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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Sequential characterizations of robust optimal solutions in uncertain convex programs via perturbation approach

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ABSTRACT. By using robust optimization approach, necessary and sufficient sequential optimality conditions without any constraint qualifications for the general convex optimization problem in the face of data uncertainty are given in terms of the ε-subdifferential. A sequential condition involving only the subdifferentials is also derived using a version of the Brøndsted-Rockafellar theorem. Consequently, sequential Lagrange multiplier condition for robust optimal solution of convex optimization problem with cone constraints in the face of data uncertainty is given. It is worth pointing out that there is no compactness assumption of uncertainty set and upper semicontinuity of functions involved in our results.

1. INTRODUCTION

Convex programs that are affected by data uncertainty have attracted attention of many researchers in the past years [2, 3, 4, 5, 6, 7, 10, 14, 15, 16, 18, 19, 21, 23, 24] and references therein. Robust optimization (see, e.g. [4, 5]) is one of the basic methodologies to treat an optimization problem against uncertain parameters in the problem by examining an optimal solution which simultaneously satisfies all possible realizations of the parameters within their prescribed uncertainty sets. Such the optimal solution is known as a robust optimal solution. Following this approach, one can get a robust optimal solution of convex optimization problems in the face of data uncertainty by solving the single convex optimization problem. As an illustration, consider the following uncertain convex optimization problem with cone constraints

$$\text{(UP)} \qquad \inf_{x\in X}\{f(x,u) : x \in C,\ g(x,v) \in -K\},$$

where C is a closed convex subset of a reflexive Banach space X, K is a closed convex cone of a Banach space Y, $\mathcal{U}$ and $\mathcal{V}$ are convex subsets of a Banach space Z, u and v are the uncertain parameters of the problem that we do not know the exact values, but we know that u (resp. v) belongs to some uncertainty set $\mathcal{U}$ (resp. $\mathcal{V}$), $f(\cdot,u) : X \to \mathbb{R}\cup\{+\infty\}$, $u \in \mathcal{U}$, is a proper lower semicontinuous convex function and $g(\cdot,v) : X \to Y$, $v \in \mathcal{V}$, is continuous and K-convex. Then the robust counterpart of the (UP) is given by

$$\text{(RP)} \qquad \inf_{x\in X}\left\{\sup_{u\in\mathcal{U}} f(x,u) : x \in C,\ g(x,v) \in -K,\ \forall v \in \mathcal{V}\right\}.$$

On the other hand, sequential forms of the Lagrange multiplier condition characterizing optimality without any constraint qualifications for convex programs in the absence

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of data uncertainty have been intensively studied [1, 13, 17, 22, 25] and the references therein because the constraint qualifications do not always hold and these conditions are a meaningful contribution for supporting the employment of different stopping criteria for practical optimization algorithms. Further study has been done for the general convex optimization problem in [8, 9] which provided some sequential Lagrange multiplier conditions in the most general setting, extending the corresponding results in the literature. In what follows, the primary aim of this paper is to investigate sequential optimality conditions without any constraint qualifications for the general convex optimization problem in the face of data uncertainty:

$$(UP_\phi) \qquad \inf_{x\in X} \phi_u(x,0),$$

where X is a reflexive Banach space, Y is a Banach space, $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$, the so-called *perturbation function*, (see, e.g. [26]) is a proper lower semicontinuous convex function and u is the uncertain parameter which belongs to the uncertainty set $\mathcal{U}$. A *robust optimal solution* of (UP_ϕ) is obtained by solving its robust counterpart (RP_ϕ) of (UP_ϕ):

$$(RP_\phi) \qquad \inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x,0).$$

It is remarkable that a sequential characterization for robust optimal solution of (UP_ϕ) is done by applying [9, Theorem 3.2 (or Theorem 3.3 also)] if the function $u \mapsto \phi_u(x,y)$ is upper semicontinuous for any $(x,y) \in X \times Y$ and the uncertainty set $\mathcal{U}$ is compact (see Remark 3.2) which, however, commonly describes in terms of a bounded set.

Motivated and inspired by the mentioned above, in this paper, we aim to establish sequential optimality conditions for robust optimal solutions of (UP_ϕ) that do not require the fulfillment of any constraint qualifications, upper semicontinuity of $u \mapsto \phi_u(x,y)$ for any $(x,y) \in X \times Y$ and compactness of the uncertainty set $\mathcal{U}$. As a consequence, we obtain sequential Lagrange multiplier condition for robust solution of convex optimization problem with cone constraints in the face of data uncertainty.

The layout of the paper is as follows. In the next section, we collect some definitions, notations and preliminary results that will be used later in the paper. Sect. 3 establishes the sequential characterizations of robust optimal solutions in terms of the ε-subdifferentials as well as subdifferentials. Then, in Sect. 4, the results established in Sect. 3 are applied in order to provide some sequential Lagrange multiplier conditions for robust solution of convex optimization problem with cone constraints in the face of data uncertainty.

2. Preliminaries

We begin this section by fixing certain notation and preliminaries of convex analysis that will be used throughout the paper. Let $(X, \|\cdot\|)$ be a reflexive Banach space, $(Y, \|\cdot\|)$ be a Banach space, with $(X^*, \|\cdot\|_*)$, $(Y^*, \|\cdot\|_*)$, respectively, their topological dual spaces. Let $\{x_n^* : n \in \mathbb{N}\}$ be a sequence in X^*. We write $x_n^* \xrightarrow{\omega^*} 0$ ($x_n^* \xrightarrow{\|\cdot\|_*} 0$) for the case when x_n^* converges to 0 in the weak* (strong) topology. For convention, we write $x_n^* \to 0$ $(n \to +\infty)$ which understand that the property holds no matter which of the two topologies (weak* or strong) is used. The following property will be frequently used in the paper: if $x_n^* \to 0$ and $x_n \to a$ $(n \to +\infty)$, then $\langle x_n^*, x_n\rangle \to 0$ $(n \to +\infty)$, where $\{x_n\} \subseteq X$, $\forall n \in \mathbb{N}$, $a \in X$, $\langle\cdot,\cdot\rangle$ denotes the corresponding linear action between the dual pairs and $x_n \to a$ $(n \to +\infty)$ means $\|x_n - a\| \to 0$ $(n \to +\infty)$, that is the convergence in the topology induced by the norm on X. We equip the space $X \times Y$ with the norm $\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$, for $(x,y) \in X \times Y$. Similarly we define the norm on $X^* \times Y^*$. For a set C in X the closure (resp. convex hull) of C is denoted by $\mathrm{cl}(C)$ (resp. $\mathrm{co}(C)$). The indicator function of C, $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$, is defined as $\delta_C(x) = 0$ if $x \in C$; otherwise, $\delta_C(x) = +\infty$.

Next we give some notions regarding functions. For an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, the effective domain and the epigraph are respectively defined by dom$f := \{x \in X : f(x) < +\infty\}$ and epi$f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. We say that f is proper if dom$f \neq \emptyset$. Moreover, if epi$f$ is closed, we say f is a lower semicontinuous function. By cl(f) (resp. $\mathrm{co}(f)$) we denote the lower semicontinuous hull (resp. convex hull) of f, namely the function of which epigraph is the closure (resp. convex hull) of epi(f) in $X \times \mathbb{R}$, that is epi(cl(f)) = cl(epi(f)) (resp. epi(co(f)) = co(epi(f))). A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $t \in [0, 1]$, $f((1-t)x+ty) \leq (1-t)f(x)+tf(y)$ for all $x, y \in X$. The function f is said to be concave whenever $-f$ is convex. As usual, for any convex function f on X, its conjugate function $f^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ is defined by $f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$. We have the so called Young-Fenchel inequality $f^*(x^*) + f(x) \geq \langle x^*, x\rangle,\ \forall x \in X,\ \forall x^* \in X^*$.

For $x \in$ dom$f$ and $\varepsilon \geq 0$ we define the ε-subdifferential of f at x by $\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon,\ \forall y \in X\}$. When $x \notin$ dom$f$ we define that $\partial_\varepsilon f(x) = \emptyset$. If $\varepsilon = 0$, the set $\partial f(x) := \partial_0 f(x)$ is then the classical subdifferential of f at x. The following characterizations of the subdifferential and ε-sudifferential of a proper function f, by means of conjugate functions will be useful in the paper (see, e.g. [26]): $x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle$ and $x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon$.

In the case of $f : X \to \mathbb{R}\cup\{+\infty\}$ is a proper lower semicontinuous and convex function, and $a \in$ dom f the epigraph of f^* can be represented as follows (see, e.g. [12, Proposition 2.1.])

$$\text{epi } f^* = \bigcup_{\varepsilon \geq 0}\{(x^*, \langle x^*, a\rangle + \varepsilon - f(a)) : x^* \in \partial_\varepsilon f(a)\}. \tag{2.1}$$

The ε-normal cone of a closed convex set C at $x \in X$ is defined by $N_C^\varepsilon(x) := \partial_\varepsilon \delta_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varepsilon,\ \forall y \in C\}$ when $x \in C$, and $N_C^\varepsilon(x) := \emptyset$ when $x \notin C$. If $\varepsilon = 0$, $N_C^0(x) := N_C(x)$ is the classical normal cone in convex analysis.

Let $K \subseteq Y$ be a nonempty closed convex cone. The dual cone of K is defined by $K^* := \{k^* \in Y^* : \langle k^*, k\rangle \geq 0,\ \forall k \in K\}$. For a vector valued function $g : X \to Y$, we say that g is K-convex if $g((1-t)x + ty) - (1-t)g(x) - tg(y) \in -K,\ \forall t \in [0, 1],\ \forall x, y \in X$.

Now, let us recall the following results which will be useful in the sequel.

Lemma 2.1. [13] *Let I be an arbitrary index set and let f_i, $i \in I$, be proper lower semicontinuous convex functions on X. Suppose that there exists $x_0 \in X$ such that $\sup_{i \in I} f_i(x_0) < +\infty$. Then epi$(\sup_{i\in I} f_i)^* = cl_{w^*}\left(co\bigcup_{i\in I} epif_i^*\right)$, where $\sup_{i\in I} f_i : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i\in I} f_i)(x) = \sup_{i\in I} f_i(x)$ for all $x \in X$.*

Lemma 2.2. [20, Theorem 2.1] *Let $h : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Define the marginal function $\eta : X \to \mathbb{R} \cup \{\pm\infty\}$ by $\eta(x) = \inf_{y\in Y} h(x, y)$. Define $Pr_{X\times\mathbb{R}}(epih) = \{(x, r),\ \exists y \in Y,\ (x, y, r) \in epih\}$. Then, we have $Pr_{X\times\mathbb{R}}(epih) \subseteq epi\eta \subseteq cl_{w^*}\left(Pr_{X\times\mathbb{R}}(epih)\right)$.*

Lemma 2.3. [19, Proposition A.1] *Let X, Y, Z be Banach spaces and let $\mathcal{U}$ be a convex subset of Z. Let $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function for any $u \in \mathcal{U}$, and $u \mapsto \phi_u(x, y)$ be a concave function for any $(x, y) \in X \times Y$. Then, $\bigcup_{u\in\mathcal{U}} epi\ \phi_u^*$ and $Pr_{X^*\times\mathbb{R}}\left(\bigcup_{u\in\mathcal{U}} epi\ \phi_u^*\right)$ are convex.*

We closed this section by recalling a version of the Brøndsted-Rockafellar theorem which was established in [11].

Theorem 2.1. [25, 11, Brøndsted-Rockafellar Theorem] *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a proper convex and lower semicontinuous function and $a \in domf$. Then for every*

$\varepsilon > 0$ *and for every* $x^* \in \partial_\varepsilon f(a)$ *there exist* $x_\varepsilon \in \text{dom} f$ *and* $x_\varepsilon^* \in \partial f(x_\varepsilon)$ *such that*

$$\|x_\varepsilon - a\| \leq \sqrt{\varepsilon},\ \|x_\varepsilon^* - x^*\|_* \leq \sqrt{\varepsilon} \text{ and } |f(x_\varepsilon) - \langle x_\varepsilon^*, x_\varepsilon - a\rangle - f(a)| \leq 2\varepsilon.$$

3. Sequential optimality conditions of robust optimal solutions

The aim of this section is to derive sequential characterizations for robust optimal solution of the problem (UP_ϕ) in terms of the ε-subdifferentials and the subdifferentials of the functions involved. In what follows, we consider the marginal function $\eta : X^* \to \mathbb{R} \cup \{\pm\infty\}$ of the conjugate function $(\sup_{u\in\mathcal{U}} \phi_u)^*$ defined by

$$\eta(x^*) := \inf_{y^* \in Y^*} \left(\sup_{u\in\mathcal{U}} \phi_u\right)^* (x^*, y^*).$$

It is worth mentioning here, as $(\sup_{u\in\mathcal{U}} \phi_u)^*$ is a lower semicontinuous convex function [26, Theorem 2.3.1(i)], that the marginal function η is convex [26, Theorem 2.1.3(v)].

We begin by deriving the following technical result, which plays a key role in establishing sequential characterizations for robust optimal solution of the problem (UP_ϕ) later in the paper.

Lemma 3.4. *Let* $\mathcal{U}$ *be a subset of a Banach space* Z*. For any* $u \in \mathcal{U}$*, let* $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Suppose that* $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x, 0) < +\infty$*. Then, for each* $a \in dom(\sup_{u\in\mathcal{U}} \phi_u(\cdot, 0))$*,* a *is a robust optimal solution of* (UP_ϕ) *if and only if* $(0, -\eta^*(a)) \in cl_{w^*}(epi(\eta))$*.*

Proof. We see that $\sup_{u\in\mathcal{U}} \phi_u$ is a lower semicontinuous convex function on $X \times Y$ because ϕ_u is a lower semicontinuous convex function for each $u \in \mathcal{U}$. Note that in view of the weak robust duality [19, Theorem 3.1.], we may assume that $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x, 0) > -\infty$. Consequently, as $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x, 0) < +\infty$, $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x, 0) \in \mathbb{R}$. Note also that $\left(\sup_{u\in\mathcal{U}} \phi_u\right)^{**}$ is a proper function on $X \times Y$ (see, e.g. [26, Theorem 2.3.4]), which in turn implies that $\sup_{u\in\mathcal{U}} \phi_u$ is proper as well. So, $\sup_{u\in\mathcal{U}} \phi_u$ is a proper lower semicontinuous convex function on $X \times Y$, which results in $\sup_{u\in\mathcal{U}} \phi_u = \left(\sup_{u\in\mathcal{U}} \phi_u\right)^{**}$. Moreover, it then follows from the definition of η that

$$\begin{aligned}
\eta^*(x) &= \sup_{x^*\in X^*} \{\langle x^*, x\rangle - \eta(x^*)\} \\
&= \sup_{x^*\in X^*} \left\{\langle x^*, x\rangle - \inf_{y^*\in Y^*} \left(\sup_{u\in\mathcal{U}} \phi_u\right)^* (x^*, y^*)\right\} \\
&= \sup_{(x^*,y^*)\in X^*\times Y^*} \left\{\langle (x^*, y^*), (x, 0)\rangle - \left(\sup_{u\in\mathcal{U}} \phi_u\right)^* (x^*, y^*)\right\} \\
&= \left(\sup_{u\in\mathcal{U}} \phi_u\right)^{**} (x, 0) = \sup_{u\in\mathcal{U}} \phi_u(x, 0)
\end{aligned} \tag{3.2}$$

for all $x \in X$. By taking into account the definition of conjugate functions together with (3.2), one can see that $a \in \text{dom}(\sup_{u\in\mathcal{U}} \phi_u(\cdot, 0))$ is a robust optimal solution of (UP_ϕ) if and only if

$$(0, -\sup_{u\in\mathcal{U}} \phi_u(a, 0)) \in \text{epi}\Big(\sup_{u\in\mathcal{U}} \phi_u(\cdot, 0)\Big)^* = \text{epi}(\eta^*)^* = \text{epi}(\text{cl}_{\omega^*} \text{co}\, \eta) = \text{cl}_{w^*}(\text{epi}(\eta)),$$

where $\text{cl}_{\omega^*} \eta$ stands for the lower semicontinuous hull of η (the closure of its epigraph is taken in weak* topology). So, we obtain the desired result.

□

With the aid of Lemma 3.4, we now establish sequential characterizations of robust optimal solution for (UP_ϕ) in terms of ε-subdifferentials.

Theorem 3.2. *Let $\mathcal{U}$ be a convex subset of a Banach space Z, $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function for any $u \in \mathcal{U}$. Suppose that $u \mapsto \phi_u(x, y)$ is a concave function for any $(x, y) \in X \times Y$ and $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x, 0) < +\infty$. Then, for each $a \in dom(\sup_{u\in\mathcal{U}} \phi_u(\cdot, 0))$, the following assertions are equivalent:*

(i) *a is a robust optimal solution of (UP_ϕ);*

(ii) *there exist $\{u_n\} \subseteq \mathcal{U}$, $\varepsilon_n \geq 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} \phi_{u_n}(a, 0)$ such that*

$$x_n^* \xrightarrow{\|\cdot\|_*} 0,\ \varepsilon_n \to 0 \text{ and } \phi_{u_n}(a, 0) \to \sup_{u\in\mathcal{U}} \phi_u(a, 0)\ (n \to +\infty);$$

(iii) *there exist $\{u_n\} \subseteq \mathcal{U}$, $\varepsilon_n \geq 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} \phi_{u_n}(a, 0)$ such that*

$$x_n^* \xrightarrow{\omega^*} 0,\ \varepsilon_n \to 0 \text{ and } \phi_{u_n}(a, 0) \to \sup_{u\in\mathcal{U}} \phi_u(a, 0)\ (n \to +\infty).$$

Proof. [(i)⇒(ii)]. Suppose that (i) holds. According to Lemma 3.4, we have that $\eta^*(x) = \sup_{u\in\mathcal{U}} \phi_u(x)$, $\forall x \in X$ and $(0, -\eta^*(a)) \in \mathrm{cl}_{w^*}(\mathrm{epi}(\eta))$. It then follows from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned}
(3.3) \qquad (0, -\eta^*(a)) &\in \mathrm{cl}_{w^*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}\left(\mathrm{epi}\ (\sup_{u\in\mathcal{U}} \phi_u)^*\right)\right) \\
&= \mathrm{cl}_{w^*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}\left(\mathrm{cl}_{w^*}\mathrm{co}\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*\right)\right) \\
&= \mathrm{cl}_{w^*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}\left(\mathrm{cl}_{w^*}\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*\right)\right) \\
&\subseteq \mathrm{cl}_{w^*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}\left(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*\right)\right).
\end{aligned}$$

In addition, since the set $\mathrm{Pr}_{X^*\times\mathbb{R}}(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*)$ is a convex set and X is a reflexive Banach space, we assert that $\mathrm{cl}_{w^*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*)\right) = \mathrm{cl}_{\|\cdot\|_*}\left(\mathrm{Pr}_{X^*\times\mathbb{R}}(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\ \phi_u^*)\right)$. This together with (3.3) and Lemma 2.1 in turn gives us that there exist $x_n^* \in X^*$, $y_n^* \in Y^*$, $r_n \in \mathbb{R}$ and $u_n \in \mathcal{U}$ such that $(x_n^*, y_n^*, r_n) \in \mathrm{epi}\phi_{u_n}^*$, $x_n^* \xrightarrow{\|\cdot\|_*} 0$ and $r_n \to -\eta^*(a)$ $(n \to +\infty)$. As $(a, 0) \in \mathrm{dom}(\sup_{u\in\mathcal{U}} \phi_u)$, we also have $(a, 0) \in \mathrm{dom}(\phi_u)$ for all $u \in \mathcal{U}$. For each $u \in \mathcal{U}$, by virtue of (2.1), we obtain that

$$(3.4) \qquad \mathrm{epi}\ \phi_u^* = \bigcup_{\varepsilon\geq 0} \{(x^*, y^*, \langle x^*, a\rangle + \varepsilon - \phi(a, 0)) : (x^*, y^*) \in \partial_\varepsilon \phi_u(a, 0)\}.$$

In view of (3.4) and the fact that $(x_n^*, y_n^*, r_n) \in \mathrm{epi}\phi_{u_n}^*$ for each positive integer n, there exists $\varepsilon_n \geq 0$ such that $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} \phi_{u_n}(a, 0)$, $r_n = \langle x_n^*, a\rangle + \varepsilon_n - \phi_{u_n}(a, 0)$, $x_n^* \xrightarrow{\|\cdot\|_*} 0$ and $r_n \to -\eta^*(a)$ $(n \to +\infty)$. So, for each $n \in \mathbb{N}$, $0 \leq \varepsilon_n = \phi_{u_n}(a, 0) - \langle x_n^*, a\rangle + r_n \leq -\langle x_n^*, a\rangle + \sup_{u\in\mathcal{U}} \phi_u(a, 0) + r_n$. This give us, passing to the limit as $n \to +\infty$, $\varepsilon_n \to 0$. Moreover, for each $n \in \mathbb{N}$, it follows from this inequality $0 \leq |\phi_{u_n}(a, 0) - \sup_{u\in\mathcal{U}} \phi_u(a, 0)| = |\langle x_n^*, a\rangle - r_n + \varepsilon_n - \sup_{u\in\mathcal{U}} \phi_u(a, 0)| \leq |\langle x_n^*, a\rangle| + |-r_n - \sup_{u\in\mathcal{U}} \phi_u(a, 0)| + \varepsilon_n$, we also have $\phi_{u_n}(a, 0) \to \sup_{u\in\mathcal{U}} \phi_u(a, 0)$ $(n \to +\infty)$. Therefore, the implication (i)⇒(ii) holds.

Clearly, (ii) implies (iii) because of $x_n^* \xrightarrow{\|\cdot\|_*} 0$ $(n \to +\infty)$ implies $x_n^* \xrightarrow{\omega^*} 0$ $(n \to +\infty)$.

[(iii)⇒(i)]. Suppose that there exist $\{u_n\} \subseteq \mathcal{U}$, $\varepsilon_n \geq 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} \phi_{u_n}(a, 0)$ such that $x_n^* \xrightarrow{\omega^*} 0$, $\phi_{u_n}(a, 0) \to \sup_{u\in\mathcal{U}} \phi_u(a, 0)$ and $\varepsilon_n \to 0$ $(n \to +\infty)$. For each $n \in \mathbb{N}$, by taking into account the definitions of the ε-subdifferential, we have $\phi_{u_n}(x,y)-\phi_{u_n}(a,0) \geq \langle x_n^*, x-a\rangle + \langle y_n^*, y\rangle - \varepsilon_n,\ \forall (x,y) \in X \times Y$, inasmuch as $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi_{u_n}(a,0)$, and further implies that $\sup_{u\in\mathcal{U}} \phi_u(x,0) - \phi_{u_n}(a,0) \geq \phi_{u_n}(x,0) - \phi_{u_n}(a,0) \geq \langle x_n^*, x-a\rangle - \varepsilon_n,\ \forall x \in X$. Passing to the limit as $n \to +\infty$, we get that $\sup_{u\in\mathcal{U}} \phi_u(x,0) - \sup_{u\in\mathcal{U}} \phi_u(a,0) \geq 0,\ \forall x \in X$, which actually means that a is a robust optimal solution of (UP_ϕ). So, (i) holds, which finishes the proof of the theorem. □

With the help of the Brøndsted-Rockafellar theorem (Theorem 2.1), we see now how the sequential characterization of robust optimal solution for (UP_ϕ) can be obtained in terms of the classical convex subdifferentials.

Theorem 3.3. *Let $\mathcal{U}$ be a convex subset of a Banach space Z, $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function for any $u \in \mathcal{U}$. Suppose that $u \mapsto \phi_u(x,y)$ be a concave function for any $(x,y) \in X \times Y$ and $\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x,0) < +\infty$. Then, for each $a \in dom(\sup_{u\in\mathcal{U}} \phi_u(\cdot,0))$, the following assertions are equivalent:*

(i) *a is a robust optimal solution of (UP_ϕ);*

(ii) *there exist $\{u_n\} \subseteq \mathcal{U}$, $(x_n, y_n) \in dom\ \phi_{u_n}$ and $(x_n^*, y_n^*) \in \partial\phi_{u_n}(x_n, y_n)$ such that*

$$\begin{cases} x_n^* \xrightarrow{\|\cdot\|_*} 0,\ x_n \to a,\ y_n \to 0\ (n \to +\infty) \text{ and} \\ \phi_{u_n}(x_n,y_n) - \langle y_n^*, y_n\rangle - \sup_{u\in\mathcal{U}} \phi_u(a,0) \to 0\ (n \to +\infty); \end{cases}$$

(iii) *there exist $\{u_n\} \subseteq \mathcal{U}$, $(x_n, y_n) \in dom\ \phi_{u_n}$ and $(x_n^*, y_n^*) \in \partial\phi_{u_n}(x_n, y_n)$ such that*

$$\begin{cases} x_n^* \xrightarrow{\omega^*} 0,\ x_n \to a,\ y_n \to 0\ (n \to +\infty) \text{ and} \\ \phi_{u_n}(x_n,y_n) - \langle y_n^*, y_n\rangle - \sup_{u\in\mathcal{U}} \phi_u(a,0) \to 0\ (n \to +\infty). \end{cases}$$

Proof. As (ii)⇒(iii) is always true, we prove just the implications (i)⇒(ii) and (iii)⇒(i).

[(i)⇒(ii)]. Suppose that (i) holds. Invoking Theorem 3.2 we find $\{u_n\} \subseteq \mathcal{U}$, $\varepsilon_n \geq 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi_{u_n}(a,0)$ such that $x_n^* \xrightarrow{\|\cdot\|_*} 0$, $\varepsilon_n \to 0$ and $\phi_{u_n}(a,0) \to \sup_{u\in\mathcal{U}} \phi_u(a,0)$ $(n \to +\infty)$. Applying Brøndsted-Rockafellar theorem allows us that for each positive integer n, there exist $(x_n, y_n) \in \text{dom}\ \phi_{u_n}$ and $(\overline{x_n^*}, \overline{y_n^*}) \in \partial\phi_{u_n}(x_n, y_n)$ such that $\|(x_n,y_n)-(a,0)\| \leq \sqrt{\varepsilon_n}$, $\|(x_n^*, y_n^*) - (\overline{x_n^*}, \overline{y_n^*})\|_* \leq \sqrt{\varepsilon_n}$ and $|\phi_{u_n}(x_n,y_n) - \langle(\overline{x_n^*}, \overline{y_n^*}), (x_n,y_n)-(a,0)\rangle - \phi_{u_n}(a,0)| \leq 2\varepsilon_n$, from which we obtain $\overline{x_n^*} - x_n^* \xrightarrow{\|\cdot\|_*} 0$, $x_n \to a$, $y_n \to 0$ $(n \to +\infty)$ and $\phi_{u_n}(x_n,y_n) - \langle\overline{x_n^*}, x_n - a\rangle - \langle\overline{y_n^*}, y_n\rangle - \phi_{u_n}(a,0) \to 0$ $(n \to +\infty)$. We also have $\overline{x_n^*} = (\overline{x_n^*} - x_n^*) + x_n^* \xrightarrow{\|\cdot\|_*} 0$ $(n \to +\infty)$ and hence $\langle\overline{x_n^*}, x_n - a\rangle \to 0$ $(n \to +\infty)$. Then, we arrive at the assertion that $\phi_{u_n}(x_n,y_n) - \langle\overline{y_n^*}, y_n\rangle - \sup_{u\in\mathcal{U}} \phi_u(a,0) = [\phi_{u_n}(x_n,y_n) - \langle\overline{x_n^*}, x_n - a\rangle - \langle\overline{y_n^*}, y_n\rangle - \phi_{u_n}(a,0)] + \langle\overline{x_n^*}, x_n - a\rangle + [\phi_{u_n}(a,0) - \sup_{u\in\mathcal{U}} \phi_u(a,0)] \longrightarrow 0$ $(n \to +\infty)$. With the notations $x_n^* := \overline{x_n^*}$ and $y_n^* := \overline{y_n^*}$, the desired result follows.

[(iii)⇒(i)]. Assume that there exist $\{u_n\} \subseteq \mathcal{U}$, $(x_n, y_n) \in \text{dom}\,\phi_{u_n}$ and $(x_n^*, y_n^*) \in \partial\phi_{u_n}(x_n, y_n)$ such that $x_n^* \xrightarrow{\omega^*} 0$, $x_n \to a$, $y_n \to 0$ $(n \to +\infty)$ and $\phi_{u_n}(x_n,y_n) - \langle y_n^*, y_n\rangle - \sup_{u\in\mathcal{U}} \phi_u(a,0) \to 0$ $(n \to +\infty)$. For each $n \in \mathbb{N}$, we have $\phi_{u_n}(x,y) \geq \phi_{u_n}(x_n,y_n) + \langle(x_n^*, y_n^*), (x - x_n, y - y_n)\rangle$, $\forall (x,y) \in X \times Y$ due to $(x_n^*, y_n^*) \in \partial\phi_{u_n}(x_n, y_n)$. In particular, it holds $\sup_{u\in\mathcal{U}} \phi_u(x,0) - \sup_{u\in\mathcal{U}} \phi_u(a,0) \geq \phi_{u_n}(x,0) - \sup_{u\in\mathcal{U}} \phi_u(a,0) \geq \phi_{u_n}(x_n,y_n) - \langle y_n^*, y_n\rangle - \sup_{u\in\mathcal{U}} \phi_u(a,0) + \langle x_n^*, x - x_n\rangle$ $\forall x \in X$. Passing to the limit as $n \to +\infty$, we get $\sup_{u\in\mathcal{U}} \phi_u(x,0) - \sup_{u\in\mathcal{U}} \phi_u(a,0) \geq 0, \forall x \in X$, so a is a minimizer of $\sup_{u\in\mathcal{U}} \phi_u(\cdot,0)$ on X, and the conclusion follows. □

Remark 3.1. In the special case when $\mathcal{U}$ is a singleton and $\phi_u \equiv \phi$, the condition that $u \mapsto \phi_u(x,y)$ is a concave function for any $(x,y) \in X \times Y$ is automatically fulfilled, and so, Theorem 3.2 and Theorem 3.3 have been investigated in [9, Theorem 3.2 and Theorem 3.3].

Remark 3.2. Let us look now at the case where the function $u \mapsto \phi_u(x,y)$ is upper semicontinuous for any $(x,y) \in X \times Y$ and $\mathcal{U}$ is a compact set. We see now how the implication (i)⇒(iii) in Theorem 3.3 will follow by applying [9, Theorem 3.3] under the same hypotheses as in Theorem 3.3. To see this, by taking

$$\phi(x,y) := \sup_{u\in\mathcal{U}} \phi_u(x,y)$$

for all $(x,y) \in X \times Y$, we first show that the set $\cup_{u\in\mathcal{U}}\text{epi}\phi_u^*$ is w^*-closed. Indeed, let $(x_n^*, y_n^*, r_n) \in \cup_{u\in\mathcal{U}}\text{epi}\phi_u^*$ be a sequence such that $(x_n^*, y_n^*, r_n) \to \omega^*(x^*, y^*, r)$ $(n \to +\infty)$ for some $(x^*, y^*, r) \in X^* \times Y^* \times \mathbb{R}$. Then, there exists $u_n \in \mathcal{U}$ such that $(x_n^*, y_n^*, r_n) \in \text{epi}\phi_{u_n}^*$. As $\mathcal{U}$ is compact, by passing to a subsequence if necessary, we may assume that $u_n \to u$ $(n \to +\infty)$ for some $u \in \mathcal{U}$. As for each $(x,y) \in X \times Y$, $\langle (x_n^*, y_n^*), (x,y)\rangle - \phi_{u_n}(x,y) \le r_n$, passing to the upper limit, we have for each $(x,y) \in X \times Y$, $\langle (x^*, y^*), (x,y)\rangle - \phi_u(x,y) \le r$ which means that $\phi_u^*(x^*, y^*) \le r$. So, $(x^*, y^*, r) \in \cup_{u\in\mathcal{U}}\text{epi}\phi_u^*$, thereby leading to the assertion that the set $\cup_{u\in\mathcal{U}}\text{epi}\phi_u^*$ is w^*-closed.

Now, we apply [9, Theorem 3.3] to assert that $a \in \text{dom}(\sup_{u\in\mathcal{U}} \phi_u(\cdot,0))$ is a minimizer of $\sup_{u\in\mathcal{U}} \phi_u(\cdot,0)$ on X if and only if there exist sequences $(x_n, y_n) \in \text{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that $x_n^* \to 0$, $x_n \to a$, $y_n \to 0$ and $\phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a,0) \to 0$ $(n \to +\infty)$. To prove the condition (iii) in Theorem 3.3, it remains to show that $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi_{u_n}(x_n, y_n)$ and $\phi(x_n, y_n) = \phi_{u_n}(x_n, y_n)$ for each $n \in \mathbb{N}$. Now, taking (2.1) into account, we see that for each $n \in \mathbb{N}$ the assertion $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ guarantees that

$$((x_n^*, y_n^*), \langle (x_n^*, y_n^*), (x_n, y_n)\rangle - \phi(x_n, y_n)) \in \text{epi}\phi^* = \text{cl}_{w^*}\text{co}(\cup_{u\in\mathcal{U}}\text{epi}\phi_u^*). \tag{3.5}$$

Keeping in mind the fact that the set $\cup_{u\in\mathcal{U}}\text{epi}\phi_u^*$ in this setting is w^*-closed and convex, the relation (3.5) gives us that there exists $u_n \in \mathcal{U}$ such that

$$((x_n^*, y_n^*), \langle (x_n^*, y_n^*), (x_n, y_n)\rangle - \phi(x_n, y_n)) \in \text{epi}\phi_{u_n}^*,$$

which further implies that there exist $\varepsilon_n \ge 0$ and $(\overline{x_n^*}, \overline{y_n^*}) \in \partial_{\varepsilon_n}\phi_{u_n}(x_n, y_n)$ such that

$$((x_n^*, y_n^*), \langle (x_n^*, y_n^*), (x_n, y_n)\rangle - \phi(x_n, y_n)) = ((\overline{x_n^*}, \overline{y_n^*}), \langle (\overline{x_n^*}, \overline{y_n^*}), (x_n, y_n)\rangle + \varepsilon_n - \phi_{u_n}(x_n, y_n)).$$

We conclude that $x_n^* = \overline{x_n^*}$, $y_n^* = \overline{y_n^*}$ and $-\phi(x_n, y_n) = \varepsilon_n - \phi_{u_n}(x_n, y_n)$. The later reduces to the following one $\varepsilon_n \le 0$ due to the definition of ϕ, and so, $\varepsilon_n = 0$. Consequently, $\phi(x_n, y_n) = \phi_{u_n}(x_n, y_n)$, thereby establishing the desired result.

4. The problem with geometric and cone constraints under data uncertainty

In this section, we consider the convex optimization problem with cone constraints in the face of data uncertainty both in the objective and constraints (UP) that given as in Section 1. We suppose in addition that $\text{dom}(\sup_{u\in\mathcal{U}} f(\cdot,u)) \cap A \neq \emptyset$ where $A := \{x \in C : g(x,v) \in -K,\ \forall v \in \mathcal{V}\}$. From now on, we are going to derive a sequential form of the Lagrange multiplier condition for a robust optimal solution of (UP) by applying Theorem 3.3 to the following perturbation function $\phi_{(u,v)} : X \times X \times Y \to \mathbb{R} \cup \{+\infty\}$, $(u,v) \in \mathcal{U} \times \mathcal{V}$,

$$\phi_{(u,v)}(x,p,q) = \begin{cases} f(x,u), & \text{if } x+p \in C \text{ and } g(x,v) - q \in -K. \\ +\infty, & \text{otherwise.} \end{cases}$$

We notice that for each $(u,v) \in \mathcal{U} \times \mathcal{V}$, $\phi_{(u,v)}$ is a proper lower semicontinuous convex function, $(u,v) \mapsto \phi_{(u,v)}(x,p,q)$ is a concave function for any $(x,p,q) \in X \times X \times Y$, and $\inf_{x\in X} \sup_{(u,v)\in\mathcal{U}\times\mathcal{V}} \phi_{(u,v)}(x,0,0) < +\infty$. Moreover, the conjugate of $\phi_{(u,v)}$ is $\phi_{(u,v)}^* : X^* \times X^* \times Y^* \to \mathbb{R} \cup \{+\infty\}$,

$$\phi_{(u,v)}^*(x^*, p^*, q^*) = \begin{cases} \delta_C^*(p^*) + (f(\cdot,u) + (-q^*g)(\cdot,v))^*(x^* - p^*), & \text{if } q^* \in -K^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

as a direct calculation shows (see, e.g. [8, p. 1019]), where, for simplicity, $(q^*g)(\cdot,v) := \langle q^*, g(\cdot,v)\rangle$ for all $q^* \in Y^*$ and $v \in \mathcal{V}$.

Theorem 4.4. *For each $a \in dom(\sup_{u\in\mathcal{U}} f(\cdot,u)) \cap A$, a is a robust optimal solution of the problem* (UP) *if and only if*

$$\begin{cases} \exists(u_n,v_n)\in\mathcal{U}\times\mathcal{V},\ \exists(x_n,\omega_n,t_n)\in dom f(\cdot,u_n)\times C\times(-K),\\ \exists(u_n^*,v_n^*,\omega_n^*,\lambda_n)\in X^*\times X^*\times X^*\times K^*,\ u_n^*\in\partial f(\cdot,u_n)(x_n),\\ v_n^*\in\partial((\lambda_n g)(\cdot,v_n))(x_n),\ \omega_n^*\in N_C(\omega_n),\ \langle\lambda_n,t_n\rangle=0,\ \forall n\in\mathbb{N},\\ u_n^*+v_n^*+\omega_n^*\to 0,\ \omega_n\to a,\ x_n\to a,\ t_n-g(x_n,v_n)\to 0\ (n\to+\infty)\ \textit{and}\\ f(x_n,u_n)-\sup_{u\in\mathcal{U}} f(a,u)+(\lambda_n g)(x_n,v_n)-\langle\omega_n^*,\omega_n-x_n\rangle\to 0\ (n\to+\infty). \end{cases}$$

Proof. In view of Theorem 3.3, $a \in \mathrm{dom}(\sup_{u\in\mathcal{U}} f(\cdot,u)) \cap A$ is a robust optimal solution of (UP) if and only if there exist $(u_n,v_n)\in\mathcal{U}\times\mathcal{V}$, $(x_n,p_n,q_n)\in \mathrm{dom}\phi_{(u_n,v_n)}$ and $(x_n^*,p_n^*,q_n^*)\in\partial\phi_{(u_n,v_n)}(x_n,p_n,q_n)$ such that $x_n^*\to 0,\ x_n\to a,\ (p_n,q_n)\to(0,0)\ (n\to+\infty)$, $\phi_{(u_n,v_n)}(x_n,p_n,q_n)-\langle(p_n^*,q_n^*),(p_n,q_n)\rangle-\sup_{(u,v)\in\mathcal{U}\times\mathcal{V}}\phi_{(u,v)}(a,0,0)\to 0\ (n\to+\infty)$. It follows from this relation $(x_n,p_n,q_n)\in\mathrm{dom}\phi_{(u_n,v_n)}$, we get that $x_n\in\mathrm{dom}f(\cdot,u_n)$, $x_n+p_n\in C$ and $g(x_n,v_n)-q_n\in -K$. In this way, we have $(x_n^*,p_n^*,q_n^*)\in\partial\phi_{u_n}(x_n,p_n,q_n)$ if and only if $\phi_{(u_n,v_n)}(x_n,p_n,q_n)+\phi^*_{(u_n,v_n)}(x_n^*,p_n^*,q_n^*)=\langle x_n^*,x_n\rangle+\langle p_n^*,p_n\rangle+\langle q_n^*,q_n\rangle \Leftrightarrow f(x_n,u_n)+\delta_C^*(p_n^*)+(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))^*(x_n^*-p_n^*)=\langle x_n^*,x_n\rangle+\langle p_n^*,p_n\rangle+\langle q_n^*,q_n\rangle$, which is equivalent to $(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))^*(x_n^*-p_n^*)+(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))(x_n)-\langle x_n^*-p_n^*,x_n\rangle+\langle -q_n^*,q_n-g(x_n,v_n)\rangle+\delta_C^*(p_n^*)-\langle p_n^*,x_n+p_n\rangle=0,\ \forall n\in\mathbb{N}$. As $q_n-g(x_n,v_n)\in K$ and $-q_n^*\in K^*$, we have $\langle -q_n^*,q_n-g(x_n,v_n)\rangle\geq 0,\ \forall n\in\mathbb{N}$. Also the Young-Fenchel's inequality yields $(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))^*(x_n^*-p_n^*)+(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))(x_n)-\langle x_n^*-p_n^*,x_n\rangle\geq 0$ and $\delta_C^*(p_n^*)-\langle p_n^*,x_n+p_n\rangle\geq 0$. Therefore, we assert that $(x_n^*,p_n^*,q_n^*)\in\partial\phi_{(u_n,v_n)}(x_n,p_n,q_n)\Leftrightarrow x_n^*-p_n^*\in\partial(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))(x_n),\ p_n^*\in\partial\delta_C(x_n+p_n)=N_C(x_n+p_n)$ and $\langle -q_n^*,q_n-g(x_n,v_n)\rangle=0,\ \forall n\in\mathbb{N}$.

On the one hand, $\phi_{(u_n,v_n)}(x_n,p_n,q_n)-\langle(p_n^*,q_n^*),(p_n,q_n)\rangle-\sup_{(u,v)\in\mathcal{U}\times\mathcal{V}}\phi_{(u,v)}(a,0,0)\to 0\ (n\to+\infty)$ is equivalent to the assertion $f(x_n,u_n)-\langle p_n^*,p_n\rangle+\langle -q_n^*,q_n\rangle-\sup_{u\in\mathcal{U}}f(a,u)\to 0\ (n\to+\infty)$. As a consequence, we obtain that $a\in\mathrm{dom}(\sup_{u\in\mathcal{U}}f(\cdot,u))\cap A$ is a robust optimal solution of (UP) if and only if

$$\begin{cases} \exists(u_n,v_n)\in\mathcal{U}\times\mathcal{V},\ (x_n,p_n,q_n)\in\mathrm{dom}f(\cdot,u_n)\times X\times Y,\\ x_n+p_n\in C,\ g(x_n,v_n)-q_n\in -K,\\ \exists(x_n^*,p_n^*,q_n^*)\in X^*\times X^*\times -K^*\ \text{such that}\\ x_n^*-p_n^*\in\partial(f(\cdot,u_n)+(-q_n^*g)(\cdot,v_n))(x_n),\ p_n^*\in N_C(x_n+p_n),\\ \langle -q_n^*,q_n-g(x_n,v_n)\rangle=0,\ \forall n\in\mathbb{N},\\ x_n^*\to 0,\ x_n\to a,\ p_n\to 0,\ q_n\to 0\ (n\to+\infty),\\ f(x_n,u_n)-\sup_{u\in\mathcal{U}}f(a,u)+\langle -q_n^*,q_n\rangle-\langle p_n^*,p_n\rangle\to 0\ (n\to+\infty). \end{cases} \tag{4.6}$$

With the following notations: $t_n:=g(x_n,v_n)-q_n$, $\omega_n:=p_n+x_n$, $\lambda_n:=-q_n^*$, $\overline{u_n^*}:=x_n^*-p_n^*$ and $\omega_n^*:=p_n^*$, $\forall n\in\mathbb{N}$, the condition (4.6) can be equivalently written as follows

$$\begin{cases} \exists(u_n,v_n)\in\mathcal{U}\times\mathcal{V},\ \exists(x_n,\omega_n,t_n)\in\mathrm{dom}f(\cdot,u_n)\times C\times(-K),\\ \exists(\overline{u_n^*},\omega_n^*,\lambda_n)\in X^*\times X^*\times K^*,\\ \overline{u_n^*}\in\partial(f(\cdot,u_n)+(\lambda_n g)(\cdot,v_n))(x_n),\ \omega_n^*\in N_C(\omega_n),\ \langle\lambda_n,t_n\rangle=0,\ \forall n\in\mathbb{N},\\ \overline{u_n^*}+\omega_n^*\to 0,\ \omega_n\to a,\ x_n\to a,\ t_n-g(x_n,v_n)\to 0\ (n\to+\infty)\ \text{and}\\ f(x_n,u_n)-\sup_{u\in\mathcal{U}}f(a,u)+(\lambda_n g)(x_n,v_n)-\langle\omega_n^*,\omega_n-x_n\rangle\to 0\ (n\to+\infty). \end{cases}$$

In addition, for each $n\in\mathbb{N}$, by the continuity of $g(\cdot,v_n)$, the subdifferential sum formula (see, e.g. [26, Theorem 2.8.7]) ensures that $\partial(f(\cdot,u_n)+(\lambda_n g)(\cdot,v_n))(x_n)=\partial f(\cdot,u_n)(x_n)+\partial((\lambda_n g)(\cdot,v_n))(x_n)$. This together with the fact that $\overline{u_n^*}\in\partial(f(\cdot,u_n)+(\lambda_n g)(\cdot,v_n))(x_n)$ in turn gives us the existence of other two sequences $u_n^*,v_n^*\in X$ such that $\overline{u_n^*}=u_n^*+v_n^*$, $u_n^*\in\partial f(\cdot,u_n)(x_n)$, and $v_n^*\in\partial((\lambda_n g)(\cdot,v_n))(x_n)$, thereby leading to the desired result. □

We close this paper with a simple example which illustrates Theorem 4.4 where the involved uncertainty sets are not compact and the standard robust type Lagrange multipliers fail to hold.

Example 4.1. Let $X := \mathbb{R}$, $Y := \mathbb{R}^2, Z := \mathbb{R}$, $K := \mathbb{R}^2_+$, $C := \mathbb{R}$, $\mathcal{U} := [1,2)$, $\mathcal{V} := (0,1]$, $f(\cdot,u) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, for each $u \in \mathcal{U}$, be defined by $f(x,u) = -u\sqrt{x} + \delta_{\mathbb{R}_+}(x)$, $\forall x \in \mathbb{R}$, and $g(\cdot,v) : \mathbb{R} \to \mathbb{R}^2$, for each $v \in \mathcal{V}$, be defined by $g(x,v) := (-1 - vx, vx)^T$, $\forall x \in \mathbb{R}$. Then, for each $u \in \mathcal{U}$, $f(\cdot,u)$ is a proper, convex and lower semicontinuous function, $g(\cdot,v)$ is K-convex and continuous function for each $v \in \mathcal{V}$, and $g(x,\cdot)$ is a K-concave function for any $x \in \mathbb{R}$. Moreover, $\sup_{u\in\mathcal{U}} f(x,u) = -\sqrt{x} + \delta_{\mathbb{R}_+}(x)$ for all $x \in \mathbb{R}$, and so $\text{dom}(\sup_{u\in\mathcal{U}} f(\cdot,u)) \cap A = [0,+\infty) \cap [-1,0] = \{0\} \neq \emptyset$. The element $a := 0$ is the (unique) minimizer of the problem (RP). Since $\partial(\sup_{u\in\mathcal{U}} f(\cdot,u))(0) = \emptyset$, we can not find $\overline{v} \in \mathcal{V}$ and $\overline{\lambda} \in K^*$ such that $0 \in \partial(\sup_{u\in\mathcal{U}} f(\cdot,u))(0) + \delta_C(0) + \partial((\overline{\lambda}g)(\cdot,\overline{v}))(0)$ and $(\overline{\lambda}g)(0,\overline{v}) = 0$. Nevertheless, as we show in the following, the sequential optimality conditions in Theorem 4.4 are satisfied. To do this, it is enough to take $u_n := 1$, $v_n := 1$, $x_n := \frac{1}{n}$, $\omega_n := 0$, $t_n := (\frac{1}{n} - 1, 0)^T$, $u_n^* := -\frac{\sqrt{n}}{2}$, $v_n^* := \frac{\sqrt{n}}{2}$, $\omega_n^* := 0$ and $\lambda_n := (0, \frac{\sqrt{n}}{2})^T$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $-\frac{\sqrt{n}}{2} \in \partial f(\cdot,u_n)(\frac{1}{n})$, $\frac{\sqrt{n}}{2} \in \partial((\lambda_n g)(\cdot,v_n))(\frac{1}{n})$, $0 \in N_{\mathbb{R}}(0)$ and $\langle \lambda_n, t_n \rangle = 0$. Further, $u_n^* + v_n^* + \omega_n^* = 0$, $x_n \to 0$, $t_n - g(x_n,v_n) = (\frac{2}{n}, \frac{1}{n})^T \to (0,0)^T$, and $f(x_n,u_n) - \sup_{u\in\mathcal{U}} f(0,u) + (\lambda_n g)(x_n,v_n) - \langle \omega_n^*, \omega_n - x_n \rangle = -\frac{1}{2\sqrt{n}} \to 0$ $(n \to +\infty)$.

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