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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Some new results of M-iteration process in hyperbolic spaces

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ABSTRACT. In this paper, we study the M-iteration process in hyperbolic spaces and prove some strong and $\triangle$-convergence theorems of this iteration process for generalized nonexpansive mappings. Moreover, we establish the weak w^2-stability and data dependence theorems for a class of contractive-type mappings by using M-iteration process. The results presented here extend and improve some recent results announced in the current literature.

1. INTRODUCTION

Let C be a nonempty subset of a metric space (X, d) and T be a self mapping on C. A point $p \in C$ is called a *fixed point* of T if $Tp = p$, and $F(T)$ denotes the set of all fixed points of T. The mapping T is said to be *contraction* if there exists $\delta \in [0, 1)$ such that $d(Tx, Ty) \leq \delta d(x, y)$ for all $x, y \in C$. The mapping T is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ and *quasi-nonexpansive* if $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and for each $p \in F(T)$.

Osilike [18] considered the mapping T satisfying the condition

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx) \tag{1.1}$$

for some $a \in [0, 1), L \geq 0$ and for all $x, y \in C$. This class of contractive-type mappings includes the classes of mappings studied by Harder and Hicks [13], Rhoades [21, 22, 23] and Osilike [19]. It is known, see Osilike [18], that the mapping T satisfying (1.1) need not have a fixed point. However, if T has a fixed point, it follows easily from (1.1) that the fixed point is unique.

Garcia-Falset et al. [7] introduced a generalization of nonexpansive mappings which in turn includes Suzuki generalized nonexpansive mappings defined in [27].

Definition 1.1. (see [7, Definiton 2]) Let T be a mapping defined on a subset C of a metric space (X, d) and $\mu \geq 1$. Then T is said to satisfy the condition (E_μ) if for all $x, y \in C$,

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y).$$

T is said to satisfy the condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

The following example shows that the class of mappings satisfying the condition (E) is larger than the class of Suzuki generalized nonexpansive mappings.

Example 1.1. (see [7, Example 1]) In the space $C([0, 1])$, consider the set

$$K := \{x \in C([0, 1]) : 0 = x(0) \leq x(t) \leq x(1) = 1\}.$$

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Take any function $g \in K$ and generate the mapping

$$F_g : K \to K,\ F_g x(t) := (g \circ x)(t) = g(x(t)).$$

Then the mapping F_g satisfies the condition (E_μ) for $\mu = 1$ but it fails to be a Suzuki generalized nonexpansive mapping.

Proposition 1.1. (see [7, Proposition 1]) *Let $T : C \to C$ be a mapping satisfying condition (E) on C. If T has some fixed point, then T is quasi-nonexpansive.*

Kohlenbach [14] introduced the concept of hyperbolic space, defined below, which plays a significant role in many branches of mathematics.

A hyperbolic space (X, d, W) is a metric space (X, d) together with a mapping $W : X \times X \times [0, 1] \to X$ satisfying

(W1) $d(z, W(x, y, \alpha)) \leq \alpha d(z, x) + (1 - \alpha)d(z, y)$,

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\, d(x, y)$,

(W3) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha)d(z, w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [30]. The concept of hyperbolic space in [14] is more restrictive than the hyperbolic type introduced by Goebel and Kirk [8] and more general than the concept of hyperbolic space defined by Reich and Shafrir [20]. The class of hyperbolic spaces in the sense of Kohlenbach [14] contains all normed linear spaces and convex subsets thereof, the Hilbert ball with the hyperbolic metric (see [9]), Cartesian products of Hilbert balls, $\mathbb{R}$-trees, Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov (see [5]).

A subset C of a hyperbolic space X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$. The following equalities hold even for the more general setting of convex metric space (see [30, Proposition 1.2]): for all $x, y \in X$ and $\alpha \in [0, 1]$,

$$d(y, W(x, y, \alpha)) = \alpha d(x, y) \quad \text{and} \quad d(x, W(x, y, \alpha)) = (1 - \alpha)d(x, y).$$

As a consequence,

$$W(x, y, 1) = x \quad \text{and} \quad W(x, y, 0) = y. \tag{1.2}$$

A hyperbolic space (X, d, W) is *uniformly convex* [25] if for any $r > 0$ and $\varepsilon \in (0, 2]$, there exists a constant $\delta \in (0, 1]$ such that, for all $u, x, y \in X$,

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$, and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ is called *a modulus of uniform convexity* if $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$. The function η is *monotone* if it decreases with r for fix ε.

Recently, Ullah and Arshad [32] introduced a new iteration process called M-iteration process in Banach spaces, as follow

$$\begin{cases} x_0 \in C, \\ z_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \quad \forall n \geq 0. \end{cases} \tag{1.3}$$

With the help of a numerical example, they showed that this iteration process is faster than Picard-S iteration [11] and S-iteration [1] for Suzuki generalized nonexpansive mappings. Very recently, Alagöz, Gündüz and Akbulut [2] proved that the iteration process (1.3) converges faster than S_n-iteration [26] with a sufficient condition and faster than Picard-S iteration [11] and S-iteration [1] for the contractive-type mappings satisfying (1.1).

Using (W3) and (1.2) in (1.3), we extend the M-iteration process in hyperbolic spaces:

$$\begin{cases} x_0 \in C, \\ z_n = W(Tx_n, x_n, \alpha_n), \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where C is a nonempty convex subset of a hyperbolic space X, T is a self mapping on C and $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

In this paper, we study the convergence, weak w^2-stability and data dependence of the iteration process (1.4) in a hyperbolic space. This paper contains four sections. In Section 2, we recollect basic definitions and a detailed overview of the fundamental results. In Section 3, we prove some strong and $\triangle$-convergence theorems of the iteration process (1.4) for the class of mappings satisfying condition (E). In Section 4, we prove the weak w^2-stability and data dependence results for the class of mappings satisfying (1.1) by using the iteration process (1.4). Our results can be viewed as refinement and generalization of several well-known results in CAT(0) and uniformly convex Banach spaces.

2. Preliminaries

Let us recall some definitions and known results in the existing literature.

Let C be a nonempty subset of a metric space (X, d) and $\{x_n\}$ be a bounded sequence in C. Consider a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \quad x \in X.$$

Then, the infimum of $r(., \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r(C, \{x_n\})$.

A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r(z, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\};$$

the set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C, \{x_n\})$. This set may be empty or a singleton or contain infinitely many points.

If the asymptotic radius and center are taken with respect to X, then these are simply denoted by $r(X, \{x_n\}) = r(\{x_n\})$ and $A(X, \{x_n\}) = A(\{x_n\})$, respectively.

It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces. The following lemma ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.1. (see [17, Proposition 3.3]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and C be a nonempty closed convex subset of X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Recall that a sequence $\{x_n\}$ in X is said to be $\triangle$*-convergent* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

Lemma 2.2. (see [15, Lemma 2.5]) *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some*

$a, b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d\left(x_n, x\right) \leq c, \quad \limsup_{n\to\infty} d\left(y_n, x\right) \leq c, \quad \lim_{n\to\infty} d\left(W\left(x_n, y_n, \alpha_n\right), x\right) = c$$

for some $c \geq 0$, then

$$\lim_{n\to\infty} d\left(x_n, y_n\right) = 0.$$

3. Some convergence results

In this section, we prove the strong and $\triangle$-convergence theorems of M-iteration process for the class of mappings satisfying condition (E) in the setting of uniformly convex hyperbolic spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \to C$ be a mapping satisfying the condition (E) on C with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence (1.4) with a real sequence $\{\alpha_n\}$ in $[a, b]$ for some $a, b \in (0,1)$. Then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. We divide our proof into three steps.

Step 1. First we prove that for each $p \in F(T)$,

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.} \tag{3.5}$$

By Proposition 1.1, we have

$$d(x_{n+1}, p) = d(Ty_n, p) \leq d(y_n, p), \tag{3.6}$$

$$d(y_n, p) = d(Tz_n, p) \leq d(z_n, p) \tag{3.7}$$

and

$$\begin{aligned} d(z_n, p) &= d(W(Tx_n, x_n, \alpha_n), p) \\ &\leq \alpha_n d(Tx_n, p) + (1-\alpha_n) d(x_n, p) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we obtain

$$d(x_{n+1}, p) \leq d(x_n, p).$$

This implies that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Step 2. Next we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.9}$$

In fact, it follows from (3.5) that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F(T)$. Let

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.10}$$

Noting

$$d(Tx_n, p) \leq d(x_n, p),$$

by (3.10) we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq c. \tag{3.11}$$

Taking the limit supremum on both sides of (3.8), we obtain

$$\limsup_{n\to\infty} d(z_n, p) \leq c. \tag{3.12}$$

By using (3.6) and (3.7), we get

$$d(x_{n+1},p) \le d(z_n,p)$$

which yields that

$$c \le \liminf_{n\to\infty} d(z_n,p). \tag{3.13}$$

From the estimates of (3.12) and (3.13), we have that $\lim_{n\to\infty} d(z_n,p) = c$. Thus, from (1.4), we obtain

$$\lim_{n\to\infty} d(W(Tx_n,x_n,\alpha_n),p) = c. \tag{3.14}$$

With the help of (3.10), (3.11), (3.14) and Lemma 2.2, we get $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Step 3. Now we are in a position to prove the $\triangle$-convergence of $\{x_n\}$. Since the sequence $\{x_n\}$ is bounded, by Lemma 2.1, it has a unique asymptotic center $A(C,\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(C,\{u_n\}) = \{u\}$. Then, by (3.9), we have

$$\lim_{n\to\infty} d(u_n,Tu_n) = 0. \tag{3.15}$$

We claim that u is a fixed point of T. Since T satisfies the condition (E), then there exists a $\mu \ge 1$ such that

$$d(u_n,Tu) \le \mu d(u_n,Tu_n) + d(u_n,u).$$

Taking the limit supremum on both sides of the above estimate and using (3.15), we have

$$\begin{aligned} r(\{u_n\},Tu) &= \limsup_{n\to\infty} d(u_n,Tu) \\ &\le \limsup_{n\to\infty} d(u_n,u) = r(\{u_n\},u). \end{aligned}$$

By the uniqueness of asymptotic center, we get $Tu = u$. Thus $u \in F(T)$. Next, we claim that the fixed point u is the unique asymptotic center for each subsequence $\{u_n\}$ of $\{x_n\}$. Assume on the contrary, that is, $x \ne u$. Since $\lim_{n\to\infty} d(x_n,u)$ exists, therefore by the uniqueness of asymptotic center, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n,u) &< \limsup_{n\to\infty} d(u_n,x) \\ &\le \limsup_{n\to\infty} d(x_n,x) \\ &< \limsup_{n\to\infty} d(x_n,u) \\ &= \limsup_{n\to\infty} d(u_n,u), \end{aligned}$$

which is a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to a fixed point of T. □

Theorem 3.2. *Let X, C, T and $\{x_n\}$ be the same as in Theorem 3.1 and C be a compact subset of X. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By (3.9), we have $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. Since T satisfies condition (E), we have

$$d(x_{n_k},Tp) \le \mu d(x_{n_k},Tx_{n_k}) + d(x_{n_k},p). \tag{3.16}$$

Then, by taking the limit on both sides of (3.16), we obtain

$$\lim_{k\to\infty} d(x_{n_k},Tp) \le \lim_{k\to\infty} d(x_{n_k},p) = 0.$$

In view of the uniqueness of the limit, we have $Tp = p$, that is $p \in F(T)$. It follows from (3.5) that $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and hence $\{x_n\}$ converges strongly to p. □

Example 3.2. Let $\mathbb{R}$ be the real line with the usual metric $|.|$ and $C = [-3, 1]$. Define a mapping $T : [-3, 1] \to [-3, 1]$ by

$$Tx = \begin{cases} \frac{|x|}{3} & x \in [-3, 1)\,, \\ -\frac{1}{3} & x = 1. \end{cases}$$

In order to see that T satisfies condition (E) on $[-3, 1]$, we consider the following (non-trivial) cases:

a) Let $x \in [-3, 0]$ and $y \in [-3, 1]\,,$ then $|x - Tx| = \frac{4}{3}|x|$ and

$$|x - Ty| \le |x| + \frac{1}{3}|y| \le \frac{4}{3}|x| + \frac{1}{3}|x - y| \le |x - Tx| + |x - y|\,.$$

b) Let $x \in [0, 1)$ and $y \in [-3, 1]\,,$ then $|x - Tx| = \frac{2}{3}|x|$ and

$$|x - Ty| \le |x| + \frac{1}{3}|y| \le \frac{4}{3}|x| + \frac{1}{3}|x - y| \le 2|x - Tx| + |x - y|\,.$$

c) Let $x = 1$ and $y \in [-3, 1)$, then $|1 - T1| = \frac{4}{3}$ and

$$|1 - Ty| = \frac{2}{3} + \frac{1 - |y|}{3} \le \frac{1}{2}|1 - T1| + \frac{1}{3}|1 - y| \le |1 - T1| + |1 - y|\,.$$

In summary, for all $x, y \in [-3, 1]\,,$

$$|x - Ty| \le 2|x - Tx| + |x - y|\,,$$

that is, the mapping T satisfies condition (E_2) on $[-3, 1]\,.$ Clearly, $F(T) = \{0\}$. Set $\alpha_n = \frac{1}{\sqrt{3n+7}}$ for all $n \ge 0$. Thus the conditions of Theorem 3.1 are fulfilled. Now the conclusions of Theorem 3.1 and Theorem 3.2 follow.

Senter and Dotson [24, p. 375] introduced the concept of condition (I) as follows.

A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \ge f(d(x, F(T))) \quad \text{for all} \quad x \in C, \tag{3.17}$$

where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}\,.$

Now we prove the strong convergence theorem using condition (I).

Theorem 3.3. *Under the assumptions of Theorem 3.1, if T satisfies condition (I), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By (3.9), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Therefore, we get that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

The rest of the proof is similar to the proof of Theorem 12 in [16] and therefore it is omitted. □

Remark 3.1. Our results generalize the corresponding results of Ullah and Arshad [32] in two ways: (i) from the class of Suzuki generalized nonexpansive mappings to the class of mappings satisfying condition (E), (ii) from uniformly convex Banach spaces to uniformly convex hyperbolic spaces.

4. THE WEAK w^2-STABILITY AND DATA DEPENDENCE RESULTS

We begin with the following lemma to shorten proofs of our theorems in this section.

Lemma 4.3. *Let C be a nonempty subset of a metric space (X, d) and $T : C \to C$ be a mapping satisfying (1.1) with unique fixed point p. Then*

$$d(Tx, p) \le ad(x, p) \quad \textit{for all } x \in C.$$

Proof. It is easily seen that

$$d(Tx, p) = d(Tp, Tx) \le ad(p, x) + Ld(p, Tp) = ad(x, p).$$

□

Now we give the strong convergence theorem of M-iteration process for a class of contractive-type mappings in a hyperbolic space.

Theorem 4.4. *Let C be a nonempty closed convex subset of a hyperbolic space X, $T : C \to C$ be a mapping satisfying (1.1) with unique fixed point p and $\{x_n\}$ be the iterative sequence (1.4) with a real sequence $\{\alpha_n\}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to p.*

Proof. By (W1), (1.4) and Lemma 4.3, we have

$$d(x_{n+1}, p) = d(Ty_n, p) \le ad(y_n, p), \tag{4.18}$$

$$d(y_n, p) = d(Tz_n, p) \le ad(z_n, p) \tag{4.19}$$

and

$$\begin{aligned} d(z_n, p) &= d(W(Tx_n, x_n, \alpha_n), p) \\ &\le \alpha_n d(Tx_n, p) + (1 - \alpha_n)d(x_n, p) \\ &\le \alpha_n ad(x_n, p) + (1 - \alpha_n)d(x_n, p) \\ &= (1 - \alpha_n(1 - a))d(x_n, p). \end{aligned} \tag{4.20}$$

Combining (4.18), (4.19) and (4.20), we obtain

$$\begin{aligned} d(x_{n+1}, p) &\le a^2(1 - \alpha_n(1 - a))d(x_n, p) \\ &\le a^2(1 - \alpha_n(1 - a))a^2(1 - \alpha_{n-1}(1 - a))d(x_{n-1}, p) \\ &\le \cdots \\ &\le \left(a^2\right)^{n+1} \prod_{k=0}^{n} (1 - \alpha_k(1 - a))d(x_0, p). \end{aligned} \tag{4.21}$$

It is well-known from the classical analysis that $1 - x \le e^{-x}$ for all $x \in [0, 1]$. Taking into account this fact together with (4.21), we get

$$d(x_{n+1}, p) \le \left(a^2\right)^{n+1} e^{-(1-a)\sum_{k=0}^{n} \alpha_k} d(x_0, p).$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $a \in [0, 1)$, therefore we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = 0.$$

Thus we obtain $x_n \to p \in F(T)$. □

Remark 4.2. The strong convergence result of the iteration process (1.3) can be obtained as a corollary from Theorem 4.4.

We say that $\{x_n\}_{n=0}^{\infty}$ is T*-stable* or *stable with respect to* T if $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point p of T, then an approximate sequence $\{y_n\}_{n=0}^{\infty}$ converges strongly to p. This notion was introduced by Urabe [33]. However, a formal definition of stability for a general iteration method is given by Harder and Hicks [13] as follows.

Definition 4.2. (see [13]) Let (X, d) be a metric space, T be a self mapping on X and $\{x_n\}_{n=0}^{\infty} \subset X$ be an iterative sequence produced by the mapping T such that

$$\begin{cases} x_0 \in X, \\ x_{n+1} = f(T, x_n), \quad \forall n \geq 0, \end{cases} \tag{4.22}$$

where x_0 is an initial approximation and f is a function. Assume that $\{x_n\}$ converges strongly to $p \in F(T)$. If for an arbitrary sequence $\{y_n\}_{n=0}^{\infty} \subset X$,

$$\lim_{n\to\infty} d\left(y_{n+1}, f(T, y_n)\right) = 0 \Longrightarrow \lim_{n\to\infty} y_n = p,$$

then the iterative sequence $\{x_n\}$ is said to be *stable with respect to* T or simply T*-stable*.

Definition 4.3. (see [6]) Let (X, d) be a metric space and let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two sequences in X. We say that these sequences are equivalent if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Timiş [31] defined the following concept of weak w^2-stability by adopting equivalent sequences instead of arbitrary sequences in Definition 4.2.

Definition 4.4. (see [31, Definition 2.4]) Let (X, d) be a metric space, T be a self mapping on X and $\{x_n\}_{n=0}^{\infty} \subset X$ be the iterative sequence given by (4.22). Suppose that $\{x_n\}$ converges strongly to $p \in F(T)$. If for any equivalent sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of $\{x_n\}$,

$$\lim_{n\to\infty} d\left(y_{n+1}, f(T, y_n)\right) = 0 \Longrightarrow \lim_{n\to\infty} y_n = p,$$

then the iterative sequence $\{x_n\}$ is said to be *weak* w^2*-stable with respect to* T.

Next we prove that the M-iteration process is weak w^2-stable with respect to T.

Theorem 4.5. *Suppose that all the conditions of Theorem 4.4 hold. Then the M-iteration process (1.4) is weak* w^2*-stable with respect to* T.

Proof. Let $\{x_n\}_{n=0}^{\infty}$ be the M-iterative sequence given by (1.4) and $\{p_n\}_{n=0}^{\infty} \subset C$ be an equivalent sequence of $\{x_n\}$. Set

$$\varepsilon_n = d(p_{n+1}, Tq_n)$$

where $q_n = Tr_n$ with $r_n = W(Tp_n, p_n, \alpha_n)$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. It follows from (1.1), (W4) and (1.4) that

$$\begin{aligned} d(p_{n+1}, p) &\leq d(p_{n+1}, x_{n+1}) + d(x_{n+1}, p) \\ &\leq d(p_{n+1}, Tq_n) + d(Tq_n, Ty_n) + d(x_{n+1}, p) \\ &\leq \varepsilon_n + ad(y_n, q_n) + Ld(y_n, Ty_n) + d(x_{n+1}, p), \end{aligned} \tag{4.23}$$

$$d(y_n, q_n) = d(Tz_n, Tr_n) \leq ad(z_n, r_n) + Ld(z_n, Tz_n) \tag{4.24}$$

and

$$\begin{aligned} d(z_n, r_n) &= d(W(Tx_n, x_n, \alpha_n), W(Tp_n, p_n, \alpha_n)) \\ &\leq \alpha_n d(Tx_n, Tp_n) + (1-\alpha_n) d(x_n, p_n) \\ &\leq \alpha_n \left[ad(x_n, p_n) + Ld(x_n, Tx_n)\right] + (1-\alpha_n) d(x_n, p_n) \\ &= (1-\alpha_n(1-a)) d(x_n, p_n) + \alpha_n L d(x_n, Tx_n). \end{aligned} \tag{4.25}$$

Combining (4.23), (4.24) and (4.25), we have

$$\begin{aligned} d(p_{n+1}, p) \leq\ & \varepsilon_n + a^2(1-\alpha_n(1-a)) d(x_n, p_n) + a^2 \alpha_n L d(x_n, Tx_n) \\ & + aLd(z_n, Tz_n) + Ld(y_n, Ty_n) + d(x_{n+1}, p). \end{aligned} \tag{4.26}$$

From Theorem 4.4, it follows that $\lim_{n\to\infty} d(x_{n+1}, p) = 0$. Hence, by Lemma 4.3, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, p) + d(p, Tx_n) \\ &\leq d(x_n, p) + ad(p, x_n) \\ &= (1+a)d(x_n, p) \end{aligned}$$

which yields that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Similarly, by using (4.19) and (4.20), we get

$$\lim_{n\to\infty} d(y_n, Ty_n) = \lim_{n\to\infty} d(z_n, Tz_n) = 0.$$

Since $\{x_n\}$ and $\{p_n\}$ are equivalent sequences, we have $\lim_{n\to\infty} d(x_n, p_n) = 0$. Now taking limit on both sides of (4.26) and then using the assumption $\lim_{n\to\infty} \varepsilon_n = 0$, it leads to $\lim_{n\to\infty} d(p_{n+1}, p) = 0$. Thus $\{x_n\}$ is weak w^2-stable with respect to T. □

Many analytical methods may fail in finding a fixed point of a mapping. Therefore, instead of computing the fixed point of the mapping, we approximate it with the help of another one whose fixed points can be easily computed. This approach is referred as to "Data Dependence" (see [3, 10, 12, 28]) and it has received a great deal of attention recently in view of its promising and interesting applications.

Now we give some information which is necessary for data dependence result of M-iteration process.

Definition 4.5. (see [4]) Let $T, \widetilde{T} : X \to X$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if for all $x \in X$ and for a fixed $\varepsilon > 0$, we have $d(Tx, \widetilde{T}x) \leq \varepsilon$.

Lemma 4.4. (see [29]) *Let $\{a_n\}$ be a non-negative sequence for which one assumes that there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,*

$$a_{n+1} \leq (1 - r_n)a_n + r_n t_n$$

is satisfied, where $r_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} r_n = \infty$ and $t_n \geq 0, \forall n \in \mathbb{N}$. Then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} t_n.$$

Next we prove the data dependence result for the M-iteration process.

Theorem 4.6. *Let X, C and T be the same as in Theorem 4.4 and $\widetilde{T} : C \to C$ be an approximate operator of T for given ε. Suppose that $\{x_n\}$ and $\{\widetilde{x}_n\}$ are two iterative sequences defined by (1.4) and*

$$\begin{cases} \widetilde{x}_0 \in C, \\ \widetilde{z}_n = W(\widetilde{T}\widetilde{x}_n, \widetilde{x}_n, \alpha_n), \\ \widetilde{y}_n = \widetilde{T}\widetilde{z}_n, \\ \widetilde{x}_{n+1} = \widetilde{T}\widetilde{y}_n, \quad \forall n \geq 0, \end{cases} \tag{4.27}$$

respectively, where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $p = Tp$ and $\widetilde{p} = \widetilde{T}\widetilde{p}$, then we have

$$d(p, \widetilde{p}) \leq \frac{(a^2 + a + 1)\varepsilon}{1 - a^2}.$$

where $a \in [0,1)$.

Proof. It follows from (1.1), (1.4) and (4.27), we get

$$\begin{aligned} d(x_{n+1},\widetilde{x}_{n+1}) &= d(Ty_n,\widetilde{T}\widetilde{y}_n) \\ &\leq d(Ty_n,T\widetilde{y}_n)+d(T\widetilde{y}_n,\widetilde{T}\widetilde{y}_n) \\ &\leq ad(y_n,\widetilde{y}_n)+Ld(y_n,Ty_n)+\varepsilon, \end{aligned} \tag{4.28}$$

$$\begin{aligned} d(y_n,\widetilde{y}_n) &= d(Tz_n,\widetilde{T}\widetilde{z}_n) \\ &\leq d(Tz_n,T\widetilde{z}_n)+d(T\widetilde{z}_n,\widetilde{T}\widetilde{z}_n) \\ &\leq ad(z_n,\widetilde{z}_n)+Ld(z_n,Tz_n)+\varepsilon \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} d(z_n,\widetilde{z}_n) &= d(W(Tx_n,x_n,\alpha_n),W(\widetilde{T}\widetilde{x}_n,\widetilde{x}_n,\alpha_n)) \\ &\leq \alpha_n d(Tx_n,\widetilde{T}\widetilde{x}_n)+(1-\alpha_n)d(x_n,\widetilde{x}_n) \\ &\leq \alpha_n d(Tx_n,T\widetilde{x}_n)+\alpha_n d(T\widetilde{x}_n,\widetilde{T}\widetilde{x}_n)+(1-\alpha_n)d(x_n,\widetilde{x}_n) \\ &\leq \alpha_n[ad(x_n,\widetilde{x}_n)+Ld(x_n,Tx_n)]+\alpha_n\varepsilon+(1-\alpha_n)d(x_n,\widetilde{x}_n) \\ &= (1-\alpha_n(1-a))d(x_n,\widetilde{x}_n)+\alpha_n Ld(x_n,Tx_n)+\alpha_n\varepsilon. \end{aligned} \tag{4.30}$$

Combining (4.28), (4.29) and (4.30), we get

$$\begin{aligned} d(x_{n+1},\widetilde{x}_{n+1}) \leq\ & a^2(1-\alpha_n(1-a))d(x_n,\widetilde{x}_n)+a^2\alpha_n Ld(x_n,Tx_n)+a^2\alpha_n\varepsilon \\ & +aLd(z_n,Tz_n)+a\varepsilon+Ld(y_n,Ty_n)+\varepsilon. \end{aligned} \tag{4.31}$$

If $a^2 \in (0,1)$, then there exists a real number $k \in (0,1)$ such that

$$a^2 = 1-k. \tag{4.32}$$

In view of (4.32) and using the facts of $\alpha_n \leq 1$ and $1-\alpha_n(1-a) \leq 1$ for all $n \in \mathbb{N}$, we can re-write (4.31) as

$$\begin{aligned} & d(x_{n+1},\widetilde{x}_{n+1}) \\ \leq\ & (1-k)d(x_n,\widetilde{x}_n) \\ & +k\frac{a^2Ld(x_n,Tx_n)+aLd(z_n,Tz_n)+Ld(y_n,Ty_n)+a^2\varepsilon+a\varepsilon+\varepsilon}{k}. \end{aligned} \tag{4.33}$$

Now define

$$\begin{aligned} a_n &= d(x_n,\widetilde{x}_n), \\ r_n &= k, \\ t_n &= \frac{a^2Ld(x_n,Tx_n)+aLd(z_n,Tz_n)+Ld(y_n,Ty_n)+a^2\varepsilon+a\varepsilon+\varepsilon}{1-a^2}. \end{aligned}$$

It is easy to check that the inequality (4.33) meets all the requirements in Lemma 4.4. Also as in the proof of Theorem 4.5, we can get

$$\lim_{n\to\infty} d(x_n,Tx_n) = \lim_{n\to\infty} d(y_n,Ty_n) = \lim_{n\to\infty} d(z_n,Tz_n) = 0. \tag{4.34}$$

Therefore, we have

$$d(p,\widetilde{p}) \leq \frac{(a^2+a+1)\varepsilon}{1-a^2}.$$

If $a^2 = 0$, from (4.31) and (4.34), we get $d(p,\widetilde{p}) \leq \varepsilon$. □

Remark 4.3. In the proof of Theorem 4.6, we can also re-write (4.31) as

$$\begin{aligned} & d(x_{n+1},\widetilde{x}_{n+1}) \\ \leq\ & (1-k)d(x_n,\widetilde{x}_n) \\ & +k\frac{a^2\alpha_n Ld(x_n,Tx_n)+aLd(z_n,Tz_n)+Ld(y_n,Ty_n)+a^2\alpha_n\varepsilon+a\varepsilon+\varepsilon}{1-a^2}. \end{aligned}$$

If the condition $\lim_{n\to\infty}\alpha_n = 0$ is added for the sequence $\{\alpha_n\}$ in the hypotheses of Theorem 4.6, then we obtain that

$$d(p,\widetilde{p}) \leq \frac{\varepsilon}{1-a}.$$

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