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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness

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ABSTRACT. In this paper, we propose fixed point results through the notion of a measure of noncompactness and give a generalization of a Darbo's fixed point theorem. We also prove some new tripled fixed point results via a measure of noncompactness for a more general class of functions. Our results generalize and extend some comparable results in the literature. Further, we apply the obtained fixed point theorems to prove the existence of solutions for a general system of non-linear functional integral equations. In the end, an example is given to illustrate the validity of our results.

1. INTRODUCTION

Fixed point theory is a crucial field in mathematics which has numerous applications in various fields of science and technology. Poincare initiated the study of fixed point theory after that Brouwer [16] established a fixed point result what has become the well-known Brouwer's fixed point theorem for finite dimensional spaces. While in 1922, Banach [14] brought his celebrated Banach contraction principle for complete metric spaces which ensures the existence and uniqueness of fixed point. Later on, in 1930, Schauder [26] extended the Brouwer's fixed point theorem to infinite dimensional spaces using the condition of compactness on a set and equivalently on the operator. On the other hand, the concept of a measure of noncompactness is a very useful tool in nonlinear functional analysis, especially in metric and topological fixed point theory. Firstly, Kuratowski [24] in 1930 defines the concept of measure of noncompactness in the following way:

$$\alpha(S) = \inf\Big\{\delta > 0 : S \subset \bigcup_{i=1}^{n} S_i \text{ with } diam(S_i) \leq \delta, 1 \leq i \leq n < \infty\Big\},$$

for a bounded set S in a metric space, where $diam(S_i)$ denotes the diameter of the set S_i, i.e. $diam(S_i) = \sup\big\{d(x,y) : x, y \in S_i\big\}$. In 1955, Darbo published a fixed point theorem [18] using the concept of a measure of noncompactness, which guarantees the existence of a fixed point for condensing operators. Darbo's theorem [18] extends both classical Banach fixed point theorem and Schauder's fixed point theorem and it has an abundance of applications on the existence of solutions of differential and integral equations. Up to now, several papers have been published on the generalization of the Darbo's fixed point theorem(for more details see [2, 3, 6, 9, 11, 17, 25]) and on the existence and behavior of solutions of nonlinear differential and integral equations (for more details see[1, 4, 5, 13, 19, 20, 21]) using the concept of a measure of noncompactness. Recently, Roshan[25] gave a generalization of Darbo's fixed point theorem and also presented some results on

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coupled fixed points. In this paper we extend Darbo's fixed point theorem and using these result to obtain the existence of tripled fixed points.

Throughout this paper, we will work in a Banach space E with the norm $\|.\|$ and the zero elements θ. Denote by $B(x, r)$ the closed ball centered at x with radius r. We use the standard notation λX and $X+Y$ to denote the algebraic operations on sets. Moreover, the symbol $\overline{X}$ stands for the closure of a set X, while coX, $\overline{co}X$ denotes the convex hull and closed convex hull of X respectively. Finally, we denote $\mathfrak{M}_E$ for the family of all bounded nonempty subsets of the space E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

2. Preliminaries

Now we recall the axiomatic definition of a measure of noncompactness.

Definition 2.1. [12] A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+ = [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

MNC1. The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$;
MNC2. $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$;
MNC3. $\mu(X) = \mu(\overline{X})$;
MNC4. $\mu(coX) = \mu(X)$;
MNC5. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, for all $\lambda \in [0, 1]$;
MNC6. If X_n is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

It follows from Definition 2.1 that the family $Ker\mu$ described in (MNC1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (MNC6) is a member of the family $Ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\mu$.

Definition 2.2. (Compact operator) [23] An operator $T : X \to Y$ is called compact if $T(S)$ is relatively compact in a Banach space Y for any bounded subset S in a Banach space X.

Theorem 2.1. (Schauder's fixed point theorem) [26] *Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : C \to C$ has one fixed point in C.*

Theorem 2.2. (Darbo's fixed point theorem) [18] *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ such that*

$$\mu(TS) \le k\mu(S),$$

for any nonempty subset S of C. Then T has a fixed point in the set C.

Definition 2.3. [15] An element (x, y) in E^2 is called a coupled fixed point of a mapping $T : E^2 \to E$ if $T(x, y) = x$ and $T(y, x) = y$.

Lemma 2.1. [8] *Suppose that $\mu_1, \mu_2, \cdots, \mu_n$ are measures of noncompactness in Banach spaces $E_1, E_2, \cdots, E_n$ respectively. Moreover, assume that the function $J : [0, \infty)^n \longrightarrow [0, \infty)$ is convex and $J(x_1, x_2, \cdots, x_n) = 0$ if and only if each $x_i = 0$ for all $i = 1, 2, \cdots, n$. Then we define a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ as follows*

$$\mu(S) = J(\mu_1(S_1), \mu_2(S_2), \cdots, \mu_n(S_n)),$$

where S_i denotes the natural projection of S into E_i for $i = 1, 2, \cdots, n$.

From now on, if S is a nonempty subset of E^d where E is a Banach space, we will write S_i for the image $\pi_i(S)$ for $i = 1, 2, \cdots, d$ where $\pi(x_1, x_2, \cdots, x_d) = x_i$, $(x_1, x_2, \cdots, x_d) \in S$. Roshan [25] gave the following class of function, let Φ be the class of all functions $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with usual order relation " $\leq$ " on $\mathbb{R}_+ \times \mathbb{R}_+$ as $(t_1, t_2) \leq (s_1, s_2) \Longleftrightarrow t_1 \leq s_1$ and $t_2 \leq s_2$, satisfying the following conditions:

Φ_1. ϕ is continuous and nondecreasing on $\mathbb{R}_+ \times \mathbb{R}_+$,
Φ_2. $\phi(t, t) < t$ for all $t > 0$,
Φ_3. $\frac{1}{2}\phi(t_1, t_2) + \frac{1}{2}\phi(s_1, s_2) \leq \phi(\frac{t_1+s_1}{2}, \frac{t_2+s_2}{2})$ with $t_i, s_i \in \mathbb{R}_+$ for $i = 1, 2$.

Theorem 2.3. [25] *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T : C \times C \longrightarrow C \times C$ be a continuous function satisfying*

$$\mu^*\big(T(S)\big) \leq \phi\big(\mu^*(S), \mu^*(S)\big),$$

for any nonempty subset S of $C \times C$, where μ^* is defined by Lemma 2.1 and $\phi \in \Phi$. Then T has fixed point.

Definition 2.4. [21] An element (x, y, z) in E^3 is called a tripled fixed point of a mapping $T : E^3 \to E$ if $T(x, y, z) = x$, $T(y, x, z) = y$ and $T(z, y, x) = z$.

Now, as a result of Lemma 2.1 we present the following examples.

Example 2.1. Let μ be a measure of noncompactness on a Banach space E, and let the function $J : [0, +\infty)^3 \to [0, +\infty)$ is convex and $J(x_1, x_2, x_3) = 0$ if and only if $x_i = 0$ for $i = 1, 2, 3$. Then

$$\mu^*(S) = J(\mu(S_1), \mu(S_2), \mu(S_3))$$

defines a measure of noncompactness in $E \times E \times E$.

Example 2.2. Let μ be a measure of noncompactness on a Banach space E, and consider a map $J(x, y, z) = x + y + z$ for any $(x, y, z) \in [0, +\infty)^3$. Then we see that J is convex and $J(x, y, z) = 0$ if and only if $x = y = z = 0$, hence all the conditions of Lemma 2.1 are satisfied. Therefore, $\mu^*(S) = \mu(S_1) + \mu(S_2) + \mu(S_3)$ defines a measure of noncompactness in the space $E \times E \times E$.

Example 2.3. Let μ be a measure of noncompactness on a Banach space E. If we define $J(x, y, z) = \max\{x, y, z\}$ for any $(x, y, z) \in [0, +\infty)^3$, then all the conditions of Lemma 2.1 are satisfied, and $\mu^*(S) = \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}$ is a measure of noncompactness in the space $E \times E \times E$.

3. Main results

In this part of the paper we define another class of functions and using them to develop some tripled fixed point results. We also consider the usual order relation " $\leq$ " on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ as follows:

$$(t_1, t_2, t_3) \leq (s_1, s_2, s_3) \Longleftrightarrow t_1 \leq s_1, t_2 \leq s_2 \quad \text{and} \quad t_3 \leq s_3.$$

Let τ be the class of all functions $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, satisfying the following conditions:

τ_1. ϕ is continuous and nondecreasing on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$,
τ_2. $\phi(t, t, t) < t$ for all $t > 0$,
τ_3. $\phi(t, s, r) = \phi(s, r, t) = \phi(r, t, s)$ and $\phi(t, s, r) = \phi(t, r, s)$ for all $t, s, r \in \mathbb{R}_+$,
τ_4. $\frac{1}{3}\phi(t_1, t_2, t_3) + \frac{1}{3}\phi(s_1, s_2, s_3) + \frac{1}{3}\phi(r_1, r_2, r_3) \leq \phi\big(\frac{t_1+s_1+r_1}{3}, \frac{t_2+s_2+r_2}{3}, \frac{t_3+s_3+r_3}{3}\big)$ for all $t_i, s_i \in \mathbb{R}_+$ for $i = 1, 2, 3$.

For example, the function $\phi(t,s,r) = c_1t + c_2s + c_3r$ in which $c_1, c_2, c_3 \in [0,1)$ having $c_1 + c_2 + c_3 < 1$, $\phi(t,s,r) = ln(1 + \frac{t+s+r}{3})$ and $\phi(t,s,r) = \frac{1}{4}(t+s+r)$ are members of τ.

Theorem 3.4. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T : C\times C\times C \longrightarrow C\times C\times C$ be a continuous function satisfying*

$$\mu^*\big(T(S)\big) \le \phi\big(\mu^*(S), \mu^*(S), \mu^*(S)\big), \tag{3.1}$$

for any nonempty subset S of $C \times C \times C$, where μ^* is defined by Example 2.1, and $\phi \in \tau$. Then T has at least one fixed point in C^3 and the set of all fixed points of T is compact.

Proof. Let $A_0 = C \times C \times C$ and define a sequence $A_n := \overline{co}T(A_{n-1}),\ n \ge 1$. We first observe that

$$\begin{aligned} \mu^*(A_{n+1}) &= \mu^*\big(\overline{co}T(A_n)\big) \\ &= \mu^*\big(T(A_n)\big) \\ &\le \phi\big(\mu^*(A_n), \mu^*(A_n), \mu^*(A_n)\big). \end{aligned} \tag{3.2}$$

Next, $A_1 = \overline{co}T(A_0) = \overline{co}T(C\times C\times C) \subset C\times C\times C = A_0$, also $A_2 = \overline{co}T(A_1) \subset \overline{co}T(A_0) = A_1$. Now if $A_n \subset A_{n-1}$, then $TA_n \subset TA_{n-1}$, which implies that

$$TA_n \cup A_{n+1} = \overline{co}(TA_n) \subset \overline{co}(TA_{n-1}) = A_n.$$

Hence we infer that $\mu^*(A_n)$ is a nonincreasing sequence of real numbers. Thus there is a number $r \ge 0$ such that $\mu^*(A_n) \to r$ as $n \to \infty$. We need to show that $r = 0$. By using (3.2) we obtain

$$\begin{aligned} r &= \lim_{n\to\infty} \mu^*(A_{n+1}) \\ &\le \lim_{n\to\infty} \phi\big(\mu^*(A_n), \mu^*(A_n), \mu^*(A_n)\big) \\ &\le \phi\big(\lim_{n\to\infty} \mu^*(A_n), \lim_{n\to\infty} \mu^*(A_n), \lim_{n\to\infty} \mu^*(A_n)\big) \\ &= \phi(r,r,r) < r, \end{aligned} \tag{3.3}$$

which is a contradiction, hence we deduce that $\mu^*(A_n) \to 0$ as $n \to \infty$ as claimed. Since $A_{n+1} \subset A_n$, so by axioms (MNC6) of Definition 2.1 we conclude that $A_\infty := \bigcap_{n=1}^{\infty} A_n$ is a nonempty, closed and convex and invariant under the mapping T and belongs to $ker\mu^*$. Consequently, Theorem 2.1 implies that T has a fixed point in A_∞. Next if F is the set of all fixed points of T, then by (3.1) we have

$$\mu^*\big(T(F)\big) \le \phi\big(\mu^*(F), \mu^*(F), \mu^*(F)\big) < \mu^*(F),$$

so from above inequality $\mu^*(F) = 0$ since $T(F) = F$. This implies that F is relatively compact. Now taking into account any convergent sequence $\{x_n\} \subset F$ and $x_n \to x^*$, we have $x^* \in A_0$ because A_0 is closed. Thus by continuity of T, $x_n = Tx_n \to Tx^*$ and $Tx^* = x^*$ which means that $x^* \in F$, i.e. F is a compact set. □

Remark 3.1. If we take, $\phi\big(\mu^*(S), \mu^*(S), \mu^*(S)\big) = k_1\mu^*(S) + k_2\mu^*(S) + k_3\mu^*(S)$, where $k_1 + k_2 + k_3 < 1$, in Theorem 3.4, then we get result of Darbo' s as in [12].

Remark 3.2. If we take, $\phi\big(\mu^*(S), \mu^*(S), \mu^*(S)\big) = k_1\beta(\mu^*(S))\mu^*(S) + k_2\beta(\mu^*(S))\mu^*(S) + k_3\beta(\mu^*(S))\mu^*(S)$, where $k_1 + k_2 + k_3 < 1$, in Theorem 3.4, we obtain Geraghty type result of Aghajani as in [7].

Remark 3.3. If we take, $\phi\big(\mu^*(S), \mu^*(S), \mu^*(S)\big) = k_1\varphi(\mu^*(S)) + k_2\varphi(\mu^*(S)) + k_3\varphi(\mu^*(S))$, where $k_1 + k_2 + k_3 < 1$, in Theorem 3.4, we get result of Aghajani as in [7].

Theorem 3.5. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T_i : C \times C \times C \longrightarrow C$ for $i = 1, 2, 3$ be a continuous function satisfying*

$$\mu\big(T_i(S_1 \times S_2 \times S_3)\big) \leq \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big), \tag{3.4}$$

for all nonempty subsets S_1, S_2, S_3 of C, where $\phi \in \tau$. Then there exists a point $(x^*, y^*, z^*) \in C^3$ such that

$$T_1(x^*, y^*, z^*) = x^*, T_2(x^*, y^*, z^*) = y^*, T_3(x^*, y^*, z^*) = z^*.$$

Proof. Consider an operator $G : C \times C \times C$ defined by

$$G(x, y, z) = \big(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)\big).$$

By Example 2.2, we have

$$\mu^*(S) = \mu(S_1) + \mu(S_2) + \mu(S_3),$$

is a measure of noncompactness in the space $E \times E \times E$. Clearly G is continuous on $C \times C \times C$. We only need to show that G has a fixed point. For this we show that G satisfies all the condition of the Theorem 3.4. Let $S \subset C^3$ we have

$$\begin{aligned}
&\mu^*\big(G(S)\big)\\
&\leq \mu^*\big(T_1(S_1 \times S_2 \times S_3) \times T_2(S_1 \times S_2 \times S_3) \times T_3(S_1 \times S_2 \times S_3)\big)\\
&= \mu\big(T_1(S_1 \times S_2 \times S_3)\big) + \mu\big(T_2(S_1 \times S_2 \times S_3)\big) + \mu\big(T_3(S_1 \times S_2 \times S_3)\big)\\
&\leq \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) + \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) + \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) \qquad (3.5)\\
&\leq 3\phi\Big(\frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}, \frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}, \frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}\Big)\\
&= 3\phi\Big(\frac{\mu^*(S)}{3}, \frac{\mu^*(S)}{3}, \frac{\mu^*(S)}{3}\Big).
\end{aligned}$$

Now from (3.5) and taking $\mu_1^* = \frac{1}{3}\mu^*$, we obtain

$$\mu_1^*\big(G(S)\big) \leq \phi\big(\mu_1^*(S), \mu_1^*(S), \mu_1^*(S)\big).$$

Hence by Theorem 3.4, G has a fixed point. □

Remark 3.4. It is observed that condition (3.4) is equivalent to the following condition:

$$\mu\big(T_i(S)\big) \leq \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) \tag{3.6}$$

for a nonempty subset S of C^3. This follows from the fact that

$$\mu\big(T_i(S)\big) \leq \mu\big(T_i(S_1 \times S_2 \times S_3)\big).$$

Remark 3.5. If we take the following function in Theorem 3.5.

$$\begin{aligned}
\phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) = k_1\varphi\Big(\max\big(\mu(S_1), \mu(S_2), \mu(S_3)\big)\Big) + k_2\varphi\Big(\max\big(\mu(S_1), \mu(S_2), \mu(S_3)\big)\Big)\\
+ k_3\varphi\Big(\max\big(\mu(S_1), \mu(S_2), \mu(S_3)\big)\Big),
\end{aligned} \tag{3.7}$$

we get result as in [22].

Corollary 3.1. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T : C \times C \times C \longrightarrow C$ be a continuous function satisfying*

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) \leq \phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big),$$

for all nonempty subsets S_1, S_2, S_3 of C, where $\phi \in \tau$. Then T has at least a tripled fixed point.

Proof. Taking $T_i = T$, for all $i = 1, 2, 3$ and $G(x, y, z) = \big(T(x, y, z), T(y, x, z), T(z, y, x)\big)$ in Theorem 3.5, we obtain the desired conclusion. □

Corollary 3.2. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Moreover assume that $T : C \times C \times C \longrightarrow C$ be a continuous function such that there exist nonnegative constants k_1, k_2, k_3 with $k_1 + k_2 + k_3 < 1$*

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) \leq k_1\mu(S_1) + k_2\mu(S_2) + k_3\mu(S_3),$$

for all nonempty subsets S_1, S_2, S_3 of C, where $\phi \in \tau$. Then T has at least a tripled fixed point.

Proof. Taking $T_i = T$, for all $i = 1, 2, 3$ and $\phi(t, s, r) = k_1 t + k_2 s + k_3 r$ in Theorem 3.5, we obtain the desired conclusion. □

Remark 3.6. If we take the following function in Corollary 3.1.

$$\begin{aligned}\phi\big(\mu(S_1), \mu(S_2), \mu(S_3)\big) = k_1\varphi\Big(\frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}\Big) + k_2\varphi\Big(\frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}\Big)\\ + k_3\varphi\Big(\frac{\mu(S_1) + \mu(S_2) + \mu(S_3)}{3}\Big),\end{aligned}$$

we get result as in [21].

Theorem 3.6. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T_i : C \times C \times C \longrightarrow C$ for $i = 1, 2, 3$ be a continuous function satisfying*

(3.8)

$$\begin{aligned}&\mu\big(T_i(S_1 \times S_2 \times S_3)\big)\\ &\leq \phi\big(\max\{\mu(S_1), \mu(S_2), \mu(S_3)\}, \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}, \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}\big),\end{aligned}$$

for all nonempty subsets S_1, S_2, S_3 of C, where $\phi \in \tau$. Then there exists $(x^*, y^*, z^*) \in C^3$ such that

$$T_1(x^*, y^*, z^*) = x^*, T_2(x^*, y^*, z^*) = y^*, T_3(x^*, y^*, z^*) = z^*.$$

Proof. To prove this theorem, we introduce an operator $G : C \times C \times C \to C$ defined by

$$G(x, y, z) = \big(T_1(x, y, z), T_2(x, y, z), T_3(x, y, z)\big).$$

Also, assume that from Example 2.3, we have

$$\mu^*(S) = \max\{\mu(S_1), \mu(S_2), \mu(S_3)\},$$

defines a measure of noncompactness in the space $E \times E \times E$. To reach the desired conclusion we only show that G has a fixed point. Thus our aim to prove all the conditions of Theorem 3.4. Let $S \subset C^3$ we have

$$\begin{aligned}&\mu^*\big(G(S)\big)\\ &\leq \mu^*\big(T_1(S_1 \times S_2 \times S_3) \times T_2(S_1 \times S_2 \times S_3) \times T_3(S_1 \times S_2 \times S_3)\big)\\ &= \max\Big\{\mu\big(T_1(S_1 \times S_2 \times S_3)\big),\ \mu\big(T_2(S_1 \times S_2 \times S_3)\big),\ \mu\big(T_3(S_1 \times S_2 \times S_3)\big)\Big\}\\ &\leq \phi\Big(\max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}, \max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}, \max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}\Big)\\ &= \phi\big(\mu^*(S), \mu^*(S), \mu^*(S)\big).\end{aligned}$$

Hence by Theorem 3.4, G has a fixed point. □

Corollary 3.3. *Let C be a nonempty, closed, bounded and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness on E. Let $T : C \times C \times C \longrightarrow C$ be a continuous function satisfying*

$$\begin{aligned}&\mu\big(T(S_1 \times S_2 \times S_3)\big)\\ &\leq \phi\big(\max\{\mu(S_1), \mu(S_2), \mu(S_3)\}, \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}, \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}\big),\end{aligned}$$

for any nonempty subsets S_1, S_2, S_3 of C, where $\phi \in \tau$. Then T has at least a tripled fixed point.

4. An application

In the following section we are going to study the application of Theorem 3.5 in the study of existence of solutions for a system of integral equation defined on the Banach spaces $BC(\mathbb{R}_+)$, consisting of all continuous real valued and bounded functions on $\mathbb{R}_+$ and equipped with the norm, $\|x\| = \sup\{x(t) : t \geq 0\}$. The measure of noncompactness [10, 12, 13] for a non negative fixed t on $\mathfrak{M}_{BC(\mathbb{R}_+)}$ is defined as follows for any bounded set

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diam X(t), \tag{4.9}$$

where $diam\, X(t) = \sup\{|x(t) - y(t)| : x, y \in X\},\ and\ X(t) = \{x(t) : x \in X\}$. To define the $\omega_0(X)$, first we need to define the modulus of continuity for any $x \in X$ and $\epsilon > 0$. The modulus of the continuity of x on the interval $[0, T]$ denoted by $\omega^T(x, \epsilon)$, i.e.

$$\omega^T(x, \epsilon) = \sup\{\big|x(t) - x(s)\big| : t, s \in [0, T], |t - s| \leq \epsilon\},$$

and let $\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : x \in X\}$, $\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X, \epsilon)$, and $\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X)$. Assume that

(i) a function $B : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded with $M_1 = \sup\{|B(t)| : t \in \mathbb{R}_+\}$;

(ii) $\xi, \eta, q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions and $\xi(t) \to \infty$ as $t \to \infty$;

(iii) a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ is continuous and there exist $\delta, \alpha > 0$ such that

$$|\psi(t_1) - \psi(t_2)| \leq \delta|t_1 - t_2|^\alpha, \tag{4.10}$$

for any $t_1, t_2 \in \mathbb{R}_+$ and moreover $\psi(0) = 0$;

(iv) functions $h : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist a nondecreasing continuous function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta(0) = 0$ and $\phi \in \tau$ such that

$$|h(t, x_1, x_2, x_3) - h(t, y_1, y_2, y_3)| \leq \frac{1}{2}\phi(|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|), \tag{4.11}$$

and

$$\begin{aligned}&|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)|\\ &\leq \frac{1}{2}\big(\phi(|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|)\big) + \theta(|x_4 - y_4|),\end{aligned} \tag{4.12}$$

for all $x_i, y_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$ and for any $t \geq 0$;

(v) moreover, the functions defined by $t \longmapsto |f(t, 0, 0, 0, 0)|$ and $t \longmapsto |h(t, 0, 0, 0)|$ are bounded on $\mathbb{R}_+$, i.e.

$$M_2 = \sup\{|f(t, 0, 0, 0, 0)| : t \in \mathbb{R}_+\} < \infty, \tag{4.13}$$

$$M_3 = \sup\{|h(t, 0, 0, 0)| : t \in \mathbb{R}_+\} < \infty; \tag{4.14}$$

(vi) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists positive r_0 satisfying

$$M_1 + \phi(r_0, r_0, r_0) + M_2 + M_3 + \theta(\delta M_4) < r_0, \tag{4.15}$$

where

$$M_4 = \sup\left\{\Big|\int_0^{q(t)} g\big(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big|^{\alpha} : t \in \mathbb{R}_+ \text{ and } x, y, z \in BC(\mathbb{R}_+)\right\}, \tag{4.16}$$

and

$$\lim_{t\to\infty}\int_0^{q(t)}\Big|g\big(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big) - g\big(t, s, u(\eta(s)), v(\eta(s)), w(\eta(s))\big)\Big|ds = 0, \tag{4.17}$$

uniformly with respect to $x, y, z, u, v, w \in BC(\mathbb{R}_+)$.

Theorem 4.7. *Suppose that (i)-(vi) hold; then the following system of integral equations*

$$\begin{cases} x(t) = B(t) + h\big(t, x(\xi(t)), y(\xi(t)), z(\xi(t))\big) + f\begin{pmatrix} t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\ \psi\Big(\int_0^{q(t)} g\big(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\end{pmatrix} \\ y(t) = B(t) + h\big(t, y(\xi(t)), x(\xi(t)), z(\xi(t))\big) + f\begin{pmatrix} t, y(\xi(t)), x(\xi(t)), z(\xi(t)), \\ \psi\Big(\int_0^{q(t)} g\big(t, s, y(\eta(s)), x(\eta(s)), z(\eta(s))\big)ds\end{pmatrix} \\ z(t) = B(t) + h\big(t, z(\xi(t)), y(\xi(t)), x(\xi(t))\big) + f\begin{pmatrix} t, z(\xi(t)), y(\xi(t)), x(\xi(t)), \\ \psi\Big(\int_0^{q(t)} g\big(t, s, z(\eta(s)), y(\eta(s)), x(\eta(s))\big)ds\end{pmatrix} \end{cases} \tag{4.18}$$

has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. Let $G : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+)$ be an operator defined by

$$G(x, y, z)(t) = B(t)+h\big(t, x(\xi(t)), y(\xi(t)), z(\xi(t))\big)+f\begin{pmatrix} t, x(\xi(t)), y(\xi(t)), z(\xi(t)), \\ \psi\Big(\int_0^{q(t)} g\big(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\end{pmatrix} \tag{4.19}$$

Moreover, the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is equipped with the following norm:

$$\|(x, y, z)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} = \|x\|_\infty + \|y\|_\infty + \|z\|_\infty. \tag{4.20}$$

We can see that the solution of (4.18) in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is equivalent to the tripled fixed point of G. To prove this, we need to satisfy all the conditions of Corollary 3.1. To follow this, first we observe that $G(x, y, z)$ is continuous function for any $(x, y, z) \in BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. Moreover, using the triangular inequality and (4.10), (4.11), (4.12),(4.13), (4.14), (4.16), (4.19) and (4.20), we obtain

(4.21)
$$\begin{aligned}
&\big|G(x,y,z)(t)\big| \\
&\le \big|B(t)\big| + \big|h\big(t,x(\xi(t)),y(\xi(t)),z(\xi(t))\big) - h(t,0,0,0)\big| + \big|h(t,0,0,0)\big| + \big|f(t,0,0,0,0)\big| \\
&\quad + \bigg|f\bigg(t,x(\xi(t)),y(\xi(t)),z(\xi(t)),\psi\Big(\int_0^{q(t)} g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\bigg) \\
&\quad - f(t,0,0,0,0)\bigg| \\
&\le M_1 + \frac{1}{2}\phi\big(|x(\xi(t))|,|y(\xi(t))|,|z(\xi(t))|\big) + M_3 + \frac{1}{2}\phi\big(|x(\xi(t))|,|y(\xi(t))|,|z(\xi(t))|\big) \\
&\quad + \theta\bigg(\bigg|\psi\Big(\int_0^{q(t)} g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big) - \psi(0)\bigg|\bigg) + M_2 \\
&\le M_1 + \phi\Big(|x(\xi(t))|,|y(\xi(t))|,|z(\xi(t))|\Big) + M_3 + M_2 \\
&\quad + \theta\Big(\delta\Big|\int_0^{q(t)} g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big|^{\alpha}\Big) \\
&\le M_1 + M_2 + M_3 + \phi(\|x\|_\infty,\|y\|_\infty,\|z\|_\infty) + \theta(\delta M_4) \le r_0.
\end{aligned}$$

Thus G is well defined and we obtain $G(\bar{B}_{r_0}\times\bar{B}_{r_0}\times\bar{B}_{r_0}) \subset \bar{B}_{r_0}$. Now we prove that $G : \bar{B}_{r_0}\times\bar{B}_{r_0}\times\bar{B}_{r_0} \to \bar{B}_{r_0}$ is continuous, for this take $(x,y,x) \in \bar{B}_{r_0}\times\bar{B}_{r_0}\times\bar{B}_{r_0}$ and $\epsilon > 0$ arbitrary. Moreover, consider $(u,v,w) \in \bar{B}_{r_0}\times\bar{B}_{r_0}\times\bar{B}_{r_0}$ such that for $\epsilon > 0$, $\|(x,y,z)-(u,v,w)\|_{\bar{B}_{r_0}\times\bar{B}_{r_0}\times\bar{B}_{r_0}} < \frac{\epsilon}{2}$, then we have

$$\begin{aligned}
&\big|G(x,y,z)(t) - G(u,v,w)(t)\big| \\
&\le \Big|h\big(t,x(\xi(t)),y(\xi(t)),z(\xi(t))\big) - h\big(t,u(\xi(t)),v(\xi(t)),w(\xi(t))\big)\Big| \\
&\quad + \bigg|f\bigg(t,x(\xi(t)),y(\xi(t)),z(\xi(t)),\psi\Big(\int_0^{q(t)} g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\bigg) \\
&\quad - f\bigg(t,u(\xi(t)),v(\xi(t)),w(\xi(t)),\psi\Big(\int_0^{q(t)} g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)ds\Big)\bigg)\bigg| \\
&\le \phi\big(|x(\xi(t))-u(\xi(t))|,|y(\xi(t))-v(\xi(t))|,|z(\xi(t))-w(\xi(t))|\big) \\
&\quad + \theta\bigg(\bigg|\psi\Big(\int_0^{q(t)} g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big) \\
&\quad - \psi\Big(\int_0^{q(t)} g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)ds\Big)\bigg|\bigg) \\
&\le \phi(\|x-u\|,\|y-v\|,\|z-w\|) + \theta\bigg(\delta\bigg|\int_0^{q(t)}\Big(g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big) \\
&\quad - g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)\Big)ds\bigg|^{\alpha}\bigg).
\end{aligned} \tag{4.22}$$

Now from (4.17) there exist $T > 0$ such that if $t > T$, then

(4.23)
$$\theta\Bigg(\delta\bigg|\int_0^{q(t)}\bigg(g\Big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\Big) - g\Big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\Big)\bigg)ds\bigg|^{\alpha}\Bigg) \le \frac{\epsilon}{2},$$

for any $x, y, z, u, v, w \in BC(\mathbb{R}_+)$. Now we notice two cases:
Case 1. If $t > T$, then from (4.22) and (4.23) we obtain

$$\big|G(x,y,z)(t) - G(u,v,w)(t)\big| \leq \phi\Big(\frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}\Big) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.24}$$

Case 2. Similarly for $t \in [0, T]$, we have

$$\begin{aligned}
&\big|G(x,y,z)(t) - G(u,v,w)(t)\big| \\
&\leq \phi\Big(\frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2}\Big) \\
&+ \theta\bigg(\delta\bigg|\int_0^{q(t)} \Big(g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big) - g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)\Big)ds\bigg|^{\alpha}\bigg) \\
&< \frac{\epsilon}{2} + \theta\big(\delta(q_T\beta(\epsilon))^{\alpha}\big),
\end{aligned} \tag{4.25}$$

where $q_T = \sup\{q(t) : t \in [0, T]\}$ and

$$\begin{aligned}
\beta(\epsilon) = \sup\bigg\{&\Big|g(t,s,x,y,z) - g(t,s,u,v,w)\Big| : t \in [0,T], s \in [0, q_T], x,y,z,u,v,w \in [-r_0, r_0], \\
&\|(x,y,z) - (u,v,w)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} < \frac{\epsilon}{2}\bigg\}.
\end{aligned} \tag{4.26}$$

Since g is continuous on $[0,T] \times [0, q_T] \times [-r_0, r_0] \times [-r_0, r_0] \times [-r_0, r_0]$ we have $\beta(\epsilon) \to 0$ as $\epsilon \to 0$, and by continuity of θ we get

$$\theta\big(\delta(q_T\beta(\epsilon))^{\alpha}\big) \to 0.$$

Finally from (4.24) and (4.25) we conclude that G is a continuous function from $\bar{B}_{r_0} \times \bar{B}_{r_0} \times \bar{B}_{r_0}$ into $\bar{B}_{r_0}$. Next we assume that X_1, X_2, X_3 are arbitrary nonempty subsets of $\bar{B}_{r_0}$ and $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leq \epsilon$. Without loss of generality let $q(t_1) \leq q(t_2)$, and for any arbitrary $(x, y, z) \in X_1 \times X_2 \times X_3$

$$\begin{aligned}
&\big|G(x,y,z)(t_1) - G(x,y,z)(t_2)\big| \\
&= \big|B(t_1) - B(t_2)\big| + \big|h\big(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2))\big) - h\big(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))\big)\big| \\
&\quad + \big|h\big(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))\big) - h\big(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))\big)\big| \\
&\quad + \bigg|f\bigg(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \psi\Big(\int_0^{q(t_2)} g\big(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg) \\
&\quad - f\bigg(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \psi\Big(\int_0^{q(t_2)} g\big(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg)\bigg| \\
&\quad + \bigg|f\bigg(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \psi\Big(\int_0^{q(t_2)} g\big(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg) \\
&\quad - f\bigg(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \psi\Big(\int_0^{q(t_2)} g\big(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg)\bigg| \\
&\quad + \bigg|f\bigg(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \psi\Big(\int_0^{q(t_2)} g\big(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg) \\
&\quad - f\bigg(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \psi\Big(\int_0^{q(t_2)} g\big(t_1, s, x(\eta(s)), y(\eta(s)), z(\eta(s))\big)ds\Big)\bigg)\bigg|
\end{aligned} \tag{4.27}$$

$$+\left|f\left(t_1,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1)),\psi\Big(\int_0^{q(t_2)}g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\right)\right.$$
$$\left.-f\left(t_1,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1)),\psi\Big(\int_0^{q(t_1)}g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\right)\right|.$$

Now we put this substitution

(4.28)

$$\begin{cases}\omega^T(B,\epsilon)=\sup\Big\{\big|B(t_1)-B(t_2)\big|:t_1,t_1\in[0,T],\ |t_1-t_2|\le\epsilon\Big\},\\ \omega^T_{r_0}(h,\epsilon)=\sup\Big\{\big|h(t_2,x,y,z)-h(t_1,x,y,z)\big|:t_1,t_2\in[0,T],\ |t_1-t_2|\le\epsilon,\ x,y,z\in[-r_0,r_0]\Big\},\\ \omega^T(\xi,\epsilon)=\sup\Big\{\big|\xi(t_1)-\xi(t_2)\big|:t_1,t_2\in[0,T],\ |t_1-t_2|\le\epsilon\Big\},\\ \omega^T\big(x,\omega^T(\xi,\epsilon)\big)=\sup\Big\{\big|x(t_1)-x(t_2)\big|:t_1,t_1\in[0,T],\ |t_1-t_2|\le\omega^T(\xi,\epsilon)\Big\},\\ U^T_{r_0}=\sup\Big\{\big|g(t,s,x,y,z)\big|:t\in[0,T],\ s\in[0,q_T],\ x,y,z\in[-r_0,r_0]\Big\},\\ K=q_T\sup\Big\{\big|g(t,s,x,y,z)\big|:t\in[0,T],\ s\in[0,q_T],x,y,z\in[-r_0,r_0]\Big\},\\ \omega^T_{r_0,K}(f,\epsilon)=\sup\Big\{\big|f(t_2,x,y,z,d)-f(t_1,x,y,z,d)\big|:t_1,t_2\in[0,T],\ |t_1-t_2|\le\epsilon,\\ \qquad\qquad x,y,z,\in[-r_0,r_0],\ d\in[-K,K]\Big\},\\ \omega^T_{r_0}(g,\epsilon)=\sup\Big\{\Big|g(t_1,s,x,y,z)-g(t_2,s,x,y,z)\Big|:t_1,t_2\in[0,T],\ |t_1-t_2|\le\epsilon,\\ \qquad\qquad x,y,z,\in[-r_0,r_0],\ s\in[0,q_T]\},\\ \omega^T(q,\epsilon)=\sup\Big\{\big|q(t_1)-q(t_2)\big|:t_1,t_1\in[0,T],\ |t_1-t_2|\le\epsilon\Big\}.\end{cases}$$

Now from (4.27) and (4.28) we obtain

$$\begin{aligned}&\Big|G(x,y,z)(t_1)-G(x,y,z)(t_2)\Big|\\ &\le\omega^T(B,\epsilon)+\frac{1}{2}\phi\big(|x(\xi(t_2))-x(\xi(t_1))|,|y(\xi(t_2))-y(\xi(t_1)),|z(\xi(t_2))-z(\xi(t_1))|\big)\\ &\quad+\omega^T_{r_0}(h,\epsilon)+\frac{1}{2}\phi\big(|x(\xi(t_2))-x(\xi(t_1))|,|y(\xi(t_2))-y(\xi(t_1)),|z(\xi(t_2))-z(\xi(t_1))|\big)\\ &\quad+\omega^T_{r_0,K}(f,\epsilon)+\theta\bigg(\bigg|\psi\Big(\int_0^{q(t_2)}g\big(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\\ &\quad-\psi\Big(\int_0^{q(t_2)}g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\bigg|\bigg)\\ &\quad+\theta\bigg(\bigg|\psi\Big(\int_0^{q(t_2)}g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\\ &\quad-\psi\Big(\int_0^{q(t_1)}g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\bigg|\bigg)\\ &\le\omega^T(B,\epsilon)+\omega^T_{r_0}(h,\epsilon)+\phi\Big(\omega^T(x,\omega^T(\xi,\epsilon)),\ \omega^T(y,\omega^T(\xi,\epsilon)),\ \omega^T(z,\omega^T(\xi,\epsilon))\Big)\\ &\quad+\omega^T_{r_0,K}(f,\epsilon)+\theta\bigg(\delta\bigg|\int_{q(t_1)}^{q(t_2)}\Big(g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)\Big)\bigg)\\ &\quad+\theta\bigg(\delta\bigg|\int_0^{q(t_2)}\Big(g\big(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)\\ &\quad-g\big(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)\Big)ds\bigg|^{\alpha}\bigg)\\ &\le\omega^T(B,\epsilon)+\omega^T_{r_0}(h,\epsilon)+\phi\Big(\omega^T(x,\omega^T(\xi,\epsilon)),\ \omega^T(y,\omega^T(\xi,\epsilon)),\ \omega^T(z,\omega^T(\xi,\epsilon))\Big)\end{aligned}\tag{4.29}$$

$$+\,\omega^T_{r_0,K}(f,\epsilon)+\theta\Big(\delta\,\big(q_T\,\omega^T_{r_0}(g,\epsilon)\big)^{\alpha}\Big)+\theta\Big(\delta\,\big(U^T_{r_0}\,\omega^T(q,\epsilon)\big)^{\alpha}\Big).$$

Since (x,y,z) is an arbitrary element of $X_1\times X_2\times X_3$

$$\begin{aligned}
&\omega^L\Big(G(X_1\times X_2\times X_3),\ \epsilon\Big)\\
&\le \omega^T(B,\epsilon)+\omega^T_{r_0}(h,\epsilon)+\phi\Big(\omega^T(X_1,\omega^T(\xi,\epsilon)),\ \omega^T(X_2,\omega^T(\xi,\epsilon)),\ \omega^T(X_3,\omega^T(\xi,\epsilon))\Big)\\
&\quad+\omega^T_{r_0,K}(f,\epsilon)+\theta\Big(\delta\,\big(q_T\,\omega^T_{r_0}(g,\epsilon)\big)^{\alpha}\Big)+\theta\Big(\delta\,\big(U^T_{r_0}\,\omega^T(q,\epsilon)\big)^{\alpha}\Big).
\end{aligned}\tag{4.30}$$

Further by the uniform continuity of f,g and h on the compact sets $[0,T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]\times[-K,K]$, $[0,T]\times[0,q_T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]$ and $[0,T]\times[-r_0,r_0]\times[-r_0,r_0]\times[-r_0,r_0]$ respectively. We get $\omega^T_{r_0,K}(f,\epsilon)\to 0$, $\omega^T_{r_0}(g,\epsilon)\to 0$ and $\omega^T_{r_0,K}(h,\epsilon)\to 0$ as $\epsilon\to 0$. Also due to the uniform continuity of ξ,q and B on $[0,T]$, we get $\omega^T(\xi,\epsilon)\to 0$, $\omega^T(q,\epsilon)\to 0$ and $\omega^T(B,\epsilon)\to 0$ as $\epsilon\to 0$. Moreover, θ is a nondecreasing continuous function with $\theta(0)=0$ and K is finite, hence we have

$$\theta\Big(\delta\,\big(q_T\,\omega^T_{r_0}(g,\epsilon)\big)^{\alpha}\Big)+\theta\Big(\delta\,\big(U^T_{r_0}\,\omega^T(q,\epsilon)\big)^{\alpha}\Big)\longrightarrow 0,\ as\ \epsilon\to 0.$$

Now taking the limit in (4.30) as $\epsilon\to 0$ we get

$$\omega^L_0\big(G(X_1\times X_2\times X_3)\big)\le\phi\big(\omega^T_0(X_1),\,\omega^T_0(X_2),\,\omega^T_0(X_3)\big),\tag{4.31}$$

also taking the limit $T\to\infty$ in (4.31) we obtain

$$\omega_0\big(G(X_1\times X_2\times X_3)\big)\le\phi\big(\omega_0(X_1),\,\omega_0(X_2),\,\omega_0(X_3)\big).\tag{4.32}$$

In addition, for arbitrary $(x,y,z),(u,v,w)\in X_1\times X_2\times X_3$ and $t\in\mathbb{R}_+$ such that

$$\begin{aligned}
&\big|G(x,y,z)(t)-G(u,v,w)(t)\big|\\
&\le\big|h\big(t,x(\xi(t)),y(\xi(t)),z(\xi(t))\big)-h\big(t,u(\xi(t)),v(\xi(t)),w(\xi(t))\big)\big|\\
&\quad+\bigg|f\bigg(t,x(\xi(t)),y(\xi(t)),z(\xi(t)),\psi\Big(\int_0^{q(t)}g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\bigg)\\
&\quad-f\bigg(t,u(\xi(t)),v(\xi(t),w(\xi(t)),\psi\Big(\int_0^{q(t)}g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)ds\Big)\bigg)\bigg|\\
&\le\frac{1}{2}\phi\Big(|x(\xi(t))-u(\xi(t))|,|y(\xi(t)))-v(\xi(t)),|z(\xi(t))-w(\xi(t))|\Big)\\
&\quad+\frac{1}{2}\phi\Big(|x(\xi(t))-u(\xi(t))|,|y(\xi(t)))-v(\xi(t)),|z(\xi(t))-w(\xi(t))|\Big)\\
&\quad+\theta\bigg(\bigg|\psi\Big(\int_0^{q(t)}g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)ds\Big)\\
&\quad-\psi\Big(\int_0^{q(t)}g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)ds\Big)\bigg|\bigg)\\
&\le\phi\big(diamX_1(\xi(t)),diamX_2(\xi(t)),diamX_3(\xi(t))\big)\\
&\quad+\theta\bigg(\delta\bigg|\int_0^{q(t)}\Big(g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big)-g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)\Big)ds\bigg|^{\alpha}\bigg).
\end{aligned}\tag{4.33}$$

Since $(x, y, z), (u, v, w)$ and t are arbitrary in above expression
(4.34)

$$\begin{aligned} & diam G(X_1 \times X_2 \times X_3) \\ & \le \phi\Big(diam X_1(\xi(t)),\ diam X_2(\xi(t)),\ diam X_3(\xi(t))\Big) \\ & \quad + \theta\bigg(\delta\bigg|\int_0^{q(t} \Big(g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big) - g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)\Big)ds\bigg|^{\alpha}\bigg) \end{aligned}$$

Taking $t \to 0$ in (4.34) and also from (4.17) we have

$$\begin{aligned} & \limsup_{t\to\infty} diam\, G(X_1 \times X_2 \times X_3)(t) \\ & \le \phi\Big(\limsup_{t\to\infty} diam X_1(\xi(t)), \limsup_{t\to\infty} diam X_2(\xi(t)),\ \limsup_{t\to\infty} diam X_3(\xi(t))\Big). \end{aligned} \tag{4.35}$$

Now from equation (4.32) and (4.35) implies that

$$\begin{aligned} & \omega_0\Big(G(X_1 \times X_2 \times X_3)\Big) + \limsup_{t\to\infty} diam\, G(X_1 \times X_2 \times X_3)(t) \\ & \le \phi\Big(\omega_0(X_1), \omega_0(X_2), \omega_0(X_3)\Big) \\ & \quad + \phi\Big(\limsup_{t\to\infty} diam X_1(\xi(t)),\ \limsup_{t\to\infty} diam X_2(\xi(t)), \limsup_{t\to\infty} diam X_3(\xi(t))\Big) \\ & \le \phi\Big(\omega_0(X_1) + \limsup_{t\to\infty} diam X_1(\xi(t)), \omega_0(X_2) + \limsup_{t\to\infty} diam X_2(\xi(t)), \\ & \qquad \omega_0(X_3) + \limsup_{t\to\infty} diam X_3(\xi(t))\Big) \end{aligned} \tag{4.36}$$

Finally, from (4.9) we get

$$\mu\big(G(X_1 \times X_2 \times X_3)\big) \le \phi\big(\mu(X_1), \mu(X_2), \mu(X_3)\big). \tag{4.37}$$

Thus by Corollary 3.1 G has atleast one tripled fixed point in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. $\square$

4.1. **An example.**

Example 4.4.

$$\begin{cases} x(t) = \frac{2e^t t^6 + 5t^6 + 10e^t t^4 + 16t^4 + 3t^2 + 6e^t + 12}{6(t^2+2)(e^t+2)(t^4+1)} + \frac{5(e^{5t}+e^{3t}+e^{2t}+e^t)}{60(e^{5t}+e^{4t}+e^t+1)} x(t) + \frac{5+11e^{t^2}}{120(1+e^{t^2})} y(t) \\ \quad + \frac{(6+11t^2)e^{t^3}+5t^2}{180(1+e^{t^3})(1+t^2)} z(t) + \int_0^t \frac{s|\sin x(t)||\sin y(t)||\sin z(t)|}{e^t(1+\sin^2 y(t))(1+\sin^2 x(t))(1+sin^2 z(t))} ds \\ y(t) = \frac{2e^t t^6 + 5t^6 + 10e^t t^4 + 16t^4 + 3t^2 + 6e^t + 12}{6(t^2+2)(e^t+2)(t^4+1)} + \frac{5(e^{5t}+e^{3t}+e^{2t}+e^t)}{60(e^{5t}+e^{4t}+e^t+1)} y(t) + \frac{5+11e^{t^2}}{120(1+e^{t^2})} x(t) \\ \quad + \frac{(6+11t^2)e^{t^3}+5t^2}{180(1+e^{t^3})(1+t^2)} z(t) + \int_0^t \frac{s|\sin x(t)||\sin y(t)||\sin z(t)|}{e^t(1+\sin^2 y(t))(1+\sin^2 x(t))(1+\sin^2 z(t))} ds \\ z(t) = \frac{2e^t t^6 + 5t^6 + 10e^t t^4 + 16t^4 + 3t^2 + 6e^t + 12}{6(t^2+2)(e^t+2)(t^4+1)} + \frac{5(e^{5t}+e^{3t}+e^{2t}+e^t)}{60(e^{5t}+e^{4t}+e^t+1)} z(t) + \frac{5+11e^{t^2}}{120(1+e^{t^2})} y(t) \\ \quad + \frac{(6+11t^2)e^{t^3}+5t^2}{180(1+e^{t^3})(1+t^2)} x(t) + \int_0^t \frac{s|\sin x(t)||\sin y(t)||\sin z(t)|}{e^t(1+\sin^2 y(t))(1+\sin^2 x(t))(1+\sin^2 z(t))} ds. \end{cases} \tag{4.38}$$

We notice that we have the special case of the integral system (4.18) with the following choices

- $h(t,x,y,z) = \frac{t^4}{3(1+t^4)} + \frac{e^t}{10(1+e^t)}x + \frac{e^{t^2}}{20(1+e^{t^2})}y + \frac{e^{t^3}}{30(1+e^{t^3})}z$,
- $h(t,x,y,z,p) = \frac{1}{2(1+e^t)} + \frac{e^{2t}}{12(1+e^{4t})}x + \frac{1}{24}y + \frac{t^2}{36(1+t^2)}z + p$,
- $g(t,s,x,y,z) = \frac{s|\sin x||\sin y||\sin z|}{e^t(1+\sin^2 y))(1+\sin^2 x)(1+\sin^2 z)}\,ds$,
- $\xi(t) = \eta(t) = \psi(t) = \theta(t) = q(t) = t$,
- $\phi(t,s,r) = \frac{t+s+r}{4}$.

To solve this system of integral equations we need to verify all the assumption of Theorem 4.7.

(1) since $B(t) = \frac{1}{1+t^2}$ is continuous on $\mathbb{R}_+$ and $M_1 = \frac{1}{2}$ assumption (i) is satisfied.
(2) we see that $\eta(t), \xi(t), q(t) = t$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$.
(3) the function $\psi(t) = t$ for $\alpha, \delta = 1$ the equation (4.10) is satisfied.
(4) we have $f(t,0,0,0,0) = \frac{1}{2(1+e^t)}$ and $h(t,0,0,0) = \frac{t^4}{3(1+t^4)}$ then we easily see that $M_2 = \frac{1}{2}$ and $M_3 = \frac{1}{3}$.

$$\begin{aligned}
&|h(t,x,y,z) - h(t,u,v,w)| \\
&\leq \Big|\frac{e^t}{10(1+e^t)}\Big||x-u| + \Big|\frac{e^{t^2}}{20(1+e^{t^2})}\Big||y-v| + \Big|\frac{e^{t^3}}{30(1+e^{t^3})}\Big||z-w| \\
&\leq \frac{1}{10}|x-u| + \frac{1}{20}|y-v| + \frac{1}{30}|z-w| \\
&\leq \frac{1}{2}\Big[\frac{1}{4}|x-u| + \frac{1}{4}|y-v| + \frac{1}{4}|z-w|\Big] \\
&\leq \frac{1}{2}\phi(|x-u|,|y-v|,|z-w|),
\end{aligned} \tag{4.39}$$

and similarly

$$\begin{aligned}
&|f(t,x,y,z,p_1) - f(t,u,v,w,p_2)| \\
&\leq \Big|\frac{e^{2t}}{12(1+e^{4t})}\Big||x-u| + \Big|\frac{1}{24}\Big||y-v| + \Big|\frac{t^2}{36(1+t^2)}\Big||z-w| + |p_1-p_2| \\
&\leq \frac{1}{12}|x-u| + \frac{1}{24}|y-v| + \frac{1}{36}|z-w| + |p_1-p_2| \\
&\leq \frac{1}{2}\Big[\frac{1}{4}|x-u| + \frac{1}{4}|y-v| + \frac{1}{4}|z-w|\Big] + |p_1-p_2| \\
&\leq \frac{1}{2}\phi(|x-u|,|y-v|,|z-w|) + \theta(|p_1-p_2|).
\end{aligned} \tag{4.40}$$

Now, we verify the assumption (vi), clearly g is continuous and

$$\begin{aligned}
g(t,s,x,y,z) - g(t,s,u,v,w) &= \Big|\frac{s|\sin x||\sin y||\sin z|}{e^t(1+\sin^2 y))(1+\sin^2 x)(1+\sin^2 z)}ds\Big| \\
&\leq \Big|\frac{s}{e^t}\Big| = \frac{s}{e^t}
\end{aligned} \tag{4.41}$$

implies that

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^t \Big|g\big(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\big) - g\big(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))\big)ds\Big| \\
&\leq \lim_{t\to\infty}\int_0^t \frac{s}{e^t}ds = \lim_{t\to\infty}\big(\frac{t^2}{2e^t}\big).
\end{aligned} \tag{4.42}$$

Finally, for remaining part of assumption (vi), for any $(x,y,z) \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$

$$M_4 = \sup \Big| \int_0^t g\Big(t, s, x\big(\eta(s)\big), y\big(\eta(s)\big), z\big(\eta(s)\big)\Big) \Big| = \frac{1}{2}$$

for $t \in \mathbb{R}_+$. Also for $M_1 = \frac{1}{2}, M_2 = \frac{1}{4}, M_3 = \frac{1}{3}$ and $M_4 = \frac{1}{2}$ we have

$$\frac{1}{2} + \phi(7,7,7) + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = 1.583 + 5.25 = 6.83 < 7.$$

Consequently, all the assumption of the Theorem 3.1 are satisfied, the system of integral equation (4.38) has at least one solution in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

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References

[1] Agarwal, R. P., Hussain, N. and Taoudi, M. A., *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal., **2012** (2012), 1–15

[2] Aghajani, A., Bana, J. and Sabzali, N., *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math. Soc. Simon Stevin, **18** (2017), No. 2, 595

[3] Aghajani, A. and Haghighi, A. S., *Existence of solutions for a system of integral equations via measure of noncompactnessm*, Novi Sad J. Math., **44** (2014), No. 1, 59–73

[4] Aghajani, A. and Jalilian, Y., *Existence and global attractivity of solutions of a nonlinear functional integral equation*, Commun. Nonlinear Sci. Numer. Simul., **15** (2010), No. 11, 3306–3312

[5] Aghajani, A. and Jalilian, Y., *Existence of nondecreasing positive solutions for a system of singular integral equations*, Mediterr. J. Math., **8** (2011), No. 4, 563–576

[6] Aghajani, A., Mursaleen, M. and Haghighi, A. S., *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci., **35** (2015), No. 3, 552–566

[7] Aghajani, A., Allahyari, R. and Mursaleen, M., *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math., **260** (2014), 68–77

[8] Akhmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E. and Sadovskii, B. N., *Measures of noncompactness and condensing operators*, Birkhuser, Basel, 1992

[9] Allahyari, R. and Aghajani, A., *Some extensions of Darbos theorem and solutions of integral equations of Hammerstein type*, International Journal of Nonlinear Analysis and Applications, **8** (2017), No. 1, 337–351

[10] Appell, J., Bana, J. and Merentes, N., *Measures of noncompactness in the study of asymptotically stable and ultimately nondecreasing solutions of integral equations*, Zeitschrift fr Analysis und ihre Anwendungen, **29** (2010), No. 3, 251–273

[11] Arab, R., *Some generalizations of Darbo fixed point theorem and its application*, Miskolc Math. Notes , **18** (2017), No. 2, 595–610

[12] Bana, J., *On measures of noncompactness in Banach spaces*, Comment. Math. Univ. Carolin., **21** (1980), No. 1, 131–143

[13] Bansa, J. and Rzepka, B., *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett., **16** (2003), No. 1, 1–6

[14] Banach, S., *Sur les oprations dans les ensembles abstraits et leur application aux quations intégrales*, Fund. Math., **3** (1992), No. 1, 133–181

[15] Bhaskar, T. G. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), No. 7, 1379–1393

[16] Brouwer, L. E. J., *Über abbildung von mannigfaltigkeiten*, Math. Ann., **71** (1911), No. 1, 97–115

[17] Cai, L. and Liang, J., *New generalizations of Darbos fixed point theorem*, Fixed Point Theory Appl., **2015** (2015), No. 1, 156

[18] Darbo, G., *Punti uniti in trasformazioni a codominio non compatto*, Rendiconti del Seminario Matematico della Universita di Padova, **24** (1955), 84–92

[19] Darwish, M. A., Henderson, J. and ORegan, D., *Existence and asymptotic stability of solutions of a perturbed fractional functional-integral equation with linear modification of the argument*, Bull. Korean Math. Soc., **48** (2011), No. 3, 539–553

[20] Dhage, B. and Bellale, S. S., *Local asymptotic stability for nonlinear quadratic functional integral equations*, Electron. J. Qual. Theory Differ. Equ., **2008** (2008), No. 10, 1–13

[21] Karakaya, V., Bouzara, N. E. H., Doğan, K. and Atalan, Y., *Existence of tripled fixed points for a class of condensing operators in Banach spaces*, The Scientific World Journal, **2014** (2014)

[22] Karakaya, V., Mursaleen, M. and Bouzara, N. E. H., *Measure of noncompactness in the study of solutions for a system of integral equations*, 2015

[23] Kreyszig, E., *Introductory functional analysis with applications*, wiley, New York, 1978

[24] Kuratowski, C., *Sur les Espaces Complets et Régulièrement Complets*, Proceedings of the Japan Academy, **30** (1930), No. 1, 912–916

[25] Roshan, J. R., *Existence of solutions for a class of system of functional integral equation via measure of noncompactness*, J. Comput. Appl. Math., **313** (2017), 129–141

[26] Schauder, J., *Der fixpunktsatz in funktionalraümen*, Studia Math., **2** (1930), No. 1, 171–180

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