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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Convergence results for fixed point iterative algorithms in metric spaces

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ABSTRACT. Let (X, d) be a metric space, $f, f_n : X \to X$, with $F_f = F_{f_n}$, $n \in \mathbb{N}$. For the fixed point equation

(1) $$x = f(x)$$

we consider the following iterative algorithm,

(2) $$x \in X,\ x_0 = x,\ x_{n+1}(x) = f_n(x_n(x)),\ n \in \mathbb{N}.$$

By definition, the algorithm (2) is convergent if,

$$x_n(x) \to x^*(x) \in F_f \text{ as } n \to \infty,\ \forall\, x \in X.$$

In this paper we give some conditions on f_n and f which imply the convergence of algorithm (2). In this way we improve some results given in [Rus, I. A., *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations*, Fixed Point Theory, **13** (2012), No. 1, 179–192]. In our results, in general we do not suppose that, $F_f \neq \emptyset$. Some research directions are formulated.

1. INTRODUCTION

In this paper we study the following two problems:

Problem A. Let (X, d) be a metric space, $f, g : X \to X$ be such that $F_f = F_g$. For the fixed point equation,

(1.1) $$x = f(x)$$

we consider the following algorithm

(1.2) $$x \in X,\ x_0 = x,\ x_{n+1}(x) = g(x_n(x)),\ n \in \mathbb{N}.$$

By definition, the algorithm (1.2) is convergent if,

$$x_n(x) \to x^*(x) \in F_f \text{ as } n \to \infty,\ \forall\, x \in X.$$

The convergence of the algorithm (1.2), when f is nonexpansive, X is a bounded, convex and closed subset of a Hilbert, Banach or metric space with a convexity structure ($g(x) = (1-\lambda)x + \lambda f(x)$, $g(x) = W(x, f(x), \lambda)$, $g(x) = G(x, f(x)), \ldots$) is studied in an impressive number of papers (see [5], [22], [29], [14], [15], [65], [18], [23], [30], [62], [55], [72], [32], [1], [57], [31], [42], [25], [46], [60], [24], . . .).

Problem B. Let (X, d) be a metric space, $f, f_n : X \to X$, $n \in \mathbb{N}$ be such that, $F_f = F_{f_n}$, $n \in \mathbb{N}$. For the fixed point equation (1.1) we consider the following iterative algorithm,

(1.3) $$x \in X,\ x_0 = x,\ x_{n+1}(x) = f_n(x_n(x)),\ n \in \mathbb{N}.$$

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By definition, the algorithm (1.3) is convergent if,

$$x_n(x) \to x^*(x) \in F_f \text{ as } n \to \infty,\ \forall\, x \in X.$$

As in the case of Problem A, the convergence of algorithm (1.3), when f is nonexpansive, X is a bounded, convex and closed subset of a Hilbert, Banach or metric space with a convexity structure and f_n are given in the terms of f and the convexity structure of such spaces, is studied in a large number of papers ([28], [5], [29], [22], [3], [8], [10], [12], [13], [16], [17], [23], [30], [37], [39], [40], [41], [43], [66], [64], [26], [36], [20], [1], [34], [67], ...).

In this paper we give some conditions on f and g, respectively on f and f_n which imply the convergence of algorithm (1.2), respectively (1.3). In this way we improve some results given in [54]. In our results, in general we do not suppose apriori that the solution of equation (1.1), F_f, is nonempty. Some research directions are formulated.

2. Preliminaries

2.1. **Notations.** Throughout this paper we use the same notations as in [54].

2.2. **Special classes of sequences in a metric space.** Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called asymptotically regular if,

$$d(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty.$$

Now, let $f : X \to X$ be an operator. The sequence $(x_n)_{n\in\mathbb{N}}$ is called f-asymptotically regular if,

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

This two notions are the basic notions in the theory of iterative algorithms (see [12], [5], [22], [34], ...).

2.3. **Weakly Picard operators in metric spaces (see** [49]**,** [51]**,** [50]**,** [56]**).** Let (X, d) be a metric space. An operator $f : X \to X$ is called a weakly Picard operator (WPO) if the sequence, $(f^n(x))_{n\in\mathbb{N}}$, converges for all $x \in X$, and its limit, $x^*(x) \in F_f$. If f is WPO and, $F_f = \{x^*\}$, then f is called Picard operator (PO).

For a WPO, $f : X \to X$, we define the limit operator, $f^\infty : X \to X$, by $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x)$. We remark that f^∞ is a retraction on the fixed point set of f, F_f.

An important class of WPO is so called, ψ-WPO. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, continuous in 0 with $\psi(0) = 0$. The WPO f is called ψ-WPO iff,

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))),\ \forall\, x \in X.$$

We call a such condition, a retraction-displacement condition.

2.4. **Some classes of operators on a metric space.** Let (X, d) be a metric space and $f : X \to X$ be an operator. Then:

(1) f is an l-contraction if $0 < l < 1$ and

$$d(f(x), f(y)) \leq l d(x, y),\ \forall\, x, y \in X;$$

(2) f is a contractive operator if,

$$d(f(x), f(y)) < d(x, y),\ \forall\, x, y \in X,\ x \neq y;$$

(3) f is nonexpansive if,

$$d(f(x), f(y)) \leq d(x, y),\ \forall\, x, y \in X;$$

(4) f is Caristi-Browder operator (see [11], [58]) if, f is continuous and there exists, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \forall\, x \in X;$$

(5) f is quasinonexpansive (see [71], [24], [47]) if $F_f \neq \emptyset$ and
$$d(f(x), x^*) \leq d(x, x^*),\ \forall\, x \in X,\ \forall\, x^* \in F_f;$$
(6) f is quasicontractive if $F_f \neq \emptyset$ and
$$d(f(x), x^*) < d(x, x^*),\ \forall\, x \in X \setminus F_f,\ x^* \in F_f;$$
(7) f is K-demicontractive (see [40], [30], [23], [41], ...) if $K < 1$, $F_f \neq \emptyset$ and
$$(d(f(x), x^*))^2 \leq (d(x, x^*))^2 + K(d(x, f(x)))^2,\ \forall\, x \in X,\ \forall\, x^* \in F_f;$$
(8) f is demicompact (see [45], [35]) if the following implication holds:
$(x_n)_{n\in\mathbb{N}}$ a bounded sequence in X such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, implies that there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ which is convergent;
(9) the fixed point for f is well posed if $F_f = \{x^*\}$ and the following implication holds:
$(x_n)_{n\in\mathbb{N}}$ in X with $d(x_n, f(x_n)) \to 0$ as $n \to \infty$ implies that $x_n \to x^*$ as $n \to \infty$;
(10) the fixed point problem for f is well posed in generalized sense if the following implication holds:
$(x_n)_{n\in\mathbb{N}}$ a sequence in X such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, implies that there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$, which is convergent to a fixed point of f.

We remark that:

(a) If f is nonexpansive operator and $F_f = \emptyset$ then f is not a quasinonexpansive operator. In our paper, in what follows, we consider the following notion of quasinonexpansivity. An operator f is quasinonexpansive if or $F_f = \emptyset$ or if $F_f \neq \emptyset$, then
$$d(f(x), x^*) \leq d(x, x^*),\ \forall\, x \in X,\ x^* \in F_f,$$
i.e., f is quasinonexpansive if
$$d(f(x), x^*) \leq d(x, x^*),\ \forall\, x \in X,\ x^* \in F_f.$$
(b) If f is continuous and demicompact then the fixed point problem for f is well posed in generalized sense.
(c) Let (X, d) be a bounded metric space, $f : X \to X$ be continuous and α_K-condensing operator, where α_K is the Kuratowski measure of noncompactness (see [53], [58], ...). Then the fixed point problem for f is well posed in generalized sense.
(d) If f is K-demicontractive with $K < 0$, then
$$-K \sum_{n=0}^{\infty} (d(f^n(x), f^{n+1}(x)))^2 \leq (d(x, x^*))^2,\ \forall\, x \in X,\ x^* \in F_f.$$
This condition implies that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is f asymptotically regular, i.e., the operator f is asymptotically regular.

3. Displacement conditions

Let (X, d) be a metric space, $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : X \to \mathbb{R}_+$. By definition, (α, β) is an admissible pair if α satisfies the following implication:
$$(t_n)_{n\in\mathbb{N}} \in \mathbb{R}_+,\ \alpha(t_n) \to 0\ \Rightarrow\ t_n \to 0 \text{ as } n \to \infty.$$
Now let $g : X \to X$ be an operator. By definition g satisfies the (α, β)-displacement condition if:

(1) (α, β) is an admissible pair;
(2) $\alpha(d(x, g(x))) \leq \beta(x) - \beta(g(x)), \forall\, x \in X$.

Here are some examples of operators which satisfy a displacement condition:

(a) If $\alpha(t)=t$, $\forall\, t\in\mathbb{R}_+$ and g is continuous then g is a Caristi-Browder operator (see [11], [58]);

(b) (F.E. Browder [13]) Let $(B,\|\cdot\|)$ be a real Banach space, X be a nonempty, closed, convex subset of B and $g:X\to X$ be an operator. The following condition appear in [13] on g:

$$\varphi(\|g(x)\|)+\psi(\|x-g(x)\|)\leq\varphi(\|x\|),\ \forall\, x\in X,$$

with, $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ continuous, strict increasing with, $\varphi(0)=\psi(0)=0$.

A such operator satisfies the (α,β)-displacement condition with, $\alpha(t)=\psi(t)$, $\beta(x)=\varphi(\|x\|)$.

(c) (Măruşter [40], Hicks-Kubicek [30]) Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, X be a nonempty subset of H and $g:X\to X$ be a K-demicontractive operator with $K<0$. Then g satisfies the (α,β)-displacement condition, with $\alpha(t)=-Kt^2$ and $\beta(x)=\|x-x^*\|^2$.

(d) In the notion of demicontractivity we suppose that the fixed point set of the operator is nonempty. We can give a type of demicontractivity without this condition in the following way.

Let (M,d) be a metric space, $X\subset M$ be a nonempty subset, $g:X\to X$ be an operator and $p\in M\setminus X$.

By definition, the operator g is K-demicontractive with respect to the point p, if $K<1$, and

$$(d(g(x),p))^2\leq(d(x,p))^2+K(d(x,g(x)))^2,\ \forall\, x\in X.$$

It is clear that if $K<0$, then g satisfies (α,β)-displacement condition with, $\alpha(t)=-Kt^2$ and $\beta(x)=(d(x,p))^2$.

For a such trick see, for example, [38], [62].

We have the following results in terms of displacement conditions.

Theorem 3.1. *Let (X,d) be a metric space and $g:X\to X$ be an operator which satisfies the (α,β)-displacement condition. Then the operator g is asymptotically regular.*

Proof. Let $x\in X$. The (α,β)-displacement condition implies that:

$$\begin{aligned}\alpha(d(x,g(x)))&\leq\beta(x)-\beta(g(x)),\\ \alpha(d(g(x),g^2(x)))&\leq\beta(g(x))-\beta(g^2(x)),\\ &\vdots\\ \alpha(d(g^n(x),g^{n+1}(x)))&\leq\beta(g^n(x))-\beta(g^{n+1}(x)),\ \forall\, n\in\mathbb{N}.\end{aligned}$$

These imply that,

$$\sum_{n=0}^{\infty}\alpha(d(g^n(x),g^{n+1}(x)))\leq\beta(x),\ \forall\, x\in X,$$

from which it follows that, g is asymptotically regular. □

Now, let we have two operators, $f,g:X\to X$ with, $F_f=F_g$. By definition g satisfies (α,β,f)-displacement condition if the pair (α,β) is admissible and

$$\alpha(d(x,f(x)))\leq\beta(x)-\beta(g(x)),\ \forall\, x\in X.$$

For a such class of operators we have:

Theorem 3.2. *If g satisfies the (α,β,f)-displacement condition, then the sequence, $(g^n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular.*

Proof. From the (α, β, f)-displacement condition we have that:

$$\alpha(d(x, f(x))) \leq \beta(x) - \beta(g(x)), \ \forall\, x \in X,$$
$$\alpha(d(g(x), f(g(x)))) \leq \beta(g(x)) - \beta(g^2(x)), \ \forall\, x \in X,$$
$$\vdots$$
$$\alpha(d(g^n(x), f(g^n(x)))) \leq \beta(g^n(x)) - \beta(g^{n+1}(x)), \ \forall\, x \in X, \ \forall\, n \in \mathbb{N}.$$

These imply that,

$$\sum_{n=0}^{\infty} \alpha(d(g^n(x), f(g^n(x)))) \leq \beta(x), \ \forall\, x \in X.$$

This condition implies that the sequence $(g^n(x))_{n \in \mathbb{N}}$ is f-asymptotically regular for all $x \in X$. □

In the next section we shall use these two results to study the Problem A.

4. THE CONVERGENCE OF THE ALGORITHM IN PROBLEM A

Let (X, d) be a metric space and $f, g : X \to X$ be two operators with $F_f = F_g$. For the fixed point equation

$$x = f(x) \tag{4.1}$$

we consider the following iterative algorithm:

$$x \in X, \ x_0 = x, \ x_{n+1}(x) = g(x_n(x)), \ n \in \mathbb{N}. \tag{4.2}$$

The problem is in which conditions on f and g the algorithm (4.2) is convergent, i.e., in which conditions on f and g, the operator g is WPO ?

For a better understanding of the Problem A we start with some examples.

Example 4.1. Let $(B, \|\cdot\|)$ be a Banach space, $X \subset B$ be a nonempty, bounded, closed and convex subset of B and $f : X \to X$ be a nonexpansive operator. For $\lambda \in]0, 1[$ let f_λ be the Krasnoselski operator, corresponding to f, defined by

$$f_\lambda(x) = (1 - \lambda)x + \lambda f(x).$$

By a Ishikawa Theorem (see [18]) the operator f_λ is asymptotically regular. But,

$$f_\lambda(x) - x = \lambda(f(x) - x), \ \forall\, x \in X.$$

This implies that the sequence $(f_\lambda^n(x))_{n \in \mathbb{N}}$ is f-asymptotically regular.

Example 4.2. Let $(B, \|\cdot\|)$ be a Banach space, $f : B \to B$ be an operator and $\lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$.

We consider the operator $f_\lambda := (1 - \lambda)1_B + \lambda f$. Then we remark that, $F_f = F_{f_\lambda}$ and f_λ is asymptotically regular if and only if the sequence, $(f_\lambda^n(x))_{n \in \mathbb{N}}$, is f-asymptotically regular.

Example 4.3. Let B be a Banach space and $f : B \to B$ be an l-Lipschitz operator. In this case, the operator f_λ (see Example 4.2), for $\lambda = \frac{1}{1+l}$ is nonexpansive.

For the Problem A we have:

Theorem 4.1. *We suppose that:*

(i) *g satisfies the (α, β)-displacement condition;*

(ii) *there exists $c > 0$ such that,*

$$d(x, g(x)) \geq c d(x, f(x)), \ \forall\, x \in X;$$

(iii) *the fixed point problem for f is well posed in generalized sense;*

(iv) *g is quasinonexpansive.*

Then g is WPO and $g^\infty(X) = F_f$.

Proof. Let $x \in X$. From Theorem 3.1, the condition (i) implies that g is asymptotically regular. Condition (ii) implies that the sequence, $(g^n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular. From (iii), there exists a subsequence $(g^{n_i}(x))_{i\in\mathbb{N}}$ such that

$$g^{n_i}(x) \to x^*(x) \in F_f \text{ as } n \to \infty.$$

So, $F_f \neq \emptyset$. In this case the condition (iv) is effectivelly and implies that the sequence $(d(g^n(x), x^*))_{n\in\mathbb{N}}$ is decreasing. So,

$$d(g^n(x), x^*) \to d \geq 0 \text{ as } n \to \infty.$$

But, $d(g^{n_i}(x), x^*(x)) \to 0$ as $n_i \to \infty$, i.e.,

$$g^n(x) \to x^*(x) \in F_f \text{ as } n \to \infty.$$

□

Theorem 4.2. *We suppose that:*

(i) *g satisfies the (α, β, f)-displacement condition;*

(ii) *the fixed point problem for f is well posed in generalized sense;*

(iii) *g is quasinonexpansive.*

Then g is WPO.

Proof. Let $x \in X$. From Theorem 3.2, condition (i) implies that, $(g^n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular. Now the proof is similar with that of Theorem 4.1. □

In what follows, we give some applications of the above results to the iterative algorithm with admissible perturbation (see [54]).

Following [54] we introduce a new class of operators which generalizes the Krasnoselski operators. Let X be a nonempty set, $G : X \times X \to X$ be an operator. We suppose that:

(A_1) $G(x, x) = x$, $\forall\, x \in X$;

(A_2) $x, y \in X$, $G(x, y) = x$ imply, $y = x$.

Let $f : X \to X$ be an operator. We consider the operator $g = f_G : X \to X$, defined by

$$f_G(x) := G(x, f(x)).$$

We remark that, $F_f = F_{f_G}$.

We call the operator f_G the admissible perturbation of f corresponding to G. For some examples of admissible perturbation in the case in which X is a subset of linear space, Hilbert space, Banach space, metric space with convexity structure, see [54]. Problem A in this case is the following:

In which conditions on $f : (X, d) \to (X, d)$ and $G : X \times X \to X$, the admissible perturbation, f_G of f is WPO ?

For some results on this problem in the case of Hilbert and Banach spaces, see: [6], [10], [72], [7], [69], [64], [68], [19], [57], [70], ...

We give a result in a metric space.

Theorem 4.3. *We suppose that:*

(i) *there exists an admissible pair (α, β) such that:*

$$\alpha(d(x, G(x, y))) \leq \beta(x) - \beta(G(x, y)),\ \forall\, x, y \in X;$$

(ii) *there exists* $c > 0$ *such that:*

$$d(x, G(x, y)) \geq cd(x, y),\ \forall\, x, y \in X;$$

(iii) *the fixed point problem for* f *is well posed in generalized sense;*
(iv) *the operator* f_G *is quasinonexpansive.*

Then, the operator f_G *is* WPO.

Proof. If, in conditions (i) and (ii) we take $y = f(x)$, then we remark that the operator f_G satisfies the conditions in Theorem 4.1. □

5. Convergence of algorithm in Problem B

We start with some remarks on the sequences in metric spaces. Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We have:

Lemma 5.1. *If there exists an admissible pair,* (α, β)*, such that,*

$$\alpha(d(x_n, x_{n+1})) \leq \beta(x_n) - \beta(x_{n+1}),\ \forall\, n \in \mathbb{N},$$

then the sequence, $(x_n)_{n\in\mathbb{N}}$*, is asymptotically regular.*

Lemma 5.2. *Let,* $f : X \to X$*, be an operator. If there exists an admissible pair,* (α, β)*, such that,*

$$\alpha(d(x_n, f(x_n))) \leq \beta(x_n) - \beta(x_{n+1}),\ \forall\, n \in \mathbb{N},$$

then the sequence, $(x_n)_{n\in\mathbb{N}}$*, is* f*-asymptotically regular.*

Now let (X, d) be a metric space and, $f, f_n : X \to X$ be operators with, $F_f = F_{f_n}$. We consider for the fixed point equation corresponding to f, the algorithm (1.3), i.e.,

$$x \in X,\ x_0 = x,\ x_{n+1}(x) = f_n(x_n(x)),\ n \in \mathbb{N}. \tag{5.1}$$

For this algorithm we have:

Theorem 5.1. *We suppose that:*

(i) *there exists an admissible pair such that,*

$$\alpha(d(x_n(x), x_{n+1}(x))) \leq \beta(x_n(x)) - \beta(x_{n+1}(x)),\ \forall\, n \in \mathbb{N},\ \forall\, x \in X;$$

(ii) $d(x_n(x), x_{n+1}(x)) \geq cd(x_n(x), f(x_n(x)))$*, with some* $c > 0$*, for all* $n \in \mathbb{N}$ *and* $x \in X$*;*
(iii) *the fixed point problem for* f *is well posed in generalized sense;*
(iv) *the operators* f_n *are quasinonexpansive.*

Then the sequence, $(x_n(x))_{n\in\mathbb{N}}$ *converges to a fixed point of* f*,* $x^*(x)$*.*

Proof. From (i), the sequence, $x_n(x)$, is asymptotically regular. The condition (ii) implies that, $(x_n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular. Condition, (iii) implies that there exists a subsequence, $(x_{n_i}(x))_{i\in\mathbb{N}}$ of $(x_n(x))_{n\in\mathbb{N}}$ which converges to a fixed point of f, $x^*(x)$. In this case, the condition (iv) is effective and we have that the sequence

$$d(x_n(x), x^*(x)) \to d \geq 0 \text{ as } n \to \infty.$$

But, $x_{n_i}(x) \to x^*(x)$ as $n \to \infty$. So, the sequence $(x_n(x))_{n\in\mathbb{N}}$ converges to $x^*(x)$. □

From the above proof it is clear that we have the following result with direct conditions on f and f_n.

Theorem 5.2. *We suppose that:*

(i) f_n *satisfies* (α, β)*-displacement condition,* $\forall\, n \in \mathbb{N}$*;*
(ii) $d(x, f_n(x)) \geq cd(x, f(x))$*, with some* $c > 0$*,* $\forall\, x \in X$*;*
(iii) *the fixed point problem for* f *is well posed in generalized sense;*
(iv) f_n *is quasinonexpansive,* $\forall\, n \in \mathbb{N}$*.*

Then, the algorithm (5.1) *is convergent.*

Proof. From (i) we have that

$$\alpha(d(x, f_n(x))) \leq \beta(x) - \beta(f_n(x)),\ \forall\, n \in \mathbb{N},\ \forall\, x \in X.$$

In this relation, instead of x we put, $x_n(x)$, and we have

$$\alpha(d(x_n(x), x_{n+1}(x))) \leq \beta(x_n(x)) - \beta(x_{n+1}(x)),\ \forall\, n \in \mathbb{N},\ \forall\, x \in X.$$

From Lemma 5.1, the sequence $(x_n(x))_{n\in\mathbb{N}}$ is asymptotically regular. From (ii), the sequence $(x_n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular. Now, see the proof of Theorem 5.1. □

In a similar way we have,

Theorem 5.3. *We suppose that:*

(i) *there exists an admissible pair* (α, β) *such that,*

$$\alpha(d(x_n(x), f(x_n(x)))) \leq \beta(x_n(x)) - \beta(x_{n+1}(x)),\ \forall\, n \in \mathbb{N},\ \forall\, x \in X;$$

(ii) *the fixed point problem for* f *is well posed in generalized sense;*

(iii) f_n *is quasinonexpansive,* $\forall\, n \in \mathbb{N}$.

Then, the sequence, $(x_n(x))_{n\in\mathbb{N}}$*, converges to a fixed point of* f.

Theorem 5.4. *We suppose that:*

(i) f_n *satisfies* (α, β, f)*-displacement condition,* $\forall\, n \in \mathbb{N}$*;*

(ii) *the fixed point problem for* f *is well posed in generalized sense;*

(iii) f_n *is quasinonexpansive,* $\forall\, n \in \mathbb{N}$.

Then, the algorithm (5.1) *is convergent.*

6. Problems

From the above considerations the following questions rise:

6.1. To construct a theory for K-demicontractive operators with $K < 0$, in a metric space. For the K-demicontractive operators in Hilbert and Banach spaces see: [40], [30], [4], [5], [17], [23], [41], [69], [6], ...

6.2. To give new metric conditions which imply asymptotic regularity of an operator, and in general, not convergence of successive approximations. A similar problem in the case of sequences.

Let (X, d) be a metric space, $g : X \to X$ be an operator and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The (α, β)-displacement condition for g (see Theorem 3.1) and (α, β)-displacement condition for $(x_n)_{n\in\mathbb{N}}$ (see Lemma 5.1), imply asymptotic regularity.

The problem is to give other conditions with these properties.

References for asymptotic regularity: [48], [14], [15], [5], [18], [56], [58], [33], [31], [42], [25], [46], [50], [62]. [59], ...

6.3. To give, in a metric space, conditions in which asymptotic regularity of an operator (sequence) implies convergence of successive approximations (sequence).

In 1945, J. Dieudonné (see [48]) has given the following result:

Let $f \in C([a, b] \times \mathbb{R}^m, \mathbb{R}^m)$ and the following Cauchy problem corresponding to f:

$$y'(x) = f(x, y(x)),\ y(a) = y_0.$$

We consider the successive approximations for this problem,

$$y_{n+1}(x) = y_0 + \int_a^x f(s, y(s))ds,\ n \in \mathbb{N}.$$

If the Cauchy problem has a unique solution, then there exists, $h \in]0, b-a[$ such that the successive approximations sequence converges uniformly to the unique solution of Cauchy problem on $[a, a+h]$, if and only if the sequence $\{y_{n+1} - y_n\}$ converges uniformly to the null function, uniformly on $[a, a+h]$.

In 1976, B.P. Hillam (see [62]) proves the following result:

A function $f \in C([0,1],[0,1])$ is weakly Picard function if and only if f is asymptotically regular.

The problem is to give similar results in a metric space.

References: [48], [9], [5], [10], [14], [15], . . .

6.4. In which conditions a nonexpansive operator is a graphic contraction ?

One basic problem in the theory of nonexpansive operators is the following:

Let (X, d) be a metric space and $f : X \to X$ be a nonexpansive operator. In which conditions f is WPO ?

For a better understanding of the relation between nonexpansive operator theory and graphic contraction theory we present the following well known results.

• Theorem of equivalent statements. Let X be a nonempty set and $f : X \to X$ be an operator. The following statements are equivalent:

(i) $F_{f^n} = F_f \neq \emptyset$;

(ii) there exists a metric d on X with respect to which f is WPO;

(iii) there exists a complete metric on X with respect to which f is a continuous graphic contraction;

(iv) $F_f \neq \emptyset$ and there exists a metric d on X with respect to which f is asymptotically regular.

• Graphic Contraction Principle. Let (X, d) be a complete metric space, $f : X \to X$ be an operator and $l \in]0, 1[$. We suppose that:

(i) $d(f^2(x), f(x)) \leq l d(x, f(x)),\ \forall\, x \in X$;

(ii) f has closed graph.

Then the operator f is WPO.

• Bernstein operators, $B_n : C[0,1] \to C[0,1]$, are nonexpansive and graphic contractions.

The Bernstein operator, $B_n : C[0,1] \to C[0,1]$, is defined by

$$B_n(f)(x) := \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1].$$

It is well known that, $\|B_n\| = 1$ and

$$\|B_n^2(f) - B_n(f)\| \leq (1 - \tfrac{1}{2^{n-1}})\|f - B_n(f)\|.$$

So, B_n is a graphic contraction and weakly Picard operator.

References: [55], [50], [44], [52], . . .

6.5. To study the stability of algorithms in Problem A and B.

For the notion of stability of an iterative algorithm see: [54], [8], [5], [10], [27], [43], [42], [2], [63], [62], . . .

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