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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

A generalization of the Pompeiu mean-value theorem to compact sets

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ABSTRACT. We generalize the Pompeiu mean-value theorem by replacing the graph of a continuous function with a compact set.

1. INTRODUCTION AND RELATED RESULTS

Let $a, b \in \mathbb{R}$, $a < b$, and $f\colon [a,b] \to \mathbb{R}$. Throughout the paper, let $(\alpha, \beta) \in (\mathbb{R} \setminus [a,b]) \times \mathbb{R}$. Denote by $L[a,b;f](x) = \dfrac{f(a)(b-x) + f(b)(x-a)}{b-a}$ the interpolating polynomial associated to f at the points a and b, and if f is differentiable at $c \in (a,b)$, let $T_1[c;f](x) = f(c) + (x-c)f'(c)$ be the Taylor polynomial associated to f at the point c.

In 1946, Pompeiu gave the following variant of the Lagrange mean-value theorem which has been extensively studied (see. e.g., [18]).

Theorem 1.1 ([13], [18, Theorem 3.1]). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) and $0 \notin [a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$\frac{a\,f(b) - b\,f(a)}{a-b} = f(c) - c\,f'(c).$$

He also gave the following geometric interpretation:

The tangent line to the graph of f at the point $(c, f(c))$, the line joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ and the y-axis intersect at the same point.

Another Pompeiu-type mean-value theorem is the following.

Theorem 1.2 ([8, 9, 10]). *Let $f : [a,b] \to \mathrm{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f has no roots in $[a,b]$ and $f(a) \neq f(b)$ then there exists a point $c \in (a,b)$ such that:*

$$\frac{af(b) - bf(a)}{f(b) - f(a)} = c - \frac{f(c)}{f'(c)}.$$

Remark 1.1. We note that, in Theorems 1.1 and 1.2, no condition is imposed on the derivative f'.

Geometrically, this means that the graph of the Taylor polynomial $T_1[c;f]$ and the graph of the Lagrange interpolation polynomial $L_1[a,b;f]$ intersect the Ox axis at the same point.

In 1948, Boggio obtained the following generalization of Pompeiu's theorem.

Theorem 1.3 ([3, Boggio]). *Let $f, g : [a,b] \to \mathbb{R}$ be two functions satisfying the conditions:*

(i) *are continuous on $[a,b]$;*

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(ii) *are differentiable on* (a, b);
(iii) $g(x) \neq 0, \forall x \in [a, b]$;
(iv) $g'(x) \neq 0, \forall x \in (a, b)$.

Then there exists a point $c \in (a, b)$ *such that*

$$\frac{g(a)f(b) - g(b)f(a)}{g(a) - g(b)} = f(c) - g(c)\frac{f'(c)}{g'(c)}.$$

Remark 1.2. Ivan ([10]) showed that the assertion of Boggio's Theorem 1.3 follows by applying Pompeiu's theorem to the function $F = f \circ g^{-1}$. See also [1, 11].

2. Main results

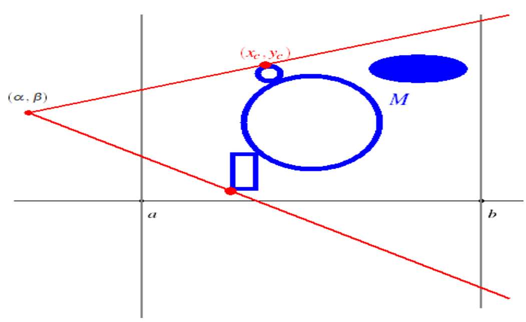
We prove that the geometric property related to the Pompeiu theorem 1.1 remains valid for certain compact sets.

Theorem 2.4. *If* $M \subset [a, b] \times \mathbb{R}$ *is a compact set in the natural topology of* $\mathbb{R}^2$*, then there exists a point* $(x_c, y_c) \in [a, b] \times \mathbb{R}$ *such that the line passing through the points* (α, β) *and* (x_c, y_c) *is an affine majorant of* M *which is exact at* (x_c, y_c)*, i.e.,*

$$y \leq \beta + \frac{y_c - \beta}{x_c - \alpha}(x - \alpha), \qquad \forall\, (x, y) \in M. \tag{2.1}$$

A similar statement can be made for the existence of an affine minorant of M.

Proof. The proof is quite simple, but the theorem generalizes certain mean-value theorems with more complicated proofs.



For definiteness, suppose that $\alpha < a$. The function $m\colon M \to \mathbb{R}$,

$$m(x, y) = \frac{y - \beta}{x - \alpha},$$

is continuous on M hence it attains its maximum at a point $(x_c, y_c) \in M$, i.e.,

$$\frac{y - \beta}{x - \alpha} \leq \frac{y_c - \beta}{x_c - \alpha}, \qquad (x, y) \in M,$$

which is nothing but (2.1), and the proof is complete. □

In particular, the set M might be an implicit curve in a plane defined as the set of zeros of a continuous function of two variables, e.g.,

$$|x| + |y| = 1, \qquad (x, y) \in \mathbb{R}^2.$$

Theorem 2.5. *If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then for any $\alpha \in \mathbb{R} \setminus [a,b]$, there exists a point $c \in (a,b)$ such that the line segment*

$$y = (x-\alpha)\frac{f(c) - L[a,b;f](\alpha)}{c-\alpha} + L[a,b;f](\alpha), \qquad x \in [a,b], \tag{2.2}$$

is an affine support for f at c.

Moreover, if f is differentiable on (a,b), then

$$\frac{f(a)(b-\alpha) + f(b)(\alpha - a)}{b-a} = (\alpha - c)f'(c) + f(c), \tag{2.3}$$

i.e.,

$$L[a,b;f](\alpha) = T_1[c;f](\alpha).$$

Proof. Suppose that $\alpha < a$. Define the continuous function $m\colon [a,b] \to \mathbb{R}$,

$$m(x) = \frac{f(x) - L[a,b;f](\alpha)}{x-\alpha}.$$

Observe that $m(a) = m(b)$. It follows that there exists an interior point of extremum $c \in (a,b)$ such that, for example, $m(x) \geq m(c)$, i.e.,

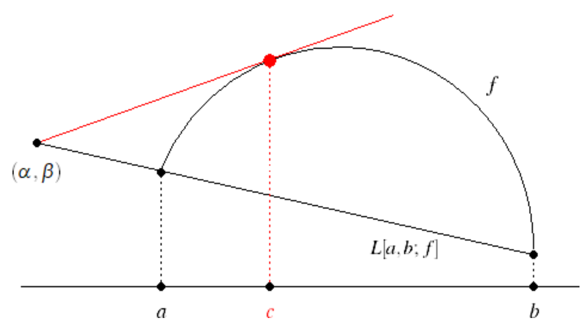
$$f(x) \geq (x-\alpha)\frac{f(c) - L[a,b;f](\alpha)}{c-\alpha} + L[a,b;f](\alpha), \qquad x \in [a,b],$$

and (2.2) is proved.

In the case when f is differentiable on (a,b), it follows that $m'(c) = 0$, hence

$$L[a,b;f](\alpha) = f(c) + (\alpha - c)f'(c).$$

and the proof is complete.



□

Remark 2.3. If $0 \notin [a,b]$, for $\alpha = 0$, Eq. (2.3) becomes the Pompeiu mean-value Theorem 1.1.

Remark 2.4. If f does not vanish on $[a,b]$ and $f(a) \neq f(b)$, for $L[a,b;f](\alpha) = 0$, i.e., $\alpha = \dfrac{bf(a) - af(b)}{f(a) - f(b)}$, Eq. (2.3) implies Theorem 1.2.

The following is a generalization of Boggio's theorem 1.3.

Theorem 2.6. *Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous such that $g(a) \neq g(b)$. Let $A \in \mathbb{R}$ be in the exterior of the interval of endpoints $g(a)$ and $g(b)$. Then there exists a point $c \in (a,b)$ such that the function*

$$x \mapsto (g(x) - A)\frac{f(c) - B}{g(c) - A} + B, \quad x \in [a,b], \tag{2.4}$$

where

$$B = \frac{f(b)(g(a) - A) + f(a)(A - g(b))}{g(a) - g(b)}, \tag{2.5}$$

is a support for f which is exact at c.

Moreover, if f and g are differentiable on (a, b), then

$$(g(c) - A)f'(c) + (B - f(c))g'(c) = 0. \tag{2.6}$$

Proof. Suppose that $g(a) < g(b)$ and $g(x) > A$, for all $x \in [a, b]$. Consider the continuous function, $m\colon [a, b] \to \mathbb{R}$,

$$m(x) = \frac{f(x) - B}{g(x) - A},$$

where B is such that $m(a) = m(b)$, i.e., B is given by (2.5). Since m is continuous and $m(a) = m(b)$, there exists a point $c \in (a, b)$ such that, for example,

$$m(x) \geq m(c), \qquad x \in [a, b],$$

i.e.,

$$f(x) \geq (g(x) - A)\frac{f(c) - B}{g(c) - A} + B, \quad x \in [a, b],$$

and (2.4) is proved.

If m is differentiable on (a, b) we deduce that $m'(c) = 0$ which is equivalent to (2.6) and the proof is complete. □

Remark 2.5. If g' does not vanish on (a, b) and $A = 0$, then Theorem 2.6 simplifies to the Boggio theorem 1.3. See also [16, 17].

Let $\gamma\colon [0, 1] \to \mathbb{R}^2$, $\gamma(t) = (u(t), v(t))$ be a parameterized differentiable regular and closed curve such that $u([0, 1]) = [a, b]$. It follows that:

- γ' does not vanish;
- $\gamma(0) = \gamma(1)$ and $\gamma'(0_+) = \gamma'(1_-)$.

The following is a generalization of Pompeiu's theorem for differentiable, regular and closed curves.

Theorem 2.7. *There exists $(x_c, y_c) \in \gamma([0, 1])$ such that the line segment*

$$y = \beta + \frac{y_c - \beta}{x_c - \alpha}(x - \alpha), \qquad x \in [a, b].$$

is an affine support for the set $\gamma([0, 1])$ which is exact at (x_c, y_c).

Proof. Let $m(t) := \dfrac{v(t) - \beta}{u(t) - \alpha}$, $t \in [0, 1]$. Since m is continuous, there exists $c \in [0, 1]$ be such that, e.g.,

$$m(t) \leq m(c), \qquad \forall t \in [0, 1].$$

Consider the line of equation

$$y - \beta = m(c)(x - \alpha). \tag{2.7}$$

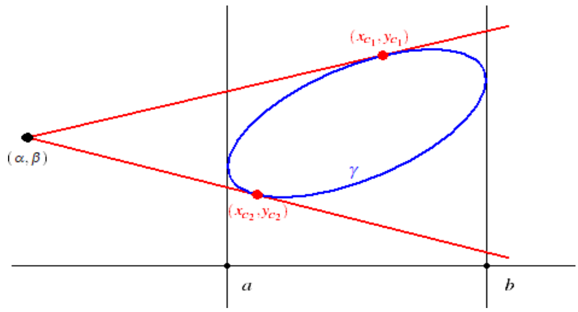
We have

$$y(u(t)) - \beta = m(c)(u(t) - \alpha) \geq m(t)(u(t) - \alpha) = v(t) - \beta, \qquad \forall t \in [0, 1],$$

i.e., $y(u(t)) \geq v(t)$, for all $t \in [0, 1]$, hence (2.7) is an affine support of $\gamma([0, 1])$. Put $x_c := u(c)$ and $y_c := v(c)$. Since γ is closed and differentiable on $[0, 1]$ we deduce that $m'(c) = 0$, hence

$$v'(c)((u(c) - \alpha) = u'(c)(v(c) - \beta),$$

i.e., (2.7) is tangent to $\gamma([0,1])$ at (x_c, y_c).



The proof is complete. □

Without claiming exhaustiveness, we also mention other works related to the Pompeiu mean-value theorem: [2], [4, 5], [7, 6], [12], [14, 15], [19, 20]. We hope that some results in the above papers may be extended to compact sets.

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