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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Coupled fixed point theorems in quasimetric spaces without mixed monotonicity

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ABSTRACT. In this paper, using the concepts of f-closed set and inverse f-closed set, we will prove some fixed point theorems for graphic contractions in complete quasimetric space. Then, as applications, coupled fixed point theorems in quasimetric spaces without the mixed monotonicity property are obtained.

1. INTRODUCTION

A nice extension of Banach's contraction principle was established by Ran and Reurings (see [28]). They combined, for an operator $f : X \to X$, the contraction condition imposed only for pairs $(x, y) \in X \times X$ of comparable elements (with respect to a partial order relation on X) with a monotonicity assumption. Various extensions of Ran-Reurings theorem were published, see [17], [18], [19], [21], [32].

Another topic of high interest in the last decades is the coupled fixed point theory, later generalized to tripled fixed point theory or to multiple fixed point theory. For coupled fixed point theory, we refer to the seminal papers of D. Guo and V. Lakshmikantham [10], T. Gnana Bhaskar and V. Lakshmikantham [9] and V. Berinde [4]. For related contributions see [2], [5], [7], [6], [14], [11], [22], [23], [24], [26] and the references therein.

In this paper, using the concepts of f-closed set and inverse f-closed set, we will prove some fixed point theorems in complete quasimetric space. Then, as applications, coupled fixed point theorems in quasimetric spaces without any mixed monotonicity property are obtained. Our theorems extend and complement some results given in [25] and [20], where the framework of a complete metric space is considered.

2. FIXED POINTS IN QUASIMETRIC SPACES

In this section, we will focus on fixed point theorems on complete quasimetric spaces. For the sake of completeness we recall the definition of a quasimetric space.

Definition 2.1. ([1, 3, 8]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a quasimetric (also called, in many papers, b-metric) with constant $s \geq 1$ if the classical axioms of the metric hold, with the following modification of the triangle inequality:

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in X.$$

A pair (X, d) with the above properties is called a quasimetric space.

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For several relevant examples of b-metrics see [3], [13], ...

The following lemma (Lemma 2.2 in see [16]) is essential in our approach. See also [30] and [31].

Lemma 2.1. *Let (X, d) be a b-metric space with constant $s \geq 1$. Then, every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X for which there exists $\gamma \in (0, 1)$ such that*

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n), \text{ for every } n \in \mathbb{N}^*,$$

is a Cauchy sequence. Moreover, the following estimation holds

$$d(x_{n+1}, x_{n+p}) \leq \frac{\gamma^n S}{1-\gamma} d(x_0, x_1), \text{ for all } n, p \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \gamma^{2i \log_\gamma s + 2^{i-1}}$.

The following abstract notion was defined in [25].

Definition 2.2. Let X be a nonempty set, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Then, $\mathbb{P}$ is said f-closed if the following implication holds:

$$(z, w) \in \mathbb{P} \text{ implies } (f(z), f(w)) \in \mathbb{P}.$$

Some examples of f-closed sets are presented in [20] and [25].

Example 2.1. 1) Let X be a nonempty set endowed with a partial order $\preceq$ and $f : X \to X$ be an increasing operator. If we define
$\mathbb{P}_1 := \{(x, y) \in X \times X : x \preceq y\}$
$\mathbb{P}_2 := \{(x, y) \in X \times X : y \preceq x\}$.
$\mathbb{P}_3 := \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$,
then $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are f-closed sets.

2) Let X be a nonempty set and denote by $\Delta := \{(x, x) \in X \times X : x \in X\}$ the diagonal of the Cartesian product $X \times X$. Let $f : X \to X$ be an edges preserving operator, see [12]. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Then $\mathbb{P}_4 := \{(x, y) \in X \times X : (x, y) \in E(G)\}$ is f-closed.

The following lemma gives the relation between an f-closed set and the monotonicity of a given operator.

Lemma 2.2. ([25]) *Let X be a nonempty set, $\mathbb{P} \subset X \times X$ such that $\Delta \subset \mathbb{P}$ and $f : X \to X$ be a given operator. We define*

$$x \preceq y \Leftrightarrow (x, y) \in \mathbb{P}.$$

Then, $\mathbb{P}$ is f-closed if and only if f is increasing with respect to $\preceq$.

If X is a nonempty set and $f : X \to X$, then $Fix(f) := \{x \in X : x = f(x)\}$.

The first main result of this paper is a fixed point theorem for graphic contractions in quasimetric spaces. The result is a generalization of several Ran-Reurings type theorems in the literature.

Theorem 2.1. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose:*
(i) $\mathbb{P}$ is f-closed;
(ii) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in \mathbb{P}$;
(iii) $\lim_{n\to\infty} f(f^n(x_0)) = f(\lim_{n\to\infty} f^n(x_0))$;

(iv) there exists $\alpha \in]0,1[$ *such that, for all* $x \in X$ *for which* $(x, f(x)) \in \mathbb{P}$*, we have*

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f *has at least one fixed point and the sequence* $x_n := f^n(x_0)$*,* $n \in \mathbb{N}$ *converges to an element* $x^* \in Fix(f)$*. Moreover,*

$$d(f^n(x_0), x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Let us denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. By (ii) and the f-closedness property of $\mathbb{P}$ we obtain that $(x_n, x_{n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$. Then, by the graphic contraction condition (iv) we obtain that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.1 we have that (x_n) is a Cauchy sequence. Using the completeness of the quasimetric space we obtain that (x_n) is convergent in (X, d). Denote by x^* its limit. By (iii), we immediately get that $x^* \in Fix(f)$. By the second conclusion of Lemma 2.1, we obtain that

$$d(x_{n+1}, x_{n+p}) \leq \frac{\alpha^n S}{1-\alpha} d(x_0, f(x_0)), \text{ for all } n, p \in \mathbb{N}.$$

Then, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\frac{1}{s} d(x_{n+1}, x^*) \leq d(x_{n+1}, x_{n+p}) + d(x_{n+p}, x^*) \leq \frac{\alpha^n S}{1-\alpha} d(x_0, f(x_0)) + d(x_{n+p}, x^*).$$

Letting $p \to \infty$ we get that $d(x_{n+1}, x^*) \leq \frac{s\alpha^n S}{1-\alpha} d(x_0, f(x_0))$, for all $n \in \mathbb{N}$. □

We present now a kind of dual definition of an f-closed set.

Definition 2.3. Let X be a nonempty set, $\mathbb{S} \subset X^2$ and $f : X \to X$ be an operator. Then, $\mathbb{S}$ is said inverse f-closed if the following implication holds:

$$(x, y) \in \mathbb{S} \text{ implies } (f(y), f(x)) \in \mathbb{S}.$$

Notice that if $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is an decreasing operator, then the sets $\mathbb{P}_i$ ($i \in \{1, 2, 3\}$) from Example 2.1 are inverse f-closed, while $\mathbb{P}_4$ isn't an inverse f-closed set. Moreover, the following lemma takes place.

Lemma 2.3. ([20]) *Let* X *be a nonempty set,* $\mathbb{S} \subset X \times X$ *such that* $\Delta \subset \mathbb{S}$ *and* $f : X \to X$ *be a given operator. We define*

$$x \preceq y \Leftrightarrow (x, y) \in \mathbb{S}.$$

Then:
(a) $\mathbb{S}$ *has the transitive property if and only if* $\preceq$ *is a preorder on* X*;*
(b) $\mathbb{S}$ *is inverse* f*-closed if and only if* f *is decreasing with respect to* $\preceq$*.*

Proof. (a) follows directly from the definition of $\preceq$. For (b) let us suppose that f is decreasing with respect to $\preceq$. We show that $\mathbb{S}$ is inverse f-closed. Indeed, take $x, y \in X$ with $(x, y) \in \mathbb{S}$. Then $x \preceq y$. By the monotonicity of f we get that $f(y) \preceq f(x)$. Thus $(f(y), f(x)) \in \mathbb{S}$. Hence, $\mathbb{S}$ is inverse f-closed. For the reverse implication, suppose that $\mathbb{S}$ is inverse f-closed and take $x, y \in X$ with $x \preceq y$. Then $(x, y) \in \mathbb{S}$, and by the inverse f-closedness property of $\mathbb{S}$, we have that $(f(y), f(x)) \in \mathbb{S}$. Thus, $f(y) \preceq f(x)$. □

We can prove now a fixed point theorem for a graphic contraction $f : X \to X$ in complete quasimetric spaces in terms of inverse f-closed sets.

Theorem 2.2. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose that the following conditions are satisfied:*

(1) $\mathbb{P}$ is inverse f-closed;

(2) there exists $x_0 \in X$ such that $(x_0, f(x_0))$ or $(f(x_0), x_0)$ belongs to $\mathbb{P}$;

(3) $\lim\limits_{n\to\infty} f(f^n(x_0)) = f(\lim\limits_{n\to\infty} f^n(x_0))$;

(4) there is $\alpha \in]0, 1[$ such that, for $x \in X$ with $\{(x, f(x)) : x \in X\} \cup \{(f(x), x) : x \in X\} \subset \mathbb{P}$, we have that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$. Moreover, we have

$$d(x_n, x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \textit{ for all } n \in \mathbb{N}, n \geq 1,$$

where $S := \sum\limits_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. Suppose, for example, that $(x_0, f(x_0)) \in \mathbb{P}$. By the inverse f-closedness property of $\mathbb{P}$ we obtain that (x_2, x_1), (x_2, x_3), . . ., (x_{2n}, x_{2n-1}), (x_{2n}, x_{2n+1}), . . . are in $\mathbb{P}$, for all $n \in \mathbb{N}^*$. Then, by the graphic contraction condition (4), we obtain that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.1 we have that (x_n) is a Cauchy sequence. By the completeness of the quasimetric space we obtain that (x_n) is convergent in (X, d). Denote by x^* its limit. By (3), we immediately get that $x^* \in Fix(f)$. The second conclusion can be obtained by a similar approach to that given in the proof of Theorem 2.1. □

By Theorem 2.2 a fixed point theorem for decreasing operators can be obtained.

Theorem 2.3. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\preceq$ be a partial order on X and $f : X \to X$ be an operator. Suppose that the following conditions are satisfied:*

(1) f is decreasing with respect to $\preceq$;

(2) there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$;

(3) $\lim\limits_{n\to\infty} f(f^n(x_0)) = f(\lim\limits_{n\to\infty} f^n(x_0))$;

(4) there is $\alpha \in]0, 1[$ such that, for $x \in X$ with $x \preceq f(x)$ or $f(x) \preceq x$, we have that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)).$$

Then f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$. Moreover, we have

$$d(x_n, x^*) \leq \frac{s\alpha^{n-1}S}{1-\alpha} d(x_0, f(x_0)), \textit{ for all } n \in \mathbb{N}, n \geq 1,$$

where $S := \sum\limits_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. Since f is decreasing with respect to $\preceq$, the set $\mathbb{P} := \{(x, y) \in X \times X : x \preceq y\}$ is inverse f-closed (by Lemma 2.3). Thus, Theorem 2.3 follows by Theorem 2.2. □

Remark 2.1. For other results for graphic contractions in complete metric spaces see [27].

3. Coupled fixed point theorems in quasimetric spaces

In this section, we will show how the above results can be applied to coupled fixed point theory.

Recall that, if X is a nonempty set and $F : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for F is a pair $(x, y) \in X \times X$ satisfying the system

$$\begin{cases} x = F(x, y) \\ y = F(y, x). \end{cases} \tag{3.1}$$

We denote by $CFix(F)$ the coupled fixed set of F. Notice also that if (x, y) is a solution of the coupled fixed point problem with $x = y$, then x is said to be a fixed point for F.

In the above context we denote

$$F^n(x, y) := F(F^{n-1}(x, y), F^{n-1}(y, x)), n \in \mathbb{N}, n \geq 1,$$

where $F^0(x, y) := x$ and $F^0(y, x) := y$.

The following definition is modeled after Kubti et all. [14].

Definition 3.4. Let (X, d) be a nonempty set and $F : X \times X \to X$ be an operator. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-closed if for all $x, y, u, v \in X$ the following implication holds:

$$(x, y, u, v) \in M \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}.$$

The following auxiliary result immediately follows by the above definitions.

Lemma 3.4. (see [20]) *Let X be a nonempty set, $F : X \times X \to X$ be a given operator and $\mathbb{M}$ be an F-closed set. Denote $Z := X \times X$. Denote*

$$\mathbb{P} := \{(z, w) \in Z \times Z : \ z = (x, y), \ w = (u, v), \ (x, y, u, v) \in \mathbb{M}\}$$

and define $T_F(x, y) := (F(x, y), F(y, x))$, for all $(x, y) \in Z$.
Then the following implication holds:

$\mathbb{M}$ is F-closed if and only if $\mathbb{P}$ is T_F-closed.

By Theorem 2.1, we obtain the following very general existence result for the coupled fixed point problem.

Theorem 3.4. *Let (X, d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator. Suppose:*
(i) $\mathbb{M}$ is F-closed;
(ii) there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$;
(iii) the following relations hold:
$(iii)_1$ $\lim\limits_{n\to\infty} F^{n+1}(x_0, y_0) = F(\lim\limits_{n\to\infty} F^n(x_0, y_0), \lim\limits_{n\to\infty} F^n(y_0, x_0))$;
$(iii)_2$ $\lim\limits_{n\to\infty} F^{n+1}(y_0, x_0) = F(\lim\limits_{n\to\infty} F^n(y_0, x_0), \lim\limits_{n\to\infty} F^n(x_0, y_0))$;
(iv) there exists $\alpha \in]0, 1[$ such that, if $(x, y, F(x, y), F(y, x)) \in \mathbb{M}$, then

$$\max\{d(F(x, y), F^2(x, y)), d(F(y, x), F^2(y, x))\} \leq \alpha \max\{d(x, F(x, y)), d(y, F(y, x))\}.$$

Then F has at least one coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0, y_0))_{n\in\mathbb{N}}$ and $(F^n(y_0, x_0))_{n\in\mathbb{N}}$ converge to x^* and y^*, respectively. Moreover, for all $n \in \mathbb{N}$, the following approximations took place

$$\max\{d(F^n(x_0, y_0), x^*), d(F^n(y_0, x_0), y^*)\} \leq K \max\{d(x_0, F(x_0, y_0)), d(y_0, F(y_0, x_0))\},$$

where $K := \frac{s\alpha^{n-1}S}{1-\alpha}$ and $S := \sum\limits_{i=1}^{\infty} \alpha^{2i \log_\alpha s + 2^{i-1}}$.

Proof. We endow $Z := X \times X$ with the quasimetric

$$\widehat{d}((x,y),(u,v)) := \max\{d(x,u), d(y,v)\}, \text{ for all } (x,y),(u,v) \in Z.$$

We consider the operator $T_F : Z \to Z$ given by

$$T_F(x,y) := (F(x,y), F(y,x)), \text{ for all } (x,y) \in Z$$

and define $\mathbb{P} := \{(z,w) \in Z \times Z : \ z = (x,y), \ w = (u,v), \ (x,y,u,v) \in \mathbb{M}\}$.

By our hypotheses, T_F satisfies the following assumptions:
(1) $\mathbb{P}$ is T_F-closed (since $\mathbb{M}$ is F-closed);
(2) there exists $z_0 := (x_0, y_0) \in Z$ such that $(z_0, T_F(z_0)) \in \mathbb{P}$;
(3) $\lim\limits_{n\to\infty} T_F(T_F^n(z_0)) = T_F(\lim\limits_{n\to\infty} T_F^n(z_0))$.
(4) for $\alpha \in]0,1[$ and for all $(z, T_F(z)) \in \mathbb{P}$, we have $\widehat{d}(T_F(z), T_F^2(z)) \leq \alpha \widehat{d}(z, T(z))$.
Since $Fix(T_F) = CFix(F)$, the conclusion follows by Theorem 2.1. □

Remark 3.2. Theorem 3.4 generalizes several results given in [25], where some particular cases of graphic contractions type operators F are considered.

In particular, the following result (which is more appropriate for applications) holds.

Theorem 3.5. *Let (X,d) be a complete quasimetric space with constant $s \geq 1$, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator. Suppose:*
(i) $\mathbb{M}$ is F-closed;
(ii) there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0,y_0), F(y_0,x_0)) \in \mathbb{M}$;
(iii) the following relations hold:
$(iii)_1$ $\lim\limits_{n\to\infty} F^{n+1}(x_0,y_0) = F(\lim\limits_{n\to\infty} F^n(x_0,y_0), \lim\limits_{n\to\infty} F^n(y_0,x_0))$;
$(iii)_2$ $\lim\limits_{n\to\infty} F^{n+1}(y_0,x_0) = F(\lim\limits_{n\to\infty} F^n(y_0,x_0), \lim\limits_{n\to\infty} F^n(x_0,y_0))$;
(iv) there exists $\alpha \in]0,1[$ such that, if $(x,y,F(x,y),F(y,x)) \in \mathbb{M}$ and $(y,x,F(y,x),F(x,y)) \in \mathbb{M}$, then

$$d(F(x,y), F^2(x,y)) \leq \alpha d(x, F(x,y)).$$

Then F has at least one coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ and $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ converge to x^* and y^*, respectively. Moreover, for all $n \in \mathbb{N}$, the following approximation formula took place:

$$\max\{d(F^n(x_0,y_0),x^*), d(F^n(y_0,x_0),y^*)\} \leq K \max\{d(x_0, F(x_0,y_0)), d(y_0, F(y_0,x_0))\},$$

where $K := \frac{s\alpha^{n-1}S}{1-\alpha}$ *and* $S := \sum\limits_{i=1}^{\infty} \alpha^{2i\log_\alpha s + 2^{i-1}}$.

Remark 3.3. Using Theorem 2.2 and Theorem 2.3, by a similar approach, we can obtain corresponding coupled fixed point theorems in terms of inverse F-closed sets.

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REFERENCES

[1] Bakhtin, I. A., *The contraction mapping principle in almost metric spaces*, Funct. Anal. Ul'yanovsk, Gos. Ped. Inst., **30** (1989), 26–37

[2] Ben-El-Mechaiekh, H., *The Ran-Reurings fixed point theorem without partial order: A simple proof*, J. Fixed Point Theory Appl., **16** (2014), 373–383

[3] Berinde, V., *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, **3** (1993), 3–9

[4] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 7347–7355

[5] Berinde, V. and Păcurar, M., *Coupled and triple fixed point theorems for mixed monotone almost contractive mappings in partially ordered metric spaces*, J. Nonlinear Convex Anal., **18** (2017), 651–659

[6] Chen, Y.-Z., *Existence theorems of coupled fixed points*, J. Math. Anal. Appl., **154** (1991), 142–150

[7] Choban, M. M. and Berinde, V., *A general concept of multiple fixed point for mappings defined on spaces with a distance*, Carpathian J. Math., **33** (2017), 275–286

[8] Czerwik, S., *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11

[9] Gnana Bhaskar, T. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393

[10] Guo, D. and Lakshmikantham, V., *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11** (1987), 623–632

[11] Harjani, J. and Sadarangani, K., *Fixed point theorems for monotone generalized contractions in partially ordered metric spaces and applications to integral equations*, J. Convex Anal., **19** (2012), 853–864

[12] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373

[13] Kirk, W. A. and Shahzad, N., *Fixed Point Theory in Distance Spaces*, Springer, New York, 2014

[14] Kutbi, M. A., Roldán, A., Sintunavarat, W., Martínez-Moreno, J. and Roldán, C., *F-closed sets and coupled fixed point theorems without the mixed monotone property*, Fixed Point Theory Appl., 2013, 2013:330, 1–11

[15] Lakshmikantham, V. and Ćirić, L., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349

[16] Miculescu, R. and Mihail, A., *New fixed point theorems for set-valued contractions in b-metric spaces*, J. Fixed Point Theory Appl., **19** (2017), 2153–2163

[17] Nieto, J. J. and Rodriguez-Lopez, R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239

[18] Nieto, J. J. and Rodriguez-Lopez, R., *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica (English Series), **23** (2007), 2205–2212

[19] O'Regan, D. and Petruşel, A., *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., **341** (2008), 1241–1252

[20] Petruşel, A., *Fixed points Vs. coupled fixed points*, J. Fixed Point Theory Appl., (2018) 20: 150. https://doi.org/10.1007/s11784-018-0630-6

[21] Petruşel, A. and Rus, I. A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134** (2005), 411–418

[22] Petruşel, A., Petruşel, G., Samet, B. and Yao, J.-C., *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to a system of integral equations and a periodic boundary value problem*, Fixed Point Theory, **17** (2016), 459–478

[23] Petruşel, A. and Petruşel, G., *A study of a general system of operator equations in b-metric spaces via the vector approach in fixed point theory*, J. Fixed Point Theory Appl., **19** (2017), 1793–1814

[24] Petruşel, A., Petruşel, G. and Samet, B., *A study of the coupled fixed point problem for operators satisfying a max-symmetric condition in b-metric spaces with applications to a boundary value problem*, Miskolc Math. Notes, **17** (2016), 501–516

[25] Petruşel, A., Petruşel, G., Xiao, Y.-B. and Yao J.-C., *Fixed point theorems for generalized contractions with applications to coupled fixed point theory*, J. Nonlinear Convex Anal., **19** (2018), 71–88

[26] Petruşel, A. and Petruşel, G., *Coupled fixed points and coupled coincidence points via fixed point theory*, Mathematical Analysis and Applications: Selected Topics (M. Ruzhansky, H. Dutta, R. P. Agarwal - Eds.), Wiley, 2018, 661–708

[27] Petruşel, A. and Rus, I. A. *Graphic contraction principle and applications*, Springer, Berlin, 2019, to appear

[28] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443

[29] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*, Cluj University Press Cluj-Napoca, 2008

[30] Suzuki, T., *Basic inequality on a b-metric space and its applications*, J. Ineq. Appl., (2017) 2017:256. https://doi.org/10.1186/s13660-017-1528-3

[31] Suzuki, T., *Fixed point theorems for single- and set-valued F-contractions in b-metric spaces*, J. Fixed Point Theory Appl., (2018) 20:35. https://doi.org/10.1007/s11784-018-0519-4

[32] Turinici, M., *Contraction maps in ordered metrical structures*, In: Pardalos P., Rassias T. (Eds.), Mathematics without Boundaries, Springer, New York, 2014, 533–575

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