$ is a bounded sequence, then*

$$\|U_n^1 f - f\| \le 2\left(3|\alpha_1(n)| + 1\right)\omega\left(f; \frac{1}{\sqrt{n}}\right),$$

where $\|\cdot\|$ is the uniform norm on the interval $[0,1]$.

Proof. It is known that the modulus of continuity $\omega(f;\delta)$ verifies

$$|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right). \tag{2.9}$$

It follows from (2.6) that

$$\begin{aligned} |\alpha(x,n)| &= |\alpha_1(n)x + \alpha_0(n)| \le |\alpha_1(n)| + |\alpha_0(n)| \\ &= |\alpha_1(n)| + \left|\frac{1-\alpha_1(n)}{2}\right| \le \frac{3}{2}|\alpha_1(n)| + \frac{1}{2}. \end{aligned} \tag{2.10}$$

We get

$$|\alpha(1-x,n)| \le \frac{3}{2}|\alpha_1(n)| + \frac{1}{2}.$$

Therefore, using Lemma 2.1, condition (2.6) and $\delta = \dfrac{1}{\sqrt{n}}$, we get

$$\begin{aligned} &\left|U_n^1(f;x) - f(x)\right| \\ &\le |\alpha(x,n)|(1-x)^{n-1}|f(0) - f(x)| + |\alpha(1-x,n)|x^{n-1}|f(1) - f(x)| \\ &+ (n-1)\sum_{k=1}^{n-1}|\alpha(x,n)|p_{n-1,k}(x)\int_0^1 p_{n-2,k-1}(t)|f(t) - f(x)|dt \\ &+ (n-1)\sum_{k=1}^{n-1}|\alpha(1-x,n)|p_{n-1,k-1}(x)\int_0^1 p_{n-2,k-1}(t)|f(t) - f(x)|dt \\ &\le \left(\frac{3}{2}|\alpha_1(n)| + \frac{1}{2}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right)\left\{(1-x)^{n-1}(nx^2+1) + x^{n-1}(n(1-x)^2+1)\right. \\ &+ (n-1)\sum_{k=1}^{n-1}p_{n-1,k}(x)\int_0^1 p_{n-2,k-1}(t)\left(n(t-x)^2+1\right)dt \\ &\left.+ (n-1)\sum_{k=1}^{n-1}p_{n-1,k-1}(x)\int_0^1 p_{n-2,k-1}(t)\left(n(t-x)^2+1\right)dt\right\} \end{aligned}$$

$$= \frac{1}{n+1}\left(3|\alpha_1(n)|+1\right)\omega\left(f;\frac{1}{\sqrt{n}}\right)\left[(-2x^2+2x+1)n+4x^2-4x+2\right]$$
$$\leq 2\left(3|\alpha_1(n)|+1\right)\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

□

Remark 2.2. If $f\in C[0,1]$, then $\lim\limits_{n\to\infty}\omega\left(f,\frac{1}{\sqrt{n}}\right)=0$. Therefore, $(U_n^1)_n$ converges uniformly on $[0,1]$ for $\alpha_1(n)$ a bounded sequence and $f\in C[0,1]$.

Theorem 2.2. *Let $\alpha_i(n), i=0,1$ be convergent sequences that satisfy the conditions (2.7) and $l_0=\lim\limits_{n\to\infty}\alpha_0(n)$. If $f''\in C[0,1]$, then*

$$\lim_{n\to\infty} n\left(U_n^1(f;x)-f(x)\right)=(1-2x)(1-l_0)f'(x)+x(1-x)f''(x),$$

uniformly on $[0,1]$.

Proof. Applying the modified genuine Bernstein-Durrmeyer operators U_n^1 to the Taylor's formula, we obtain

$$U_n^1(f;x)-f(x)=U_n^1(t-x;x)f'(x)+\frac{1}{2}U_n^1\left((t-x)^2;x\right)f''(x)$$
$$+U_n^1\left(\xi(t,x)(t-x)^2;x\right),$$

where $\xi\in C[0,1]$ and $\lim\limits_{t\to x}\xi(t,x)=0$.

Using the Cauchy-Schwarz inequality, we get

$$nU_n^1\left(\xi(t,x)(t-x)^2;x\right)\leq\sqrt{U_n^1\left(\xi^2(t,x);x\right)}\sqrt{n^2U_n^1\left((t-x)^4;x\right)}.$$

From Lemma 2.2 we have $\lim\limits_{n\to\infty}n^2U_n^1\left((t-x)^4;x\right)=12x^2(1-x)^2$. Since $\xi^2(x,x)=0$ and $\xi^2(\cdot,x)\in C[0,1]$, by Remark 2.2, we obtain

$$\lim_{n\to\infty}U_n^1\left(\xi^2(t,x);x\right)=0$$

uniformly with respect to $x\in[0,1]$. Therefore,

$$\lim_{n\to\infty}nU_n^1\left(\xi(t,x)(t-x)^2;x\right)=0.$$

Applying Lemma 2.2, the theorem is proved. □

Now, we extend the results from Theorem 2.2 when the sequences $\alpha_i(n)$, $i=0,1$, satisfy the conditions (2.8), namely the operator U_n^1 is nonpositive.

Theorem 2.3. *Let $\alpha_i(n), i=0,1$ be bounded convergent sequences which satisfy* (2.8) *and $l_0=\lim\limits_{n\to\infty}\alpha_0(n)$. If $f\in C[0,1]$ and f'' exists at a certain point $x\in[0,1]$, then we have*

$$\lim_{n\to\infty} n\left(U_n^1(f;x)-f(x)\right)=(1-2x)(1-l_0)f'(x)+x(1-x)f''(x), \tag{2.11}$$

Moreover the relation (2.11) *holds uniformly on* $[0,1]$ *if* $f''\in C[0,1]$.

Proof. Applying the modified genuine Bernstein-Durrmeyer operators U_n^1 to the Taylor's formula, we get

$$U_n^1(f;x)-f(x)=U_n^1(t-x;x)f'(x)+\frac{1}{2}U_n^1\left((t-x)^2;x\right)f''(x)+U_n^1\left(\xi(t,x)(t-x)^2;x\right),$$

where $\xi\in C[0,1]$ and $\lim\limits_{t\to x}\xi(t,x)=0$. It is sufficient to show that

$$\lim_{n\to\infty}nU_n^1(\xi(t,x)(t-x)^2;x)=0. \tag{2.12}$$

Since the operators U_n^1 are not positive linear operators we can not use Cauchy-Schwarz inequality and we will introduce new techniques in order to prove the theorem.

Let $\varepsilon > 0$ be given. There exist a $\delta > 0$ such that if $|t-x| < \delta$ then $|\xi(t,x)| < \varepsilon$. We denote

$$K_1 = \left\{ i : \left| \frac{i}{n} - x \right| < \delta, i = 0,1,2,\cdots,n \right\},\ K_2 = \left\{ i : \left| \frac{i}{n} - x \right| \geq \delta, i = 0,1,2,\cdots,n \right\}.$$

The boundedness of the sequences $\alpha_i(n)$, $i = 0,1$ implies that these is a constant $C > 0$ such that $|\alpha(x,n)| < C$.

Let $k \in K_1$. Hence $|\xi(t,x)| < \varepsilon$. Therefore we get

$$\begin{aligned}(2.13)\qquad &\left|U_n^1(\xi(t,x)(t-x)^2;x)\right| \leq C(1-x)^{n-1}|\xi(0,x)|x^2 + Cx^{n-1}|\xi(1,x)|(1-x)^2 \\ &+ C(n-1)\sum_{k=1}^{n-1}\{p_{n-1,k}(x) + p_{n-1,k-1}(x)\}\int_0^1 p_{n-2,k-1}(t)|\xi(t,x)|(t-x)^2 dt \\ &\leq C\varepsilon\left\{(n-1)\sum_{k=1}^{n-1}[p_{n-1,k}(x) + p_{n-1,k-1}(x)]\int_0^1 p_{n-2,k-1}(t)(t-x)^2 dt\right. \\ &\left.+(1-x)^{n-1}x^2 + x^{n-1}(1-x)^2\right\} = \frac{2\varepsilon C}{n(n+1)}\left\{2nx(1-x) + (1-2x)^2\right\}.\end{aligned}$$

Let $k \in K_2$. We denote $M = \sup\limits_{0\leq t\leq 1}|\xi(t,x)|(t-x)^2$. Then

$$|\xi(t,x)|(t-x)^2 \leq \frac{M}{\delta^4}\left(\frac{k}{n} - x\right)^4$$

Moreover, the below upper bound is obtained

$$\begin{aligned}(2.14)\qquad &\left|U_n^1(\xi(t,x)(t-x)^2;x)\right| \\ &\leq \frac{MC}{\delta^4}\left\{\sum_{k=1}^{n-1}[p_{n-1,k}(x)+p_{n-1,k-1}(x)]\left(\frac{k}{n}-x\right)^4 + (1-x)^{n-1}x^4 + x^{n-1}(1-x)^4\right\} \\ &= \frac{CM}{n^4\delta^4}\left\{6x^2(1-x)^2n^2 + 4x(1-x)(13x^2-13x+3)n + (1-2x)^2(12x^2-12x+1)\right\}.\end{aligned}$$

Using (2.13) and (2.14), it follows

$$\begin{aligned}&\left|U_n^1(\xi(t,x)(t-x)^2;x)\right| \leq \frac{2\varepsilon C}{n(n+1)}\left\{2nx(1-x) + (1-2x)^2\right\} \\ &+ \frac{CM}{n^4\delta^4}\left\{6x^2(1-x)^2n^2 + 4x(1-x)(13x^2-13x+3)n + (1-2x)^2(12x^2-12x+1)\right\}.\end{aligned}$$

Therefore, the last inequality leads to (2.12) and the theorem is proved. □

3. Better rate of convergence

In the following, we will improve the previous results considering a new genuine Bernstein-Durrmeyer operators that have order of approximation $O\left(n^{-2}\right)$ defined as

$$\begin{aligned}(3.15)\qquad U_n^2(f;x) &= (n-1)\sum_{k=1}^{n-1}p_{n,k}^2(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt \\ &+ \beta(x,n)(1-x)^{n-2}f(0) + \beta(1-x,n)x^{n-2}f(1).\end{aligned}$$

where

$$(3.16)\qquad p_{n,k}^2(x) = \beta(x,n)p_{n-2,k}(x) + \gamma(x,n)p_{n-2,k-1}(x) + \beta(1-x,n)p_{n-2,k-2}(x)$$

and

$$\beta(x,n) = \beta_2(n)x^2 + \beta_1(n)x + \beta_0(n),\ \gamma(x,n) = \gamma_0(n)x(1-x),$$

where $\beta_i(n), i = 0,1,2$ and $\gamma_0(n)$ are unknown sequences. For $\beta_2(n) = \beta_0(n) = 1$, $\beta_1(n) = -2$, $\gamma_0(n) = 2$ we obtained the classical genuine-Bernstein-Durrmeyer operators.

In the following, we suppose $U_n^2(e_0;x) = 1$, namely

$$(2\beta_2(n) - \gamma_0(n))\,x^2 - (2\beta_2(n) - \gamma_0(n))\,x + 2\beta_0(n) + \beta_1(n) + \beta_2(n) = 1.$$

This yields

$$2\beta_2(n) - \gamma_0(n) = 0,\ 2\beta_0(n) + \beta_1(n) + \beta_2(n) = 1.$$

The above relations lead to

$$U_n^2(e_1;x) = x + \frac{2}{n}\left[2\left(\beta_0(n) - 1\right)x - \beta_0(n) + 1\right];$$

$$U_n^2(e_2;x) = x^2 + \frac{2}{n(n+1)}\left[4\beta_0(n)nx^2 - 2\beta_0(n)nx - 8\beta_0(n)x^2 + \beta_2(n)x^2 \right.$$
$$\left. - 5nx^2 + 10\beta_0(n)x - \beta_2(n)x + 3nx + 7x^2 - 3\beta_0(n) - 9x + 3\right].$$

In order to have $\lim\limits_{n\to\infty} U_n^2(e_i;x) = x^i$, $i = 0,1,2$ we suppose the sequences $\beta_0(n)$ and $\beta_2(n)$ to verify the conditions

$$\lim_{n\to\infty}\frac{\beta_0(n)}{n} = 0 \text{ and } \lim_{n\to\infty}\frac{\beta_2(n)}{n^2} = 0.$$

We consider the case $\beta_0(n) = 1$ and $\beta_2(n) = n$. Thus $\beta_1(n) = -n-1$ and $\gamma_0(n) = 2n$. For this particular case the modified genuine Bernstein-Durrmeyer operator becomes

$$\tilde{U}_n^2(f;x) = (n-1)\sum_{k=1}^{n-1}\tilde{p}_{n,k}^2(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt$$
$$+ \left[nx^2 - (n+1)x + 1\right](1-x)^{n-2}f(0) + \left[nx^2 - (n-1)x\right]x^{n-2}f(1), \tag{3.17}$$

where

$$\tilde{p}_{n,k}^2(x) = \left[nx^2 - (n+1)x + 1\right]p_{n-2,k}(x) + 2nx(1-x)p_{n-2,k-1}(x)$$
$$+ \left[nx^2 - (n-1)x\right]p_{n-2,k-2}(x).$$

Denote $\tilde{m}_{n,k}^2(x) := \tilde{U}_n^2(e_k;x)$, $\tilde{\mu}_{n,k}^2(x) := \tilde{U}_n^2\left((t-x)^k;x\right)$, $k = 0,1,\ldots$. the moments and central moments, respectively for the modified genuine Bernstein-Durrmeyer operator $\tilde{U}_n^2$.

Lemma 3.3. *The following statements hold:*

i) $\tilde{m}_{n,0}^2(x) = 1$;
ii) $\tilde{m}_{n,1}^2(x) = x$;
iii) $\tilde{m}_{n,2}^2(x) = x^2 + \dfrac{2x(1-x)}{n(n+1)}$.

Lemma 3.4. *The following statements hold:*

i) $\tilde{\mu}_{n,2}^2(x) = \dfrac{2x(1-x)}{n(n+1)}$,
ii) $\tilde{\mu}_{n,3}^2(x) = -\dfrac{6x(1-x)(1-2x)(n-2)}{n(n+1)(n+2)}$,
iii) $\tilde{\mu}_{n,4}^2(x) = -\dfrac{12x^2(1-x)^2n}{(n+1)(n+2)(n+3)} + O\left(n^{-3}\right)$,

iv) $\tilde{\mu}^2_{n,5}(x) = \dfrac{240x^2(1-x)^2(2x-1)n}{(n+1)(n+2)(n+3)(n+4)} + O\left(n^{-4}\right)$,

v) $\tilde{\mu}^2_{n,6}(x) = -\dfrac{240x^3(1-x)^3n^2}{(n+1)(n+2)(n+3)(n+4)(n+5)} + O\left(n^{-4}\right)$.

Theorem 3.4. *If $f \in C^6[0,1]$ and $x \in [0,1]$, then*

$$\tilde{U}^2_n(f;x) - f(x) = O\left(n^{-2}\right).$$

Proof. Applying the modified genuine Bernstein-Durrmeyer operators $\tilde{U}^2_n$ to the Taylor's formula, we obtain

$$\tilde{U}^2_n(f;x) - f(x) = \sum_{k=1}^{6} f^{(k)}(x)\tilde{U}^2_n\left((t-x)^k;x\right) + \tilde{U}^2_n\left(\xi(t,x)(t-x)^6;x\right),$$

where $\lim\limits_{t\to x}\xi(t,x) = 0$. We have

$$\begin{aligned}\tilde{U}^2_n(f;x) &= (n-1)\left(nx^2-(n+1)x+1\right)\sum_{k=1}^{n-2}p_{n-2,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt \\ &+ 2n(n-1)x(1-x)\sum_{k=1}^{n-1}p_{n-2,k-1}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt \\ &+ (n-1)\left(nx^2-(n-1)x\right)\sum_{k=2}^{n-1}p_{n-2,k-2}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt \\ &+ \left(nx^2-(n+1)x+1\right)(1-x)^{n-2}f(0) + \left(nx^2-(n-1)x\right)x^{n-2}f(1).\end{aligned} \tag{3.18}$$

Let $\varepsilon > 0$ be given. There exist $\delta > 0$ such that if $|t-x| < \delta$, then $|\xi(t,x)| < \varepsilon$. We denote

$$K_1 = \left\{i : \left|\frac{i}{n} - x\right| < \delta, i = 0,1,2,\cdots,n\right\},\ K_2 = \left\{i : \left|\frac{i}{n} - x\right| \geq \delta, i = 0,1,2,\cdots,n\right\}.$$

Let $k \in K_1$. Since $|\xi(t,x)| < \varepsilon$, from (3.18) we obtain

$$\begin{aligned}\left|\tilde{U}^2_n\left(\xi(t,x)(t-x)^6;x\right)\right| &\leq \frac{\varepsilon n}{4}\left\{(n-1)\sum_{k=1}^{n-2}p_{n-2,k}(x)\int_0^1 p_{n-2,k-1}(t)(t-x)^6dt\right. \\ &+ 2(n-1)\sum_{k=1}^{n-1}p_{n-2,k-1}(x)\int_0^1 p_{n-2,k-1}(t)(t-x)^6dt \\ &+ \left.(n-1)\sum_{k=2}^{n-1}p_{n-2,k-2}(x)\int_0^1 p_{n-2,k-1}(t)(t-x)^6dt + (1-x)^{n-2}x^6 + x^{n-2}(1-x)^6\right\} \\ &\leq \frac{120\varepsilon x^3(1-x)^3(1+x-x^2)n^3}{(n+1)(n+2)(n+3)(n+4)(n+5)} + O\left(n^{-3}\right).\end{aligned} \tag{3.19}$$

Let $k \in K_2$. We denote $M = \sup\limits_{0\leq t\leq 1}|\xi(t,x)|(t-x)^6$. Then

$$|\xi(t,x)|\,(t-x)^6 \leq \frac{M}{\delta^6}\left(\frac{k}{n} - x\right)^6.$$

Moreover, the below upper bound is obtained

$$(3.20)\quad \left|\tilde{U}_n^2\left(\xi(t,x)(t-x)^6;x\right)\right| = \frac{Mn}{4\delta^6}\left\{\sum_{k=1}^{n-2} p_{n-2,k}(x)\left(\frac{k}{n}-x\right)^6\right.$$
$$+2\sum_{k=1}^{n-1} p_{n-2,k-1}(x)\left(\frac{k}{n}-x\right)^6+\sum_{k=2}^{n-1} p_{n-2,k-2}(x)\left(\frac{k}{n}-x\right)^6$$
$$\left.+(1-x)^{n-2}x^6+x^{n-2}(1-x)^6\right\} = \frac{15Mx^3(1-x)^3}{n^2\delta^6}+O\left(n^{-3}\right)=O\left(n^{-2}\right).$$

From (3.19) and (3.20) the proof of the theorem is completed. □

In the following, we will improve the previous results considering a new genuine Bernstein-Durrmeyer operators that have order of approximation $O\left(n^{-3}\right)$ defined as

$$(3.21)\quad U_n^3(f;x) = (n-1)\sum_{k=1}^{n-1} p_{n,k}^3(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt$$
$$+\beta(x,n)(1-x)^{n-4}f(0)+\beta(1-x,n)x^{n-4}f(1),$$

where

$$p_{n,k}^3(x) = \beta(x,n)p_{n-4,k}(x)+\gamma(x,n)p_{n-4,k-1}(x)+\delta(x,n)p_{n-4,k-2}(x)$$
$$+\gamma(1-x,n)p_{n-4,k-3}(x)+\beta(1-x,n)p_{n-4,k-4}(x)$$

and

$$\beta(x,n) = \beta_4(n)x^4+\beta_3(n)x^3+\beta_2(n)x^2+\beta_1(n)x+\beta_0(n),$$
$$\gamma(x,n) = \gamma_4(n)x^4+\gamma_3(n)x^3+\gamma_2(n)x^2+\gamma_1(n)x+\gamma_0(n),$$
$$\delta(x,n) = \delta_0(n)(x(1-x))^2.$$

Note that $\beta_i(n)$, $\gamma_i(n)$, $i=0,1,\ldots,4$ and $\delta_0(n)$ are some unknown sequences. Denote $\tilde{U}_n^3$ the operator (3.21) with

$$\beta_0(n)=1, \beta_1(n)=-4-\frac{4}{3}n, \beta_2(n)=5+\frac{10}{3}n+\frac{1}{2}n^2, \beta_3(n)=-n^2-2n-2,$$
$$\beta_4(n)=\frac{1}{2}n^2, \gamma_0(n)=0, \gamma_1(n)=4+\frac{7}{3}n,\ \gamma_2(n)=-\frac{19}{3}n-2n^2-8,$$
$$\gamma_3(n)=4n^2+4n+4, \gamma_4(n)=-2n^2,\ \delta_0(n)=3n^2.$$

Let $\tilde{\mu}_{n,k}^3(x) := \tilde{U}_n^3\left((t-x)^k;x\right)$, $k=0,1,\ldots$ the central moments of $\tilde{U}_n^3$.

Lemma 3.5. *The modified genuine Bernstein-Durrmeyer operator $\tilde{U}_n^3$ verify:*

i) $\tilde{\mu}_{n,1}^3(x)=\tilde{\mu}_{n,2}^3(x)=\tilde{\mu}_{n,3}^3(x)=0$;

ii) $\tilde{\mu}_{n,4}^3(x)=\dfrac{4x(1-x)(39x^2-39x+10)}{\prod_{k=1}^3(n+k)}+O\left(n^{-4}\right)$;

iii) $\tilde{\mu}_{n,5}^3(x)=\dfrac{120(1-2x)x^2(1-x)^2n}{\prod_{k=1}^4(n+k)}+O\left(n^{-4}\right)$;

iv) $\tilde{\mu}_{n,6}^3(x)=\dfrac{120x^3(1-x)^3n^2}{\prod_{k=1}^5(n+k)}+O\left(n^{-4}\right)$;

v) $\tilde{\mu}_{n,7}^3(x)=\tilde{\mu}_{n,8}^3(x)=O\left(n^{-4}\right)$;

vi) $\tilde{\mu}_{n,9}^3(x)=\tilde{\mu}_{n,10}^3(x)=O\left(n^{-5}\right)$.

The asymptotic order of approximation of $\tilde{U}_n^3$ to f when n goes to infinity is given in the following result:

Theorem 3.5. *If $f \in C^{10}[0,1]$ and $x \in [0,1]$, then $\tilde{U}_n^3(f;x) - f(x) = O\left(n^{-3}\right)$.*

4. Numerical Example

Some numerical examples with illustrative graphics have been added to validate the theoretical results and also compare the rate of convergence by using Maple algorithms.

Example 4.1. Let $g(x) = \sin(4\pi x) + 4\sin\left(\frac{1}{4}\pi x\right)$, $n = 10$, $\alpha_0(n) = \frac{n-1}{2n}$ and $\alpha_1(n) = \frac{1}{n}$. The convergence of the modified genuine Bernstein-Durrmeyer operators is illustrated in Figure 1. Let $\varepsilon_n(f;x) = |g(x) - U_n(g;x)|$ and $\varepsilon_n^i(g;x) = \left|g(x) - \tilde{U}_n^i(g;x)\right|$, $i = 1,2,3$ be the error of approximation for genuine Bernstein-Durrmeyer operators and the modified genuine Bernstein-Durrmeyer operators, respectively. The error of approximation is illustrated in Figure 2 and for this particular case the approximation by the modified genuine Bernstein-Durrmeyer operators $\tilde{U}_n^i$, $i = 1,2,3$ is better using classical genuine Bernstein-Durrmeyer operator U_n.

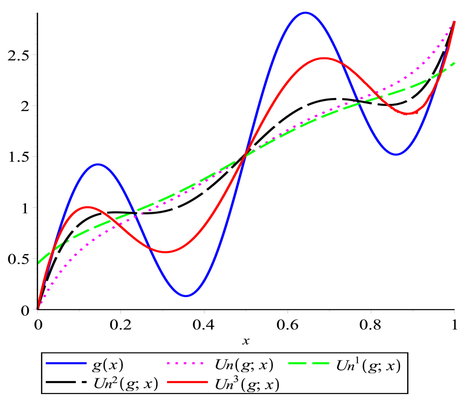

FIGURE 1. Convergence of the modified genuine Bernstein-Durrmeyer operators

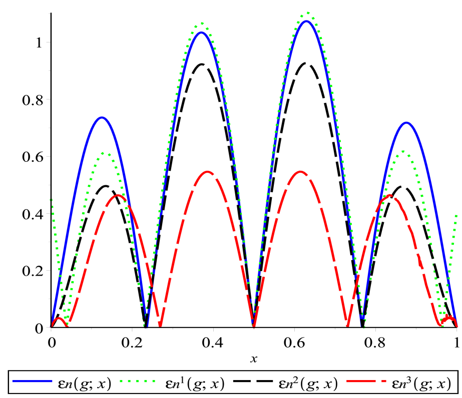

FIGURE 2. Error of approximation by the modified genuine Bernstein-Durrmeyer operators

Example 4.2. Let $g(x) = \left|x - \frac{1}{4}\right|\cos(4\pi x)$. For $n = 10$, $\alpha_0(n) = \frac{n-1}{2n}$ and $\alpha_1(n) = \frac{1}{n}$, the convergence of the modified genuine Bernstein-Durrmeyer operators to $g(x)$ is shown in Figure 3. The errors of approximation ε_n and ε_n^i, $i = 1,2,3$ are illustrated in Figure 4.

Example 4.3. *Let $g(x) = \left(x - \frac{1}{4}\right)\sin(2\pi x)$. The behaviours of the approximations $U_n(g;x)$, $\tilde{U}_n^i(g;x)$, $i = 1,2,3$ and their errors $\varepsilon_n(g;x)$, $\varepsilon_n^i(g;x)$ for $n = 5,7,10$, $\alpha_0(n) = \frac{n-1}{2n}$, $\alpha_1(n) = \frac{1}{n}$ are illustrated in the Figures 5-12.*

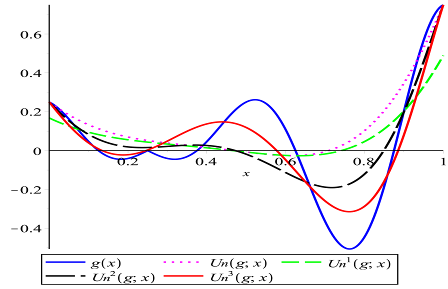

FIGURE 3. Convergence of the modified genuine Bernstein-Durrmeyer operators

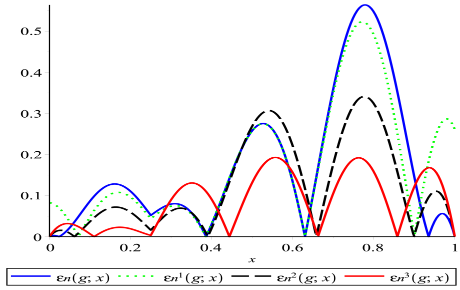

FIGURE 4. Error of approximation ε_n and ε_n^i, $i = 1, 2, 3$

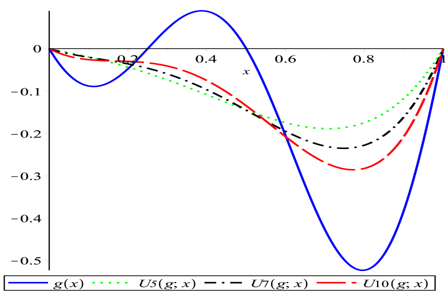

FIGURE 5. Convergence of the genuine Bernstein-Durrmeyer operators U_n

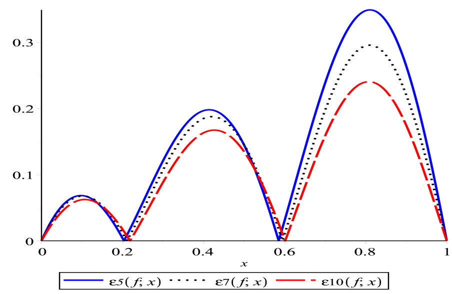

FIGURE 6. Error of approximation ε_n

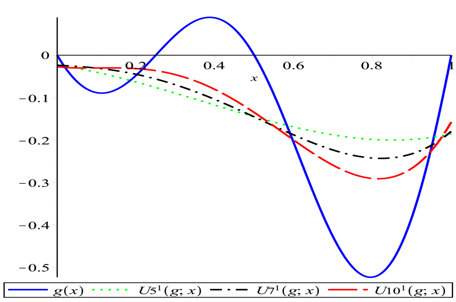

FIGURE 7. Convergence of the modified genuine Bernstein-Durrmeyer operators U_n^1

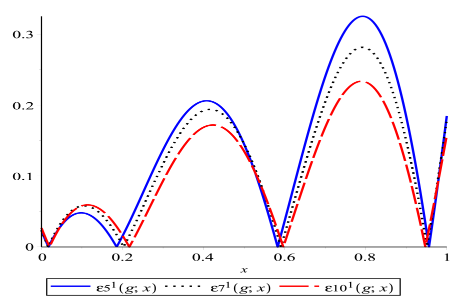

FIGURE 8. Error of approximation ε_n^1

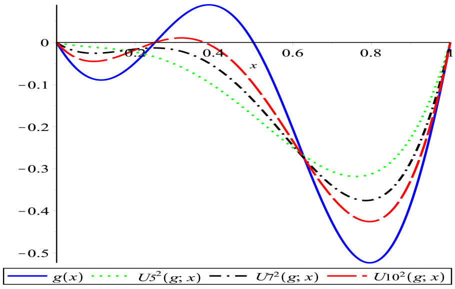

FIGURE 9. Convergence of the modified genuine Bernstein-Durrmeyer operators U_n^2

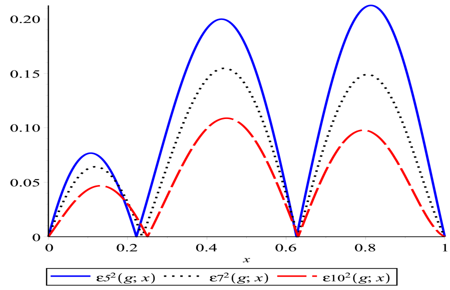

FIGURE 10. Error of approximation ε_n^2

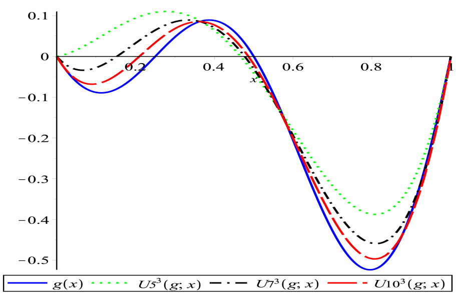

FIGURE 11. Convergence of the modified genuine Bernstein-Durrmeyer operators U_n^3

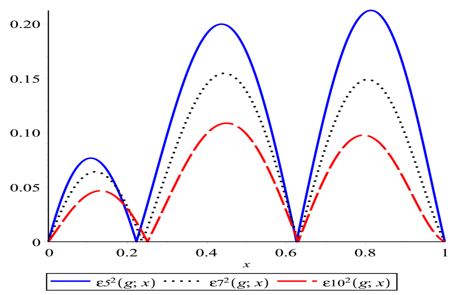

FIGURE 12. Error of approximation ε_n^3

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