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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} *anniversary*

Intersection theorems with applications in set-valued equilibrium problems and minimax theory

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ABSTRACT. In this paper, we obtain three intersection theorems that can be considered versions of Theorem 3.1 from the paper [Agarwal, R. P., Balaj, M. and O'Regan, D., *Intersection theorems with applications in optimization*, J. Optim. Theory Appl., **179** (2018), 761–777]. As will be seen, there are two major differences between the hypotheses of the above mentioned theorem and those of our results. Applications of the main results are considered in the last two sections of the paper.

1. INTRODUCTION

An intersection theorem establishes sufficient conditions such that for a given set-valued mapping S between two sets X and Y, endowed with an adequate topological and/or algebraic structure, the intersection $\bigcap_{x\in X} S(x)$ to be nonempty. Intersection theorems are powerful tools for proving existence results in mathematics. For example, existence problems in optimization and nonlinear functional analysis can be solved using an intersection theorem, and regarding an intersection point as a fixed point, a coincidence point, an equilibrium point, a saddle point etc.

If X is a subset of a vector space E, a set-valued mapping S from X into E is called a KKM mapping, if for any finite subset A of X, $\operatorname{conv} A \subseteq S(A)$ (here, the standard notation $\operatorname{conv} A$ designates the convex hull of A). In 1961, Ky Fan [12] extended the famous Knaster-Kuratowski-Mazurkiewicz (simply, KKM) principle to arbitrary topological vector spaces obtaining a remarkable intersection theorem. Fan's result, known today as the Fan-KKM theorem, states that if E is a Hausdorff topological vector space and $S : X \subseteq E \rightrightarrows E$ is a closed-valued KKM mapping, such that $S(x_0)$ is compact for at least one $x_0 \in X$, then $\bigcap_{x\in X} S(x) \neq \emptyset$. The importance of the Fan-KKM theorem is due to its wide range of applications in nonlinear analysis, optimization and other fields of mathematics. This fact motivated Sehie Park [23] to introduce the concept of KKM theory, in which, the notion of KKM mapping plays a central role. Subsequently, there have been introduced concepts more general than that of KKM mapping, by means of which significant generalizations of some results from nonlinear analysis and optimization were obtained.

We need to recall below one of these new concepts.

Definition 1.1. (see [7], [1]) Let X be a convex set in a vector space, Y be a nonempty set and $S, T : X \rightrightarrows Y$ two set-valued mappings. We say that S is a weak KKM mapping w.r.t. T if for each nonempty finite subset A of X and any $x \in \operatorname{conv} A$, $T(x) \cap S(A) \neq \emptyset$.

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The starting point in our investigations is a recent intersection theorem obtained by Agarwal et al. in [1]. This theorem reads as follows:

Theorem 1.1. *Let X be a nonempty compact and convex set and Z be a nonempty convex set, each in a topological vector space. Let $S, T : X \rightrightarrows Z$ be nonempty-valued set-valued mappings that satisfy the following conditions:*

(i) *S is a weak KKM mapping w.r.t. T;*
(ii) *S is closed set-valued mapping with convex values and convex cofibers;*
(iii) *T has compact convex values;*
(iv) *for each $x \in X$, the set $\{y \in X : T(y) \cap S(x) \neq \emptyset\}$ is closed.*

Then, there exists an $x_0 \in X$ such that $T(x_0) \cap \bigcap_{x\in X} S(x) \neq \emptyset$.

Our aim in this paper is to establish three versions of the above theorem, in which the domains of the set-valued mappings S and T are distinct convex sets and instead of condition (i), T is required to be a convex set-valued mapping.

The paper is organized as follows. In Section 2, are recalled some basic definitions about set-valued mappings. The promised generalizations of Theorem 1.1 are established in Section 3. Applications of the main results are considered in the last two sections.

2. Basic concepts

We recall in this section some notions and results concerning set-valued mappings, needed in the paper. Given set-valued mapping $S : X \rightrightarrows Y$, we denote by $\operatorname{Gr} S$ its graph, that is, $\operatorname{Gr} S = \{(x, y) \in X \times Y : y \in S(x)\}$ and by S^* the set-valued mapping $S^* : Y \rightrightarrows X$ defined by $S^*(y) = \{x \in X : y \notin S(x)\}$. S^* is called the dual of S and its values are called the cofibers of S. The following lemma is a particular case of Proposition 3 in [9].

Lemma 2.1. *Let X be a nonempty convex set in a vector space and Y be a nonempty set. A set-valued mapping $S : X \rightrightarrows Y$ has convex cofibers if and only if $S(\mathrm{conv}\, A) \subseteq S(A)$, for each nonempty finite subset A of X (that is, S is a KKM mapping w.r.t. itself).*

Let X be a convex subset of a vector space and E be a vector space. A set-valued mapping $T : X \rightrightarrows E$ is said to be convex if

$$\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda)x_2),$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. In other words, T is convex if its graph is a convex set. It is easy to prove that a convex set-valued mapping is convex-valued.

If X and Y are topological spaces, a set-valued mapping $T : X \rightrightarrows Y$ is said to be: (i) lower semicontinuous, if for any open subset G of Y the set $\{x \in X : T(x) \cap G \neq \emptyset\}$ is open; (ii) upper semicontinuous, if for any open subset G of Y the set $\{x \in X : T(x) \subseteq G\}$ is open; (iii) continuous, if it is both upper and lower semicontinuous; (iv) closed, if its graph is a closed subset of $X \times Y$; (v) compact if $T(X)$ is contained in a compact subset of Y.

Lemma below collects two known results needed in the next sections.

Lemma 2.2. *Let $T : X \rightrightarrows Y$ be a set-valued mapping between two topological spaces.*

(i) *If Y is Hausdorff and compact, T is closed if and only if it is upper semicontinuous and closed-valued.*
(ii) *If T is upper semiconinuous and compact-valued, then $T(K)$ is compact whenever K is a compact subset of X.*

From now on, all topological (vector) spaces are assumed to be Hausdorff. For a subset A of a topological vector space, the standard notations $\operatorname{cl} A$, $\operatorname{int} A$ will designate the closure, and, respectively, the interior of A.

3. Main results

As we promised in Section 1, we establish here versions of Theorem 1.1 in which S and T have different domains and the assumption as S to be a weak KKM mapping w.r.t. T is dropped. The proof of the first result relies on the following lemma.

Lemma 3.3. (see [2, Lemma 3.1]) *Let X be a nonempty and convex set and Y be a nonempty, compact and convex set, each in a topological vector space. If $P : X \rightrightarrows Y$ is a closed mapping with nonempty convex values and convex cofibers, then $\bigcap_{x\in X} P(x) \neq \emptyset$.*

Theorem 3.2. *Let X, Y and Z be three nonempty convex sets, each in a topological vector space such that Y is compact. Assume that $S : X \rightrightarrows Z$, $T : Y \rightrightarrows Z$ are two closed set-valued mappings with nonempty values that satisfy the following conditions:*

(ii) *for each $x \in X$, there exists $y \in Y$ such that $T(y) \cap S(x) \neq \emptyset$;*
(ii) *T is convex and compact;*
(iii) *S has convex values and convex cofibers.*

Then, there exists an $y_0 \in X$ such that $T(y_0) \cap \bigcap_{x\in X} S(x) \neq \emptyset$.

Proof. We divide the proof into two steps.

Step 1. We prove first that there exists $y_0 \in Y$ such that $T(y_0) \cap S(x) \neq \emptyset$ for all $x \in X$. To this aim, let us consider the set-valued mapping $P : X \rightrightarrows Y$ defined by

$$P(x) = \{y \in Y : T(y) \cap S(x) \neq \emptyset\}.$$

We show that the mapping P is closed. Let $(x, y) \in \operatorname{cl}(\operatorname{Gr} P)$ and $\{(x_t, y_t)\}$ be a net in $\operatorname{Gr} P$ converging to (x, y). Then, for each index t, there exists $z_t \in T(y_t) \cap S(x_t)$. From (ii), the net $\{z_t\}$ has a subnet $\{z_{t_\alpha}\}$ converging to a point $z \in Z$. Since S and T are closed set-valued mappings, $z \in T(y) \cap S(x)$. Hence $(x, y) \in \operatorname{Gr} P$.

Let $x \in X$, $y_1, y_2 \in P(x)$ and $\lambda \in [0, 1]$. For each $i \in \{1, 2\}$ there is $z_i \in T(y_i) \cap S(x)$. As $S(x)$ is a convex set and T is a convex mapping,

$$\lambda z_1 + (1-\lambda)z_2 \in T\big(\lambda y_1 + (1-\lambda)y_2\big) \cap S(x),$$

hence $\lambda y_1 + (1-\lambda)y_2 \in P(x)$. Thus P is convex-valued.

Consider now an arbitrary $y \in Y$ and $x_1, x_2 \in P^*(y)$. Since S has convex cofibers, from Lemma 2.1, $S(\lambda x_1 + (1-\lambda)x_2) \subseteq S(x_1) \cup S(x_2)$ for all $\lambda \in [0, 1]$. Thus,

$$T(y) \cap S(\lambda x_1 + (1-\lambda)x_2) \subseteq T(y) \cap S(x_1) \cup T(y) \cap S(x_2) = \emptyset,$$

whence $\lambda x_1 + (1-\lambda)x_2 \in P^*(y)$. Consequently, the cofibers of P are convex sets.

From Lemma 3.3, there exists $y_0 \in \bigcap_{x\in X} P(x)$. Then, $T(y_0) \cap S(x) \neq \emptyset$ for all $x \in X$.

Step 2. Define now the set-valued mapping $Q : X \rightrightarrows T(y_0)$ by $Q(x) = T(y_0) \cap S(x)$. By the previous step, Q has nonempty values. As $T(y_0)$ and the values of S are convex sets, so will be the values of Q. One can easily see that Q is a closed mapping. For each $z \in T(y_0)$,

$$Q^*(z) = \{x \in X : z \notin T(y_0) \cap S(x)\} = \{x \in X : z \notin S(x)\} = S^*(z).$$

Consequently, the cofibers of Q are convex. From Lemma 3.3,

$$\emptyset \neq \bigcap_{x\in X} Q(x) = T(y_0) \cap \bigcap_{x\in X} S(x).$$

□

Remark 3.1. When $T(y) = Z$ for all $y \in Y$, Theorem 3.2 reduces to Corollary 2.2 in [3].

We give below a simple example where Theorem 3.2 is applicable, while Theorem 1.1 is not.

Example 3.1. Let $X = Y = [0,1]$ and $T, S : [0,1] \rightrightarrows \mathbb{R}$ be defined as follows

$$T(x) = \left[1 - x, \sqrt{1 - x^2}\,\right], \quad S(x) = [0, x].$$

From Definition 1.1 it follows immediately that if the set-valued mapping S would be weak KKM w.r.t. T, then $T(x) \cap S(x) \neq \emptyset$ for all $x \in [0,1]$. But, for any $x \in [0, \frac{1}{2}[$, $T(x) \cap S(x) = \emptyset$, hence S is not a weak KKM mapping w.r.s. T. Therefore, Theorem 1.1 cannot be applied.

We show that the set-valued mapping T is convex. Let $x_1, x_2 \in [0,1]$, $y_1 \in T(x_1), y_2 \in T(x_2)$ and $\lambda \in [0,1]$. Since the function $x \to \sqrt{1-x^2}$ is concave, we have

$$\begin{aligned} 1 - \big(\lambda x_1 + (1-\lambda)x_2\big) &= \lambda(1 - x_1) + (1-\lambda)(1 - x_2) \leq \lambda y_1 + (1-\lambda)y_2 \\ &\leq \lambda\sqrt{1 - x_1^2} + (1-\lambda)\sqrt{1 - x_2^2} \leq \sqrt{1 - \big(\lambda x_1 + (1-\lambda)x_2\big)^2}. \end{aligned}$$

Hence $\lambda y_1 + (1-\lambda)y_2 \in T(\lambda x_1 + (1-\lambda)x_2)$. Clearly, the other assumptions of Theorem 3.2 are also satisfied. By checking directly, one sees that $y_0 = 1$ is the unique point for which $T(y_0) \cap \bigcap_{x \in X} S(x) \neq \emptyset$.

The following "dual" version of Lemma 3.3 is Lemma 3.2 in [2].

Lemma 3.4. *Let X be a nonempty compact and convex set and Y be a nonempty convex set, each in a topological vector space. If $P : X \rightrightarrows Y$ is a set-valued mapping with open graph, nonempty convex values and convex cofibers, then $\bigcap_{x \in X} P(x) \neq \emptyset$.*

Based on Lemma 3.4 we establish the following open version of Theorem 3.2.

Theorem 3.3. *Let X, Y and Z be three nonempty convex sets, each in a topological vector space, so that X is compact. Let $S : X \rightrightarrows Z, T : Y \rightrightarrows Z$ be two set-valued mappings with nonempty values. Assume that:*

(i) *for each $x \in X$, there exists $y \in Y$ such that $T(y) \cap S(x) \neq \emptyset$;*
(ii) *T is convex and lower semicontinuous;*
(iii) *S has open graph, convex values and convex cofibers.*

Then, there exists an $y_0 \in X$ such that $T(y_0) \cap \bigcap_{x \in X} S(x) \neq \emptyset$.

Proof. The proof is similar with that of Theorem 3.2, using Lemma 3.4 instead of Lemma 3.3. Consider first the set-valued mapping $P : X \rightrightarrows Y$ defined by

$$P(x) = \{y \in Y : T(y) \cap S(x) \neq \emptyset\}.$$

We have already seen that P has nonempty convex values and convex cofibers. Lemma 3.4 will be applicable as soon as we show that P has open graph. Take an arbitrary $(x_0, y_0) \in \mathrm{Gr}\, P$ and choose an $z_0 \in T(y_0) \cap S(x_0)$. As $\mathrm{Gr}\, S$ is open, there exist a neighborhood U of x_0 in X and a neighborhood W of z_0 in Z such that $U \times W \subseteq \mathrm{Gr}\, S$. Since T is lower semicontinuous, there is a neighborhood V of y_0 in Y such that $T(y) \cap W \neq \emptyset$ for all $y \in V$.

To prove that $\mathrm{Gr}\, P$ is open, it suffices to show that $U \times V \subseteq \mathrm{Gr}\, P$. Let $(x,y) \in U \times V$. If $z \in T(y) \cap W$, then $(x,z) \in \mathrm{Gr}\, S$, hence $z \in T(y) \cap S(x)$ and thus $(x,y) \in \mathrm{Gr}\, P$. Consequently, $U \times V \subseteq \mathrm{Gr}\, P$.

Applying Lemma 3.4, we get a point $y_0 \in Y$ such that $T(y_0) \cap S(x) \neq \emptyset$ for all $x \in X$.

Next, as in the proof of Theorem 3.2, the set-valued mapping $Q : X \rightrightarrows T(y_0)$ defined by $Q(x) = T(y_0) \cap S(x)$ has nonempty convex values and convex cofibers. Since $\mathrm{Gr}\, Q = \mathrm{Gr}\, S \cap \big(X \times T(y_0)\big)$, it follows that the graph of Q is open in $X \times T(y_0)$. The desired conclusion follows now by Lemma 3.4 applied to the set-valued mapping Q. $\square$

Since any set-valued mapping with open graph is lower semicontinuous the following question arises naturally: does Theorem 3.3 remain true if we ask that S to be only lower semicontinuous? A result in this direction is established in the next theorem.

Theorem 3.4. *Let X and Y be nonempty compact convex subsets of two topological vector spaces and Z be a nonempty convex set in a locally convex topological vector space E. Assume that $S : X \rightrightarrows Z, T : Y \rightrightarrows Z$ satisfy the following conditions:*

(i) *for each $x \in X$, there exists $y \in Y$ such that $T(y) \cap S(x) \neq \emptyset$;*

(ii) *T is convex, continuous and with nonempty compact values;*

(iii) *S is lower semicontinuous, with nonempty closed convex values and convex cofibers;*

Then, there exists an $y_0 \in Y$ such that $T(y_0) \cap \bigcap_{u\in X} S(u) \neq \emptyset$.

Proof. Let $\mathcal{B}$ be a basis of open convex neighborhoods of E. For every $B \in \mathcal{B}$, consider the set-valued mapping $S_B : X \rightrightarrows Z$ defined by

$$S_B(u) = \big(S(u) + B\big) \cap Z.$$

From [17, Lemma 1], the graph of S_B is open in $X \times Z$ and clearly S_B is convex-valued.

For $z \in Z$,

$$S^*_B(z) = \{x \in X : z \notin S(x) + B\}$$
$$= \{x \in X : \forall b \in B, z - b \notin S(x)\} = \bigcap \{S^*(z - b) : b \in B \cap (z - Z)\}.$$

Since the cofibers of S are convex, so will the cofibers of S_B.

From Theorem 3.3, for each $B \in \mathcal{B}$, there are $y_B \in Y$ and $z_B \in T(y_B) \cap \bigcap_{x\in X} S_B(x)$. As T is upper semicontinuous and compact-valued, $T(Y)$ is a compact subset of Z. Since $Y \times T(Y)$ is a compact set, we may assume, without loss of generality, that the net $\{(y_B, z_B)\}_{B\in\mathcal{B}}$ converges to $(y_0, z_0) \in Y \times T(Y)$. By Lemma 2.2, T is a closed mapping. Consequently, $z_0 \in T(y_0)$. Let $x \in X$ be arbitrarily fixed. For every $B \in \mathcal{B}$, since $z_B \in S_B(x)$, there exists $b_B \in B$ such that $z_B - b_B \in S(x)$. Since the net $\{z_B - b_B\}$ converges to z_0, and $S(x)$ is a closed set, $z_0 \in S(x)$. Thus, $z_0 \in T(y_0) \cap \bigcap_{x\in X} S(x)$. □

4. Set-valued equilibrium problems

Let X, Y and Z be nonempty convex subsets of three topological vector spaces such that Y is compact, E be a topological vector space, C be a closed convex cone with nonempty interior in E and $P : Y \rightrightarrows Z$, $F : X \times Y \times Z \rightrightarrows E$ be set-valued mappings with nonempty values. As applications of the intersection theorems from the previous section, we establih further existence criteria of the solutions for the following four types of set-valued equilibrium problems:

(SVEP-1) Find $y_0 \in Y$ and $z_0 \in P(y_0)$ such that $F(x, y_0, z_0) \subseteq C$ for all $x \in X$.

(SVEP-2) Find $y_0 \in Y$ and $z_0 \in P(y_0)$ such that $F(x, y_0, z_0) \nsubseteq -\text{int}\, C$ for all $x \in X$.

(SVEP-3) Find $y_0 \in Y$ and $z_0 \in P(y_0)$ such that $F(x, y_0, z_0) \subseteq \text{int}\, C$ for all $x \in X$.

(SVEP-4) Find $y_0 \in Y$ and $z_0 \in P(y_0)$ such that $F(x, y_0, z_0) \nsubseteq -C$ for all $x \in X$.

There is a rich literature dedicated to the existence of solutions for problems (SVEP-1) and (SVEP-2) or their various generalizations (see, for instance, [5]- [15]), but the mentioned papers deal only with the case when $X = Z$. To the best of our knowledge, problems (SVEP-3) and (SVE-4) have not been studied until now.

Since the proofs of the existence theorems for problems (SVEP-1) $\div$ (SVEP-4) are similar, we prefer to study first the existence of solution for a variational relation problem. Recall that variational relation problems were introduced by Luc in [21] as general models for a large class of problems from nonlinear analysis and applied mathematics. Given three sets X, Y and Z, a relation R between their elements is represented as a nonempty subset of the product space $X \times Y \times Z$. Adopting Luc's terminology, we say that $R(x, y, z)$ holds, if $(x, y, z) \in R$.

Theorem 4.5. *Let X, Y and Z be three nonempty convex sets, each in a topological vector space such that Y is compact, $P : Y \rightrightarrows Z$ be a closed, compact and convex set-valued mapping with nonempty values and R be a relation linking elements $x \in X$, $y \in Y$ and $z \in Z$. Assume that:*

(i) *for any $x \in X$, there exists $(y, z) \in$ Gr P such that $R(x, y, z)$ holds;*
(ii) *the set $\{(x, y, z) \in X \times Y \times Z : R(x, y, z)$ holds$\}$ is closed in $X \times Y \times Z$;*
(iii) *for each $x \in X$, the set $\{(y, z) \in Y \times Z : R(x, y, z)$ holds$\}$ is convex;*
(iv) *for each $(y, z) \in Y \times Z$, the set $\{x \in X : R(x, y, z)$ does not hold$\}$ is convex.*

Then, there exist $y_0 \in Y$ and $z_0 \in P(y_0)$, such that $R(x, y_0, z_0)$ holds for all $x \in X$.

Proof. Consider the set-valued mappings $T : Y \to Y \times Z$, $S : X \to Y \times Z$ defined by

$$T(y) = \{y\} \times P(y), \quad S(x) = \{(y, z) \in Y \times Z : R(x, y, z) \text{ holds}\}.$$

Since P is closed, compact and convex, so is T. The set-valued S is closed (by (ii)), has convex values (from (iii)) and convex cofibers (from (iv)). Moreover, by (i), for any $x \in X$ there exists $y \in Y$ such that $T(y) \cap S(x) \neq \emptyset$. By Theorem 3.2, there exists $(y_0, z_0) \in T(y_0) \cap \bigcap_{x\in X} S(x)$. This means that $z_0 \in P(y_0)$ and $R(x, y_0, z_0)$ holds for all $x \in X$. □

In a similar manner, from Theorem 3.3, one obtains

Theorem 4.6. *Let X, Y and Z be convex sets in three topological vector spaces so that X is compact. The conclusion of Theorem 4.5 remains true if the set-valued mapping P is lower semicontinuous and convex and the relation R satisfies assumptions (i), (iii) and (iv) of Theorem 4.5 and condition (ii') below*

(ii') *the set $\{(x, y, z) \in X \times Y \times Z : R(x, y, z)$ holds$\}$ is open in $X \times Y \times Z$*

Remark 4.1. The conclusion of Theorems 4.5 and 4.6 is the same as that of Theorem 4.3 in [8], but the assumptions in the mentioned result are different.

Further, we establish existence results for the aforementioned set-valued equilibrium problems. But before, we need to recall some concepts of cone continuity and cone convexity.

Let C be a closed convex cone in the topological vector space E. According to Definition 7.1 in [22], a set-valued mapping $F : X \to 2^E$ is said to be:

(i) lower C-continuous at x_0, if for each $e \in F(x_0)$ and any neighborhood V of e, there is a neighborhood U of x_0 such that $F(x) \cap (V + C) \neq \emptyset$ for all $x \in U$;
(ii) upper C-continuous at x_0, if for each neighborhood V of $F(x_0)$, there is a neighborhood U of x_0 such that $F(x) \subseteq V + C$ for all $x \in U$;
(iii) lower C-continuous (respectively, upper C- continuous), if it is lower C- continuous (respectively, upper C- continuous) at every point $x \in X$.

Let X be a convex set in a vector space, E be a vector space and C be convex cone in E. A set-valued mapping $F : X \to 2^E$ is said to be:

(i) C-concave (see [10]) if

$$F(\lambda x_1 + (1 - \lambda)x_2) \subseteq \lambda F(x_1) + (1 - \lambda)F(x_2) + C,$$

for all $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$;

(ii) C-quasiconvex (see [14]) if for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ there is an index $i \in \{1, 2\}$ such that

$$F(x_i) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

Theorem 4.7. *Problem (SVEP-1) has at least a solution whenever the convex set Y is compact, the set-valued mapping P is closed, compact and convex and F satisfies the following conditions:*

(i) *for each $x \in X$, there exists $(y, z) \in$ Gr P such that $F(x, y, z) \subseteq C$;*

(ii) *F is lower (- C)-continuous*
(iii) *the set valued-mapping $F(x,\cdot,\cdot)$ is C-concave;*
(iv) *for each $(y,z)\in Y\times Z$, the set valued mapping $F(\cdot,y,z)$ is C -quasiconvex.*

Proof. We intend to apply Theorem 4.5 when the relation R is defined by

$$R(x,y,z) \text{ holds iff } F(x,y,z)\subseteq C.$$

Note that condition (i) is nothing other than the condition similarly noted in Theorem 4.5. We prove that the set

$$M := \{(x,y,z)\in X\times Y\times Z : F(x,y,z)\subseteq C\}$$

is closed in $X\times Y\times Z$. Let $(x,y,z)\in$ cl M and $\{(x_t,y_t,z_t)\}$ be a net in M converging to (x,y,z). Fix arbitrarily an $e\in F(x,y,z)$ and a neighborhood V of the origin of E. Since F is lower (- C)-continuous there exists an index t_0 such that for every $t\geq t_0$ we have

$$F(x_t,y_t,z_t)\cap(e-V-C)\neq\emptyset.$$

As $(x_t,y_t,z_t)\in M$, $F(x_t,y_t,z_t)\subseteq C$, hence $C\cap(e-V-C)\neq\emptyset$. It follows that

$$e\in C+V+C=C+V.$$

Since V has been an arbitrary neighborhood of 0_E, $e\in$ cl $C=C$. It follows that $F(x,y,z)\subseteq C$, hence $(x,y,z)\in M$.

Let $x\in X$, $(y_1,z_1),(y_2,z_2)\in\{(y,z)\in Y\times Z : F(x,y,z)\subseteq C\}$ and $\lambda\in[0,1]$. Since $F(x,\cdot,\cdot)$ is a C-concave mapping

$$F(x,\lambda y_1+(1-\lambda)y_2,\lambda z_1+(1-\lambda)z_2)\subseteq\lambda F(x,y_1,z_1)+(1-\lambda)F(x,y_2,z_2,)+C\subseteq C.$$

Thus, condition (iii) in Theorem 4.5 is fulfilled.

Now, we show that for any $(y,z)\in Y\times Z$, the set $\{x\in X : F(x,y,z)\not\subseteq C\}$ is convex. Assume, by way of contradiction, that for some $(y,z)\in Y\times Z$ there are $x_1,x_2\in X$ and $\lambda\in]0,1[$ such that $F(x_i,y,z)\not\subseteq C$ $(i=1,2)$ and $F(\lambda x_1+(1-\lambda)x_2,y,z)\subseteq C$. As $F(\cdot,y,z)$ is a C -quasiconvex mapping, for some index $i\in\{1,2\}$ the following inclusion holds:

$$F(x_i,y,z)\subseteq F(\lambda x_1+(1-\lambda)x_2,y,z)+C\subseteq C+C=C;\ \text{a contradiction.}$$

Now, the desired conclusion follows from Theorem 4.5. □

Theorem 4.8. *Assume that Y is compact, P is closed, compact and convex and F satisfies the following conditions:*

(i) *for each $x\in X$, there exists $(y,z)\in$ Gr P such that $F(x,y,z)\not\subseteq -$ int C;*
(ii) *F is compact-valued and upper (- C)-continuous;*
(iii) *for each $x\in X$, the set-valued mapping $F(x,\cdot,\cdot)$ is (- C)-quasiconvex;*
(iv) *for each $(y,z)\in Y\times Z$, $F(\cdot,y,z)$ is (- C)-concave.*

Then, problem (SVEP-2) has solution.

Proof. We show that each of conditions (ii)– (iv) implies the condition similarly noted in Theorem 4.5, when the relation R is defined by

$$R(x,y,z) \text{ holds iff } F(x,y,z)\not\subseteq -\operatorname{int} C.$$

Let $M := \{(x,y,z)\in X\times Y\times Z : F(x,y,z)\not\subseteq -\operatorname{int} C\}$ and $\{(x_t,y_t,z_t)\}$ be a net in M converging to $(x,y,z)\in X\times Y\times Z$. Let V be a neighborhood of the origin of E. Because F is upper $(-C)$-continuous, there exists an index t_0 such that for every $t\geq t_0$ we have $F(x_t,y_t,z_t)\subseteq F(x,y,z)+V-C$. As $(x_t,y_t,z_t)\in M$, $F(x_t,y_t,z_t)\not\subseteq -\text{int}\,C$. Thus, $F(x,y,z)+V-C\not\subseteq -\text{int}\,C$. Consequently, for every neighborhood V of the origin of E, there exist $f_V\in F(x,y,z)$ and $e_V\in V$, such that

$$f_V+e_V\notin -\text{int}\,C. \tag{4.1}$$

Since $F(x,y,z)$ is a compact set, without loss of generality, we may assume that the net $\{f_V\}$ converges to a point $f \in F(x,y,z)$. Then, the net $\{f_V + e_V\}$ also converges to f. From (4.1) we get $f \notin -\text{int}\, C$, hence $(x,y,z) \in M$. Thus, the set M is closed in $X \times Y \times Z$.

We now prove that for each $x \in X$ the set $\{(y,z) \in Y \times Z : F(x,y,z) \not\subseteq -\text{int}\, C\}$ is convex. We argue by contradiction. Assume that there exist $x \in X$, $(y_1,z_1),(y_2,z_2) \in Y \times Z$ and $\lambda \in]0,1[$ such that $F(x,y_i,z_i) \not\subseteq -\text{int}\, C$ $(i=1,2)$ and $F(x,\lambda y_1+(1-\lambda)y_2,\lambda z_1+(1-\lambda)z_2) \subseteq -\text{int}\, C$. As the mapping $F(x,\cdot,\cdot)$ is (- C)-quasiconvex, for some index $i \in \{1,2\}$ the following inclusion holds:

$$F(x,y_i,z_i) \subseteq F(x,\lambda y_1+(1-\lambda)y_2,\lambda z_1+(1-\lambda)z_2)-C \subseteq -\text{int}\, C - C = -\text{int}\, C; \text{ a contradiction.}$$

Let $(y,z) \in Y \times Z$ and $x_1,x_2 \in X$ such that $F(x_i,y,z) \subseteq -\text{int}\, C$ $(i=1,2)$. From (iv), for every $\lambda \in [0,1]$,

$$F(\lambda x_1+(1-\lambda)x_2,y,z) \subseteq \lambda F(x_1,y,z)+(1-\lambda)F(x_2,y,z)-C \subseteq -\text{int}\, C,$$

hence for every $(y,z) \in Y \times Z$ the set $\{x \in X : F(x,y,z) \subseteq -C\}$ is convex. Thus, Theorem 4.5 can be applied to get the desired conclusion. □

In a similar manner, from Theorem 4.6, can be derived existence criteria for the solutions of problems (SVEP-3) and (SVEP-4).

Theorem 4.9. *Problem (SVEP-3) has solution if the convex set X is compact, the set-valued mapping P is lower semicontinuous and convex and the following conditions hold:*

(i) *for each $x \in X$, there exists $(y,z) \in$ Gr P such that $F(x,y,z) \subseteq -$ int C;*
(ii) *F is upper C-continuous;*
(iii) *for each $x \in X$, the set-valued mapping $F(x,\cdot,\cdot)$ is C-concave;*
(iv) *for each $(y,z) \in Y \times Z$, $F(\cdot,y,z)$ is C-quasiconvex.*

Proof. Consider relation R defined as follows

$$R(x,y,z) \text{ holds iff } F(x,y,z) \subseteq C.$$

Let $(x_0,y_0,z_0) \in X \times Y \times Z$ such that $F(x_0,y_0,z_0) \subseteq \text{int}\, C$. Then, $\text{int}\, C$ is an open neighborhood of $F(x_0,y_0,z_0)$. As F is upper C-continuous, there is a neighborhood U of (x_0,y_0,z_0) in $X \times Y \times Z$ such that for each $(x,y,z) \in U$,

$$F(x,y,z) \subseteq \text{int}\, C + C = \text{int}\, C,$$

hence the set $\{(x,y,z) \in X \times Y \times Z : F(x,y,z) \subseteq \text{int}\, C\}$ is open in $X \times Y \times Z$. Consequently, condition (ii') in Theorem 4.6 is fulfilled. Following the same lines as in the proof of Theorem 4.7 one can easily check that al the other assumptions of Theorem 4.6 are satisfied. Applying this theorem we obtain the desired conclusion. □

Theorem 4.10. *Problem (SVEP-4) has solution if X is compact, P is lower semicontinuous and convex and F satisfies the following conditions:*

(i) *for each $x \in X$, there exists $(y,z) \in$ Gr P such that $F(x,y,z) \not\subseteq -C$;*
(ii) *F is lower C-continuous;*
(iii) *for each $x \in X$, the set-valued mapping $F(x,\cdot,\cdot)$ is (- C)-quasiconvex;*
(iii) *for each $(y,z) \in X \times Y$, $F(\cdot,y,z)$ is (- C)-concave.*

Proof. Following the same arguments as in the previous proofs one can easily show that each of assumptions $(i) \div (iv)$ implies the condition similarly noted in Theorem 4.6, when relation R is defined by

$$R(x,y,z) \text{ holds iff } F(x,y,z) \not\subseteq -C.$$

For instance, from the first part of the proof of Theorem 4.7 one sees that under assumption (i) the set $\{(x,y,z) \in X \times Y \times Z : F(x,y,z) \subseteq -C\}$ is closed in $X \times Y \times Z$, and thus the set $\{(x, y, z) \in X \times Y \times Z : F(x, y, z) \nsubseteq -C\}$ is open. Theorem 4.6 leads to the needed conclusion. □

5. Minimax theorems

Let X, Y, Z be convex sets in three topological vector spaces, $T : Y \rightrightarrows Z$ be a convex set-valued mapping with nonempty values and f be a real function defined on $X \times Z$. We are interested to find sufficient conditions such that

$$\inf_{x\in X} \sup_{z\in T(Y)} f(x,z) = \sup_{z\in T(Y)} \inf_{x\in X} f(x,z). \tag{5.2}$$

Since the inequality $\sup_{z\in T(Y)} \inf_{x\in X} f(x,z) \leq \inf_{x\in X} \sup_{z\in T(Y)} f(x,z)$ holds for any function $f : X \times Z \to \mathbb{R}$, equality (5.2) is equivalent with

$$\inf_{x\in X} \sup_{z\in T(Y)} f(x,z) \leq \sup_{z\in T(Y)} \inf_{x\in X} f(x,z).$$

In the proofs of the next theorems, we may assume that $\inf\limits_{x\in X} \sup\limits_{z\in T(Y)} f(x,z) > -\infty$, because the above inequality is trivial in contrary case.

Theorem 5.11. *Assume that the convex set Y is compact, the set-valued mapping T is closed and compact and the function f satisfies the following conditions:*

(i) *is upper semicontinuous on $X \times Z$;*
(ii) *for each $x \in X$, $f(x, \cdot)$ is quasiconcave;*
(iii) *for each $z \in Z$, $f(\cdot, z)$ is quasiconvex.*

Then, $\inf_{x\in X} \max_{z\in T(Y)} f(x,z) = \max_{z\in T(Y)} \inf_{x\in X} f(x,z)$.

Proof. First, let us justify why we can replace, in the conclusion of the theorem, $\sup_{z\in T(Y)}$ with $\max_{z\in T(Y)}$. From Lemma 2.2, it follows readily that the set $T(Y)$ is compact. Since f is upper semicontinuous on $X \times Z$, for each $x \in X$, $f(x, \cdot)$ is also an upper semicontinuous function of z on Z and therefore its maximum $\max_{z\in T(y)} f(x,z)$ on the compact set $T(Y)$ exists. Then, by Lemma 2.41 in [4], the function $z \longrightarrow \inf_{x\in X} f(x,z)$ is upper semicontinuous and therefore its maximum $\max_{z\in T(Y)} \inf_{x\in X} f(x,z)$ on the compact set $T(Y)$ exists.

Set

$$m := \inf_{x\in X} \max_{z\in T(Y)} f(x,z), \tag{5.3}$$

and define the set-valued mapping $S : X \rightrightarrows Z$ by

$$S(x) = \{z \in Z : f(x,z) \geq m\}.$$

The mapping S is closed (by (i)), has convex values (by (ii)) and convex cofibers (by (iii)). Moreover, from (5.3), it follows that for each $x \in X$ there is $y \in Y$ such that $S(x) \cap T(y) \neq \emptyset$.

By Theorem 3.2, there exists $y_0 \in Y$ such that $T(y_0) \cap \bigcap_{x\in X} S(x) \neq \emptyset$. Thus, $f(x, z_0) \geq m$ for some $z_0 \in T(y_0)$ and for all $x \in X$. Consequently,

$$\max_{z\in T(X)} \inf_{x\in X} f(x,z) \geq \inf_{x\in X} f(x, z_0) \geq m.$$

Thus the proof is complete. □

When $Y \equiv Z$ and T is the identity mapping on Z, Theorem 5.11 reduces to the following

Corollary 5.1. *Let X be a nonempty convex set and Z a nonempty compact convex set, each in a topological vector space. If $f : X \times Z \to \mathbb{R}$ satisfies condition (i) $\div$ (iii) from Theorem 5.11, then* $\inf_{x\in X} \max_{z\in Z} f(x,z) = \max_{z\in Z} \inf_{x\in X} f(x,z)$.

Remark 5.2. The above corollary can be regarded as a version of the well-known Sion's minimax theorem ([24]). The unique difference consists in the fact that in Sion's result f is assumed lower semicontinuous in the first variable and upper semicontinuous in the second one, while in Corollary 5.1, f is upper semicontinuous on $X \times Z$.

Theorem 5.12. *Assume that X is compact, T is lower semicontinuous and f is lower semicontinuous on $X \times Z$ and satisfies conditions (ii) and (iii) of Theorem 5.11. Then, the minimax equality (5.2) holds.*

Proof. For $\lambda < \inf_{x\in X} \sup_{z\in T(Y)} f(x,z)$ arbitrarily fixed, consider the set-valued mapping $S : X \rightrightarrows Z$ defined by

$$S(x) = \{z \in Z : f(x,z) \geq \lambda\}.$$

It follows readily by hypotheses that S has open graph, convex values and open fibers. Moreover, taking into account the choice of λ, for each $x \in X$ there exists $y \in Y$ such that $S(x) \cap T(y) \neq \emptyset$.

By Theorem 3.3, there exists $y_0 \in Y$ such that $T(y_0) \cap \bigcap_{x\in X} S(x) \neq \emptyset$. Thus, $f(x,z_0) \geq \lambda$ for some $z_0 \in T(y_0)$ and for all $x \in X$. Consequently,

$$\sup_{z\in T(Y)} \inf_{x\in X} f(x,z) \geq \inf_{x\in X} f(x,z_0) \geq \lambda.$$

As λ was an arbitrary real number less than $\inf_{y\in Y} \sup_{z\in T(y)} f(y,z)$, we infer that $\sup_{z\in T(Y)} \inf_{x\in X} f(x,z) \geq \inf_{x\in X} \sup_{z\in T(Y)} f(x,z)$. □

Recall that a real function h defined on a convex set X is said to be strictly quasiconcave if

$$\min(h(x_1), h(x_2)) < h(\lambda x_1 + (1-\lambda)x_2),$$

for each $x_1, x_2 \in X$, $x_1 \neq x_2$, and $\lambda \in]0,1[$.

Theorem 5.13. *The minimax equality (5.2) holds when X and Y are compact convex sets, Z is a nonempty convex set in a locally convex topological vector space, T is continuous and compact-valued and f satisfies the following conditions:*

(i) *for each $x \in X$, $f(x,\cdot)$ is upper semicontinuous and either concave or strict quasiconcave;*
(ii) *for each $z \in Z$, $f(\cdot,z)$ is lower semicontinuous and quasiconvex.*

Proof. As in the previous proof, for an arbitrary

$$\lambda < \inf_{x\in X} \max_{z\in T(Y)} f(x,z) \tag{5.4}$$

we consider the set-valued mapping $S : X \rightrightarrows Z$ defined by $S(x) = \{z \in Z : f(x,z) \geq \lambda\}$.

The conclusion follows from Theorem 3.4 as soon as we prove that S is lower semicontinuous. Suppose to the contrary that S is not lower semicontinuous. This means that there exist a point $x_0 \in X$ and a set $G \subseteq Z$, open relative to Z, such that $S(x_0) \cap G \neq \emptyset$ and for any open neighborhood V of x_0 there exists $x_V \in V$ satisfying

$$S(x_V) \cap G = \emptyset. \tag{5.5}$$

Fix a point $z_0 \in S(x_0) \cap G$. Hence $z_0 \in G$ and $f(x_0,z_0) \geq \lambda$. From (5.4), there exists a point $z_1 \in T(Y)$ such that $f(x_0,z_1) > \lambda$. As G is an open neighborhood of z_0, we can find $\alpha \in]0,1[$ such that $z_\alpha = \alpha z_0 + (1-\alpha)\, z_1 \in G$. If $f(x_0,\cdot)$ is concave, we have

$$f(x_0,z_\alpha) \geq \alpha f(x_0,z_0) + (1-\alpha) f(x_0,z_1) > \lambda.$$

Otherwise, assume that f is strictly quasiconcave in the first variable. Then we have

$$f(x_0,z_\alpha) > \min\{f(x_0,z_0), f(x_0,z_1)\} > \lambda.$$

Therefore, in both cases

$$f(x_0, z_\alpha) > \lambda. \tag{5.6}$$

From (5.5), we infer that for each open neighborhood V of x_0, $z_\alpha \notin S(x_V)$, hence $f(x_V, z_\alpha) < \lambda$. As the function $f(\cdot, z_\lambda)$ is lower semicontinuous, it follows that $f(x_0, z_\alpha) \leq \lambda$, which contradicts (5.6). Thus S is lower semicontinuous. □

REFERENCES

[1] Agarwal, R. P., Balaj, M. and O'Regan, D., *Intersection theorems with applications in optimization*, J. Optim. Theory Appl., **179** (2018), 761–777

[2] Agarwal, R. P., Balaj, M. and O'Regan, D., *Common fixed point theorems in topological vector spaces via intersection theorems*, J. Optim. Theory Appl., **173** (2017), 443–458

[3] Agarwal, R. P., Balaj, M. and O'Regan, D., *An intersection theorem for set-valued mappings*, Appl. Math., **58** (2013), 269–278

[4] Aliprantis, C. D., Border, K. C., *Infinite Dimensional Analysis. A hitchhiker's guide*, Springer, Berlin, 2006

[5] Ansari, Q. H., Konnov, I. V. and Yao, J. C., *On generalized vector equilibrium problems*, Nonlinear Anal., **47** (2001), 543–554

[6] Ansari, Q. H., Farajzadeh, A. P. and Schaible, S., *Existence of solutions of strong vector equilibrium problems*, Taiwanese J. Math., **16** (2012), 165–178

[7] Balaj, M., *Weakly G-KKM mappings, G-KKM property, and minimax inequalities*, J. Math. Anal. Appl., **294** (2004), 237–245

[8] Balaj, M., *Three types of variational relation problems*, Taiwanese J. Math., **17** (2013), 47–61

[9] Balaj, M., Luc, D. T., *On mixed variational relation problems*, Comput. Math. Appl., **60** (2010), 2712–2722

[10] Borwein, J., *Multivalued convexity and optimization: a unified approach to inequality and equality constraints*, Math. Programming, **13** (1977), 183–199

[11] Chiang, Y., Chadli, O. and Yao, J. C., *Generalized vector equilibrium problems with trifunctions*, J. Global Optim., **30** (2004), 135–154

[12] Fan, K., *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142** (1961), 305–310

[13] Fu, J. Y. and Wan, A. H., *Generalized vector equilibrium problems with set-valued mappings*, Math. Methods Oper. Res., **56** (2002), 259–268

[14] Hou, S. H., Yu, H. and Chen, G. Y., *On vector quasi-equilibrium problems with set-valued maps*, J. Optim. Theory Appl., **119** (2003), 485–498

[15] Junlouchai, P. and Plubtieng, S., *Existence of solutions for generalized variational inequality problems in Banach spaces*, Nonlinear Anal., **74** (2011), 999–1004

[16] Kien, B. T., Wong, N. C. and Yao, J. C., *Generalized vector variational inequalities with star-pseudomonotone and discontinuous operators*, Nonlinear Anal., **68** (2008), 2859–2871

[17] Kim, W. K., *A fixed point theorem in a Hausdorff topological vector space*, Comment. Math. Univ. Carolin., **36** (1995), 33–38

[18] Konnov, I. V. and Yao, J. C., *On the generalized vector variational inequality problem*, J. Math. Anal. Appl., **206** (1997), 42–58

[19] Lin, L. J. and Chen, H. L., *The study of KKM theorems with applications to vector equilibrium problems with implicit vector variational inequalities problems*, J. Global Optim., **32** (2005), 135–157

[20] Li, X. B. and Li, S. J., *Existence of solutions for generalized vector quasi-equilibrium problems*, Optim. Lett., **4** (2010), 17–28

[21] Luc, D. T., *An abstract problem in variational analysis*, J. Optim. Theory Appl., **138** (2008), 65–76

[22] Luc, D. T., *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, 319. Springer-Verlag, Berlin, 1989

[23] Park, S., Some coincidence theorems on acyclic multifunctions and applications to KKM theory, in *Fixed point theory and applications*, World Sci. Publ., River Edge, NJ. 1992, 248–277

[24] Sion, M., *On general minimax theorems*, Pacific J. Math., **8** (1958), 171–176

[25] Yu, J. and Yang, H., *Existence of solutions for generalized variational inequality problems*, Nonlinear Anal., **71** (2009), 2327–2330

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