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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} *anniversary*

Approximation of solutions of Hammerstein equations with monotone mappings in real Banach spaces

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ABSTRACT. Let E be a uniformly convex and uniformly smooth real Banach space with dual space, E^*. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. An iterative algorithm is constructed and the sequence of the algorithm is proved to converge strongly to a solution of the Hammerstein equation $u + KFu = 0$. This theorem is a significant improvement of some important recent results which were proved in L_p spaces, $1 < p \leq 2$ under the assumption that F and K are *bounded*. This restriction on K and F have been dispensed with even in the more general setting considered here. Finally, a numerical experiment is presented to illustrate the convergence of the sequence of the algorithm which is found to be much faster, in terms of the number of iterations and the computational time than the convergence obtained with existing algorithms.

1. INTRODUCTION

Let E be a real Banach space with a strictly convex dual space, E^*. Consider on E the Hammerstein equation

$$(I + KF)u = 0, \tag{1.1}$$

where, $F : E \to E^*$ is a nonlinear mapping and $K : E^* \to E$ is a linear map, such that $R(F) \subset D(K)$. If Ω denotes a domain of σ-finite measure dy in R^N, and $\kappa : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable real-valued functions on Ω, one can define a linear integral operator K by $Kv := \displaystyle\int_\Omega \kappa(\cdot, y)v(y)dy$ and an operator F by the *Nemitskyi* or *superposition* operator given by $Fu := f\big(\cdot, u(\cdot)\big)$ to obtain equation (1.1).

Numerous problems in differential equation, optimal control, automation and network systems can, as a rule, be modeled as a Hammerstein equation (see, e.g., Pascali and Sburlan [35]).

Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., Brezis and Browder [4, 5], Browder and Gupta [7], Chepanovich [8], De Figueiredo and Gupta [24]).

Let $A : D(A) \subset E \to E$ be a mapping. A is called *accretive* if for each $u, v \in D(A)$, there exists $j(u - v) \in J(u - v)$ such that $\langle Au - Av, j(u - v)\rangle \geq 0$, where $J : E \to 2^{E^*}$ is the *normalized duality* map defined, for each $x \in E$, by $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|\}$. The map A is called *m-accretive* if, it is accretive and, in addition, the graph of A is not properly contained in the graph of any other accretive mapping. In Hilbert spaces, accretive mappings are called *monotone*. The accretive mappings were

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introduced independently in 1967 by Browder [3] and Kato [30]. Interest in such mappings stems mainly from their firm connection with *evolution equations* (see e.g., Berinde [2], Chidume [9], Gobel and Reich [27] and the references contained in them).

A mapping $B : D(B) \subset E \to E^*$ is called *monotone* if for all $x, y \in E$, $\langle Bx - By, x - y\rangle \geq 0$. The map B is called *maximal monotone* if, in addition, $R(J + \lambda B)$ is E^*, for all $\lambda > 0$. Monotone mappings were studied in Hilbert spaces by Zarantonello [41], Minty [32], and a host of other authors. Interest in such mappings stems from their usefulness in applications, (in particular, monotone mappings are useful in convex optimization problems see, e.g., Chidume and Bello [19]).

In general, equations of Hammerstein type are nonlinear and thus, there is no closed form solutions of such equations. Consequently, methods for approximating such equations are of interest. Several attempts have been made to approximate solutions of equations of Hammerstein type.

An early method was that used by Brezis and Browder [6] in a special case where one of the operators is *angle bounded* (see e.g., Pascali and Sburlan, [35]). They proved strong convergence of a suitable defined *Galerking approximation* to a solution of (1.1), (see e.g., Brezis and Browder [6]).

The first *iterative methods* for approximating solutions of Hammerstein equations, in real Banach spaces more general than Hilbert spaces, as far as we know, were obtained by Chidume and Zegeye [14] (see also Chidume [9], Chapter 13).

Let X be a real Banach space and $F, K : X \to X$ be *accretive-type* mappings. Let $E := X \times X$. Then, defined $T : E \to E$ by $T[u, v] = [Fu - v, Kv + u],\ for\ [u, v] \in E$. We note that $T[u, v] = 0 \ \Leftrightarrow u$ solves (1.1) and $v = Fu$. With this, they were able to obtain *strong convergence* of an iterative algorithm defined in the cartesian product space E to a solution of the Hammerstein equation (1.1). Extensions of these early results of Chidume and Zegeye [14] were obtained by several authors (see, e.g., Chidume and Zegeye [13, 15], Chidume and Djitte [21, 22], Chidume and Ofoedu [12], Chidume and Shehu [10, 11, 20], Zegeye and Molanza [42], Shehu [37], Minjibir and Mohammed [33] and the references contained in them).

In 2013, Djitte and Sene [26] proved strong convergence theorem for the following explicit iterative algorithm in uniformly smooth real Banach spaces.

Theorem 1.1. *Let E be a uniformly smooth real Banach space and $K, F : E \to E$ be bounded and accretive mappings with $R(F) = D(K) = E$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively from arbitrary points $u_1, v_1 \in E$ as follows:*

$$\begin{cases} u_{n+1} &= u_n - \lambda^2(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1)\big), \\ v_{n+1} &= v_n - \lambda_n^2(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1)\big), \end{cases} \tag{1.2}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1)$ satisfying some appropriate conditions. Suppose that $u + KFu = 0$ has a solution u^*, then the sequence $\{u_n\}$ converges to u^*.

In 2016, Chidume and Idu [16], proved strong convergence theorem for the following explicit iterative algorithm in uniformly convex uniformly smooth real Banach spaces.

Theorem 1.2. *Let E be a uniformly convex and uniformly smooth real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps, respectively. For arbitrary $(u, v) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively, by*

$$\begin{cases} u_{n+1} &= J^{-1}\big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju)\big),\ n \geq 1, \\ v_{n+1} &= J\big(J^{-1}v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J^{-1}v_n - J^{-1}v)\big),\ n \geq 1. \end{cases} \tag{1.3}$$

Assume that the equation $u + KFu = 0$ *has a solution. Then, the sequences* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* u^* *and* v^**, respectively, where* u^* *is a solution of (1.1) with* $v^* = Fu^*$.

Recently, Uba *et al.* [40], introduced a *new* coupled iterative algorithm and proved the following strong convergence theorem.

Theorem 1.3. *Let* $E = L_p$, $1 < p \leq 2$. *Let* $F : E \to E^*$ *and* $K : E^* \to E$ *be monotone and bounded maps. For* $(u_0, v_0) \in E \times E^*$*, define the sequences* $\{u_n\}$ *and* $\{v_n\}$ *in* E *and* E^**, respectively by*

$$\begin{cases} u_{n+1} &= J^{-1}(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n), \; n \geq 0, \\ v_{n+1} &= J(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n), \; n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ *and* θ_n *are acceptably paired sequences in* $(0,1)$*. Assume that the equation* $u + KFu = 0$ *has a solution. Then, the sequences* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* u^* *and* v^**, respectively, where* u^* *is the solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

It is our purpose in this paper to prove a significant improvement of Theorem 1.3. We extend Theorem 1.3 to uniformly convex and uniformly smooth real Banach spaces and, at the same time, dispense with the requirement in Theorem 1.3 that the mappings K and F be bounded. In particular, our Theorem is applicable in L_p spaces, $1 < p < \infty$, thereby providing an iterative algorithm which converges strongly to a solution of the Hammerstein equation (1.1) in L_p spaces, $1 < p < \infty$, and without requiring that F and K be bounded, as is imposed in Theorem 1.3. Furthermore, our theorem improves and compliments Theorems 1.2 and 1.1 see Remark 4.5 below.

2. Preliminaries

In this section, we present definitions of some terms, and results that will be needed in the proof of our main theorem.

Definition 2.1. Let E be a smooth real Banach space. The Lyapounov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2, \; \forall \, u, v \in E. \tag{2.5}$$

It was introduced by Alber and has been studied by many authors (see, e.g., Alber [1]; Chidume *et al.* [17, 18]; Kamimura and Takahashi [29]; Nilsrakoo and Saejung [34]; and the references contained in them). It is easy to see that from the definition of ϕ,

$$(\|u\| - \|v\|)^2 \leq \phi(u, v) \leq (\|u\| + \|v\|)^2, \forall u, v \in E. \tag{2.6}$$

Definition 2.2. Let E be a normed linear space, consider the map $W : E \times E^* \to \mathbb{R}$ defined by $W(u, u^*) = \|u\|^2 - 2\langle u, u^*\rangle + \|u^*\|$, $\forall u \in E, u^* \in E^*$. Observe that $W(u, u^*) = \phi(u, J^{-1}u^*), \forall u \in E, u^* \in E^*$.

Lemma 2.1 (Alber and Ryazantseva, [1]). *Let* E *be a reflexive strictly convex and smooth Banach space with* E^* *as its dual. Then, for each* $u \in E$ *and* $u^*, v^* \in E^*$*, we have*

$$W(u, u^*) + 2\langle J^{-1}u^* - u, v^*\rangle \leq W(u, u^* + v^*). \tag{2.7}$$

Lemma 2.2 (Alber and Ryazantseva, [1]). *Let* E *be a reflexive strictly convex and smooth Banach space with dual space* E^**. Let* $V : E \times E \to \mathbb{R}$ *be defined by* $V(u, v) = \frac{1}{2}\phi(v, u)$*. Then,* $\forall u, v, s \in E$,

$$V(u, v) - V(s, u) \geq \langle s - v, Ju - Js\rangle, \text{ i.e., } \phi(v, u) - \phi(u, s) \geq 2\langle s - v, Ju - Js\rangle,$$

and also, $V(u, v) \leq \langle u - v, Ju - Jv\rangle$.

Lemma 2.3 (Chidume and Idu, [16]). *Let E be a smooth real Banach space with dual space E^*. Let $\phi : E \times E \to \mathbb{R}$ be the Lyapounov functional. Then, $\phi(v,u) = \phi(u,v) - \langle u+v, Ju - Jv\rangle + 2(\|u\|^2 - \|v\|^2),\ \forall u,v \in E$.*

Lemma 2.4 (Alber and Ryazantseva, [1]). *Let E be a uniformly convex Banach space. Then, for any $r > 0$ and any $u, v \in E$ such that $\|u\| \le r, \|v\| \le r$, the following inequality holds: $\langle u-v, Ju-Jv\rangle \ge (2L)^{-1}\delta_E(c_2^{-1}\|u-v\|)$, where $c_2 = 2\max\{1,r\},\ 1 < L < 1.7$. Define*

$$D := 4rL\sup\big\{\|Ju - Jv\| : \|x\| \le r, \|y\| \le r\big\} + 1. \tag{2.8}$$

Lemma 2.5 (Alber and Ryazantseva, [1]). *Let E be a uniformly convex Banach space. Then, for any $r > 0$ and any $u, v \in E$ such that $\|u\| \le r, \|v\| \le r$, the following inequality holds: $\langle u-v, Ju-Jv\rangle \ge (2L)^{-1}\delta_{E^*}(c_2^{-1}\|Ju-Jv\|)$, where $c_2 = 2\max\{1,r\},\ 1 < L < 1.7$.*

Lemma 2.6 (Rockafellar, [39]; see also, Pascali and Sburlan, [35]). *A monotone mapping $T : E \to 2^{E^*}$ is locally bounded at the interior points of its domain.*

Lemma 2.7 (Reich, [36]). *Let E^* be a real strictly convex dual space with a Fréchet differentiable norm, and let A be a maximal monotone operator from E to E^* such that $A^{-1}0 \ne \emptyset$. Let $s \in E^*$ be arbitrary but fixed. For each $\rho > 0$ there exists a unique $u_\rho \in E$ such that $Ju_\rho + \rho A u_\rho \ni s$. Furthermore, u_ρ converges strongly to a unique point $p \in A^{-1}0$.*

Corollary 2.1. *From Lemma 2.7, setting $\rho_n := \frac{1}{\theta_n}$, where $\theta_n \to 0$ as $n \to \infty$, $z = j(v)$ for some $j(v) \in J(v), v \in E, y_n := \Big(j + \frac{1}{\theta_n}A\Big)^{-1} z$, we obtain: $Ay_n = \theta_n(j(v) - j(y_n)),\ for\ some\ j(y_n) \in J(y_n)$. Furthermore, $y_n \to y^* \in A^{-1}0$, where $A : E \to E^*$ is maximal monotone (see, Chidume and Idu,* [16]*).*

Remark 2.1. Let $r > 0$ such that $\|v\| \le r, \|y_n\| \le r$, for all $n \ge 1$. The following estimates will be needed in the sequel.

$$\|y_{n-1} - y_n\| \le c_2\delta_E^{-1}\bigg(\frac{\theta_{n-1} - \theta_n}{\theta_n}D\bigg), \tag{2.9}$$

$$\|Jy_{n-1} - Jy_n\| \le c_2\delta_{E^*}^{-1}\bigg(\frac{\theta_{n-1} - \theta_n}{\theta_n}D\bigg), \tag{2.10}$$

where D is the constant defined in equation (2.8) and δ_E denotes the modulus of convexity of a normed space E (see, e.g., Lindenstrauss and Tzafriri [31], Chidume [9]), $\{y_n\}$ and $\{\theta_n\}$ are as defined in Corollary 2.1 (see, Chidume and Idu, [16], Remark 1).

Lemma 2.8 (Kamimura and Takahashi, [29]). *Let E be a uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded and $\phi(u_n, v_n) \to 0$ then $\|u_n - v_n\| \to 0$.*

Lemma 2.9 (Xu, [38]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\sigma_n)a_n + \sigma_n b_n + c_n,\ n \ge 1, \tag{2.11}$$

where $\{\sigma_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the conditions:

$$(i)\ \ \{\sigma_n\} \subset [0,1],\ \sum_{n=1}^{\infty}\sigma_n = \infty; \qquad (ii)\ \limsup_{n\to\infty} b_n \le 0; \qquad (iii)\ c_n \ge 0,\ \sum_{n=1}^{\infty} c_n < \infty.$$

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.10 (Chidume and Idu, [16]). *Let X, Y be real uniformly convex and uniformly smooth spaces. Then $E = X \times Y$ is uniformly convex and uniformly smooth.*

Lemma 2.11 (Chidume and Idu, [16]). *Let E be a uniformly convex and uniformly smooth real Banach Banach space and $F : E \to E^*, K : E^* \to E$ be maximal monotone. Define $A : E \times E^* \to E^* \times E$ by $A[u, v] = [Fu - v, Kv + u]$, $\forall [u, v] \in E \times E^*$. Then, A is maximal monotone.*

Remark 2.2. The following estimates (see, Uba *et al.* [40], Remark 2) will be needed in the sequel.

$$Jy_n + \frac{1}{\theta_n}(Fy_n - y_n^*) = 0, \ \forall n \geq 1, \ and \tag{2.12}$$

$$J_* y_n^* + \frac{1}{\theta_n}(Ky_n^* + y_n) = 0, \ \forall n \geq 1, \tag{2.13}$$

Remark 2.3. Let $y_n \to u^*$ and $y_n^* \to v^*$. From Lemma 2.7 we have that $[y_n, y_n^*]$ converges to a point in $A^{-1}0$. This implies that $[u^*, v^*] \in A^{-1}0$. Consequently, $A[u^*, v^*] = 0$, that is $Fu^* - v^* = 0$ and $Kv^* + u^* = 0$. Hence, $v^* = Fu^*$ and $u^* + KFu^* = 0$.

Definition 2.3. If $A : E \to 2^{E^*}$ is monotone with $0 \in Int\ D(A)$, then A is *quasi-bounded*, i.e., if for any $M > 0$ there exists $C > 0$ such that $(y, v) \in G(A)$, $\langle y, v \rangle \leq M\|y\|$ and $\|y\| \leq M$ implies $\|v\| \leq C$ (see I. Cioranescu [23], p. 176).

Lemma 2.12. *Let E be a real normed space with dual space E^*. Any monotone map $A : D(A) \subset E \to E^*$ with $0 \in IntD(A)$ is quasi-bounded.*

3. Main result

In Theorem 3.4 below, the sequences $\{\alpha_n\}$ and $\{\theta_n\}$ are in $(0, 1)$ and are assumed to satisfy the following conditions:

(i) $\delta_E^{-1}(\alpha_n M_0) \leq \theta_n \gamma_0;\ \alpha_n M_1 \leq \theta_n \gamma_0,$

(ii) $\delta_{E^*}^{-1}(\alpha_n M_0^*) \leq \theta_n \gamma_0;\ \alpha_n M_1^* \leq \theta_n \gamma_0,$

for all $n \geq 1$ and for some constants, $M_0, M_0^*, M_1, M_1^*, \gamma_0 > 0$.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$u_{n+1} = J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n\Big), \tag{3.14}$$

$$v_{n+1} = J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n\Big). \tag{3.15}$$

Assume that the equation $u + KFu = 0$ has a solution u^*, with $v^* = Fu^*$. Then, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Proof. To show that the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, set $w_n = (u_n, v_n), w^* = (u^*, v^*) \in W = E \times E^*$, where u^* is a solution (1.1) with $v^* = Fu^*$. Define $\Phi : W \times W \to \mathbb{R}$ by $\Phi(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2)$, where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. Let W be endowed with norm $\|(u, v)\|_W = \Big(\|u\|_E^2 + \|v\|_{E^*}^2\Big)^{\frac{1}{2}}$. It suffices to show that $\{w_n\}$ is bounded. We show this by induction. Let $w_1 \in W$. Then there exists $r > 0$ such that $\|w^*\|_W \leq \frac{r}{4}$ and $\Phi(w^*, w_1) \leq \frac{r}{4}$. Let $B := \big\{w = (u, v) \in W : \Phi(w^*, w) \leq r\big\}$. It suffices to show that $\Phi(w^*, w_n) \leq r$, for all $n \geq 1$. Let $w \in B$ and $\theta \in (0, 1)$. Then, $\Phi(w^*, w) \leq r$ *i.e.*, $\phi(u^*, u) + \phi(v^*, v) \leq r$.Therefore, $\phi(u^*, u) \leq r$ and $\phi(v^*, v) \leq r$. Now, using inequality (2.6), $\phi(u^*, u) \leq r \Rightarrow \|u\| \leq \|u^*\| + \sqrt{r}$. Since F is also locally bounded at u, there exists $k_1 > 0$ such that $\langle u, Fu \rangle \leq k_1\|u\|$. Define $\sigma := \max\{k_1, \|u^*\| + \sqrt{r}\}$. Hence, $\langle u, Fu \rangle \leq \sigma\|u\|$ and $\|u\| \leq \sigma$. By Lemma 2.12, F is quasi-bounded. Thus, there exists $\tau_1 > 0$ such that $\|Fu\| \leq \tau_1,\ \forall(u,v) \in B$. Similarly, there exists $\tau_2 > 0$ such that $\|Kv\| \leq \tau_2,\ \forall(u,v) \in B$. Define:

$$M_1 = \sup\Big\{\|Fu - v + \theta Ju\|\Big\} + 1; \qquad M_2 = \sup\Big\{\|u - u^*\|\Big\} + 1;$$
$$M_1^* = \sup\Big\{\|Kv + u + \theta J^{-1}v\|\Big\} + 1; \qquad M_2^* = \sup\Big\{\|v - v^*\|\Big\} + 1;$$

Let $M := \max\left\{c_2M_1, c_2M_1^*, M_2, M_2^*\right\}$, $\gamma_0 := \min\Big\{1, \frac{r}{16M}\Big\}$. Then, for $n = 1$, by construction $\Phi(w^*, w_1) \leq r$. Assume $\Phi(w^*, w_n) \leq r$, for some $n \geq 1$, i.e., $\phi(u^*, u_n) + \phi(v^*, v_n) \leq r,\ for\ some\ n \geq 1$. We show that $\Phi(w^*, w_{n+1}) \leq r$. For contradiction, suppose $r < \Phi(w^*, w_{n+1})$. Observe that $\|u_{n+1} - u_n\| = \|J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_nJu_n\Big) - J^{-1}(Ju_n)\|$. Now, using Lemma 2.4 and recurrence relation (3.14), we have

$$\begin{aligned}(2L)^{-1}\delta_E(c_2^{-1}\|u_{n+1} - u_n\|) &\leq \langle Ju_{n+1} - Ju_n, u_{n+1} - u_n\rangle \\ &\leq \|Ju_{n+1} - Ju_n\|\|u_{n+1} - u_n\| \leq \alpha_nM_1\|u_{n+1} - u_n\|.\end{aligned}$$

(3.16) Thus, $$\|u_{n+1} - u_n\| \leq c_2\delta_E^{-1}(\alpha_nM_0),\ for\ some\ M_0 > 0.$$

Similarly, using Lemma 2.4 and recurrence relation (3.15), we obtain

$$\|v_{n+1} - v_n\| \leq c_2\delta_{E^*}^{-1}(\alpha_nM_0^*),\ for\ some\ M_0^* > 0. \tag{3.17}$$

Now, using recurrence relation (3.14), Lemma 2.1, and inequality (3.16), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &= V(u^*, Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_nJu_n) \\ &\leq V(u^*, Ju_n) - 2\langle u_{n+1} - u^*, \alpha_n(Fu_n - v_n) + \alpha_n\theta_nJu_n\rangle \qquad (3.18)\\ &= \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_nJu_n\rangle \\ &\quad - 2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n + \theta_nJu_n\rangle \\ &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_nJu_n\rangle \\ &\quad + 2\alpha_n\|u_{n+1} - u_n\|\|Fu_n - v_n + \theta_nJu_n\| \\ &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_nJu_n\rangle + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0).\end{aligned}$$

Observe that by monotonicity of F and the fact that $v^* = Fu^*$, we have $\langle u_n - u^*, Fu_n - v_n + \theta_nJu_n\rangle \geq \langle u_n - u^*, v^* - v_n + \theta_nJu_n\rangle$. Thus, substituting this in inequality (3.18), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n + \theta_nJu_n\rangle + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0) \\ &= \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\theta_n\langle u_n - u^*, Ju_n - Ju_{n+1}\rangle \\ &\quad - 2\alpha_n\theta_n\langle u_n - u^*, Ju_{n+1}\rangle + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0) \qquad (3.19)\end{aligned}$$

Using Lemma 2.2, we have $-2\alpha_n\theta_n\langle u_n - u^*, Ju_{n+1}\rangle \leq \alpha_n\theta_n\|u^*\|^2 - \alpha_n\theta_n\phi(u^*, u_{n+1})$. Substituting this in inequality (3.19), we obtain

$$\begin{aligned}\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\theta_n\langle u_n - u^*, Ju_n - Ju_{n+1}\rangle \\ &\quad + \alpha_n\theta_n\|u^*\|^2 - \alpha_n\theta_n\phi(u^*, u_{n+1}) + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0) \qquad (3.20)\\ &\leq \phi(u^*, u_n) - \alpha_n\theta_n\phi(u^*, u_{n+1}) + \alpha_n\theta_n\|u^*\|^2 + 2\alpha_n\theta_n\|u_n - u^*\|\|Ju_n - Ju_{n+1}\| \\ &\quad + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle \\ &\leq \phi(u^*, u_n) - \alpha_n\theta_n\phi(u^*, u_{n+1}) + \alpha_n\theta_n\|u^*\|^2 + 2\alpha_n\theta_nM_2(\alpha_nM_1) \\ &\quad + 2\alpha_nc_2M_1\delta_E^{-1}(\alpha_nM_0) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle.\end{aligned}$$

Similarly, using recurrence relation (3.15), Lemma 2.1, inequality (3.17), monotonicity of K, the fact that $Kv^* = -u^*$ and Lemma 2.2, we obtain

$$\begin{aligned}\phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - \alpha_n\theta_n\phi(v^*, v_{n+1}) + \alpha_n\theta_n\|v^*\|^2 + 2\alpha_n\theta_n M_2^*(\alpha_n M_1^*)\\ &\quad + 2\alpha_n c_2 M_1 \delta_E^{-1}(\alpha_n M_0^*) - 2\alpha_n\langle v_n - v^*, u_n - u^*\rangle. \end{aligned} \tag{3.21}$$

Thus, adding inequalities (3.20) and (3.21), we obtain

$$\begin{aligned} r < \Phi(w^*, w_{n+1}) &= \phi(u^*, u_{n+1}) + \phi(v^*, v_{n+1})\\ &\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 + 2\alpha_n\theta_n M_2(\alpha_n M_1)\\ &\quad + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) + 2\alpha_n\theta_n M_2^*(\alpha_n M_1^*) + 2\alpha_n c_2 M_1 \delta_E^{-1}(\alpha_n M_0^*)\\ &\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 + 2\alpha_n\theta_n^2 M_2\gamma_0\\ &\quad + 2\alpha_n\theta_n\gamma_0 c_2 M_1 + 2\alpha_n\theta_n^2 M_2^*\gamma_0 + 2\alpha_n\theta_n\gamma_0 c_2 M_1^*\\ &\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 + 2\alpha_n\theta_n M\gamma_0\\ &\quad + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0\\ &\leq r - \alpha_n\theta_n r + \frac{3}{4}\alpha_n\theta_n r = r - \frac{1}{4}\alpha_n\theta_n r < r. \end{aligned}$$

This is a contradiction. Hence, $\Phi(w^*, w_{n+1}) \leq r$. Thus, $\Phi(w^*, w_n) \leq r$, for all $n \geq 1$. Consequently, we have $\phi(u^*, u_n) \leq r$ and $\phi(v^*, v_n) \leq r$, for all $n \geq 1$. Therefore, using inequality (2.6), we deduce that $\{u_n\}$ and $\{v_n\}$ are bounded. □

In Theorem 3.5 below, $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}\alpha_n\theta_n = \infty$, (ii) $\delta_E^{-1}(\alpha_n M_0) \leq \theta_n^2\gamma_0$, (iii) $\delta_{E^*}^{-1}(\alpha_n M_0^*) \leq \theta_n^2\gamma_0$,

(iv) $\delta_E^{-1}(\eta_n) \to 0$; $\delta_{E^*}^{-1}(\eta_n) \to 0$, (v) $\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n} \to 0$; $\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n} \to 0$,

where $\eta_n = \left(\frac{\theta_{n-1}-\theta_n}{\theta_n}D\right)$.

We now prove our main Theorem.

Theorem 3.5. *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$\begin{aligned} u_{n+1} &= J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n\Big),\\ v_{n+1} &= J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n\Big), \end{aligned} \tag{3.22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying conditions (i)-(v). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. Using Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \phi(y_n, u_{n+1}) &= V(y_n, Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n)\\ &\leq V(y_n, Ju_n) - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle\\ &= \phi(u_n, y_n) - 2\langle u_n + y_n, Ju_n - Jy_n\rangle + 2(\|u_n\|^2 - \|y_n\|^2)\\ &\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle \end{aligned} \tag{3.23}$$

Observe that

$$\begin{aligned}\phi(u_n, y_n) &= V(u_n, Jy_n) = V(u_n, Jy_{n-1} + Jy_n - Jy_{n-1})\\ &\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle.\end{aligned}$$

Thus, substituting this in inequality (3.23), and using Lemmas 2.3 and 2.2 we obtain

$$\begin{aligned}\phi(y_n, u_{n+1}) &\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle\\ &\quad + 2(\|u_n\|^2 - \|y_n\|^2) - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle \qquad (3.24)\\ &= \phi(y_{n-1}, u_n) + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle + 2(\|y_{n-1}\|^2 - \|y_n\|^2)\\ &\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle\\ &\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle\\ &\quad - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle\\ &\quad - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle\\ &\leq \phi(y_{n-1}, u_n) + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle + 2(\|y_{n-1}\|^2 - \|y_n\|^2)\\ &\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle\\ &\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle\\ &\quad - \alpha_n\theta_n\phi(y_{n-1}, u_n) - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle\\ &= (1 - \alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle\\ &\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle\\ &\quad \underline{-2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle} - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle\\ &\quad \underline{-2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle} - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle\end{aligned}$$

We now estimate the underlined terms. Using equation (2.12) and the fact that F is monotone, we obtain

$$\begin{aligned}&-2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle\\ &\quad = -2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle - 2\alpha_n\langle u_n - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle\\ &\qquad - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle + 2\alpha_n\langle u_n - y_n, Fy_n - y_n^*\rangle\\ &\quad \leq -2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle\\ &\qquad - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle.\end{aligned}$$

Thus, substituting this in inequality (3.24), and using inequalities (2.9), (2.10) and (3.16), we obtain

$$\begin{aligned}\phi(y_n, u_{n+1}) &\leq (1 - \alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle\\ &\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle \qquad (3.25)\\ &\quad - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle\\ &\quad - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle\\ &\quad - 2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle\\ &\leq (1 - \alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2N_1(\|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|)\\ &\quad + 2\alpha_n\theta_n N_2(\|u_{n+1} - u_n\| + \|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|)\\ &\quad + 2\alpha_n N_3\|u_{n+1} - u_n\| + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle\end{aligned}$$

$$\leq (1-\alpha_n\theta_n)\phi(y_{n-1},u_n) + 2N_1\Big(c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big) + 2\alpha_n\theta_n N_2\Big(c_2\delta_E^{-1}(\alpha_n M_0)$$
$$+ c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big) + 2\alpha_n N_3 c_2\delta_E^{-1}(\alpha_n M_0) + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle$$
$$\leq (1-\alpha_n\theta_n)\phi(y_{n-1},u_n) + \alpha_n\theta_n\widehat{N}\Bigg(c_2\delta_E^{-1}(\alpha_n M_0) + c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)$$
$$+ \theta_n c_2\gamma_0 + c_2\delta_E^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big) + c_2\delta_{E^*}^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big)\Bigg) + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle,$$

for some $N_1, N_2, N_3 > 0$, and $\widehat{N} = \max\{N_1, N_2, N_3\}$. Similarly, using Lemmas 2.1, 2.3 and 2.2, equation (2.13), we obtain

$$\phi(y_n^*, v_{n+1}) \leq (1-\alpha_n\theta_n)\phi(y_{n-1}^*, v_n) + 2(\|y_{n-1}^*\|^2 - \|y_n^*\|^2) + 2\langle y_{n-1}^* - y_n^*, J^{-1}v_n - J^{-1}y_{n-1}\rangle$$
$$+2\langle y_n^* + v_n, J^{-1}y_n - J^{-1}y_{n-1}^*\rangle - 2\langle y_n^* - v_n, J^{-1}y_{n-1}^* - J^{-1}y_n^*\rangle$$
$$-2\alpha_n\theta_n\langle v_{n+1} - v_n, J^{-1}v_n\rangle - 2\alpha_n\theta_n\langle y_{n-1}^* - y_n^*, J^{-1}v_n\rangle - 2\alpha_n\theta_n\langle y_n^* - y_{n-1}^*, J^{-1}y_{n-1}^*\rangle$$
$$-2\alpha_n\theta_n\langle v_n - y_n^*, J^{-1}y_{n-1} - J^{-1}y_n^*\rangle - 2\alpha_n\langle v_{n+1} - v_n, Kv_n + u_n\rangle + 2\alpha_n\langle v_n - y_n^*, y_n - u_n\rangle.$$

Thus, using inequalities (2.9), (2.10) and (3.17), we obtain

$$\phi(y_n^*, v_{n+1}) \leq (1-\alpha_n\theta_n)\phi(y_{n-1}^*, v_n) + \alpha_n\theta_n\widehat{N}^*\Bigg(c_2\delta_{E^*}^{-1}(\alpha_n M_0^*) + c_2\delta_{E^*}^{-1}(\eta_n) + c_2\delta_E^{-1}(\eta_n)$$
$$+ \theta_n c_2\gamma_0 + c_2\delta_{E^*}^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big) + c_2\delta_E^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big)\Bigg) + 2\alpha_n\langle v_n - y_n^*, y_n - u_n\rangle, \tag{3.26}$$

for some $\widehat{N}^* > 0$. Let $p_n = (y_n, y_n^*)$, adding inequalities (3.25) and (3.26) we obtain

$$\Phi(p_n, w_{n+1}) \leq (1-\alpha_n\theta_n)\Phi(p_{n-1}, w_n) + \alpha_n\theta_n N\Bigg(c_2\delta_E^{-1}(\alpha_n M_0) + 2c_2\delta_E^{-1}(\eta_n) + 2c_2\delta_{E^*}^{-1}(\eta_n)$$
$$+ 2\theta_n c_2\gamma_0 + 2c_2\delta_E^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big) + 2c_2\delta_{E^*}^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big) + c_2\delta_{E^*}^{-1}(\alpha_n M_0^*)\Bigg), \tag{3.27}$$

where $N = \max\{\widehat{N}, \widehat{N}^*\}$. Now, setting $a_n = \Phi(p_{n-1}, w_n);\ \sigma_n = \alpha_n\beta_n;\ c_n \equiv 0$ and

$$b_n := N\Bigg(c_2\delta_E^{-1}(\alpha_n M_0) + 2c_2\delta_E^{-1}(\eta_n) + 2c_2\delta_{E^*}^{-1}(\eta_n) + 2\theta_n c_2\gamma_0 + 2c_2\delta_E^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big)$$
$$+ 2c_2\delta_{E^*}^{-1}\Big(\frac{\eta_n}{\alpha_n\theta_n}\Big) + c_2\delta_{E^*}^{-1}(\alpha_n M_0^*)\Bigg),$$

inequality (3.27) becomes $a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n,\ n \geq 1$. It follows from Lemma 2.9 that $\Phi(p_{n-1}, w_n) \to 0$, as $n \to \infty$. By Lemma 2.8, we have $\|w_n - p_{n-1}\|_W \to 0$. Consequently, $\|u_n - y_{n-1}\| \to 0$. Furthermore, using Remark 2.3, since $[y_n, y_n^*] \to [u^*, v^*] \in A^{-1}0$, we have that $\{u_n\}$ converges to a solution of the Hammerstein equation (1.1) with $v^* = Fu^*$. This completes the proof. $\square$

Remark 3.4. Real sequences that satisfy the hypothesis of above theorem are $\alpha_n = (n+1)^{-a}$ and $\theta_n = (n+1)^{-b}$ with $0 < b < a$ and $a + b < 1$.

4. Numerical illustration

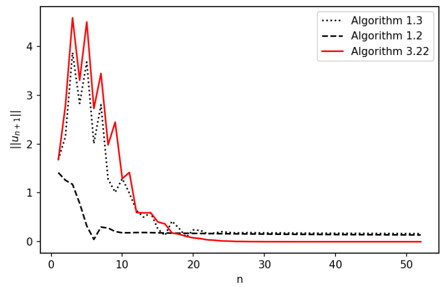
In this section, we present a numerical example to compare the convergence of a sequence generated algorithms (1.3) and (1.2), and our algorithm, algorithm (3.22).

Example 4.1. In Theorems 1.2, 1.1 and 3.5 set $E = \mathbb{R}^2$, $E^* = \mathbb{R}^2$,

$$Fu = (u_1 + u_2 + \sin u_1, -u_1 + u_2 + \sin u_2), \qquad Kv = (v_1 + v_2, v_1 + v_2).$$

Then, it is easy to see that F and K are monotone and the vector $u^* = (0,0)$ is the only solution of the equation $u + KFu = 0$. In algorithms (1.3) and (3.22), we take $\alpha_n = \lambda_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{5}}}$, $n = 1, 2, \cdots$, and in algorithm (1.2), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2, 1.1 and 3.5, respectively. Choosing $u_1 = (1,0)$, $v_1 = (1,1)$, $n = 5000$ and using a tolerance of 10^{-8} we obtain the following iterates. And in the graph below, y-axis represents the values of $\|u_{n+1} - \mathbf{0}\|$ while the x-axis represents the number of iterations n.

	Alg. (1.3)	Alg. (1.2)	Alg. (3.22)
CPU time	0.43 sec	0.49 sec	0.21 sec
No. iter.	$\|u_{k+1}\|$	$\|u_{k+1}\|$	$\|u_{k+1}\|$
1	1.6817	1.4142	1.6817
10	1.2912	0.1825	1.2932
30	0.1811	0.1566	0.0008
52	0.1656	0.1385	8.8×10^{-9}
5000	0.0703	0.0465	-



Remark 4.5. Observe that in this experiment and with the specified tolerance, the sequence of our iteration process converges after 52 iterations, whereas, after 5,000 iterations the sequences of algorithms (1.3) and (1.2), with this given tolerance, are yet to converge. From the results obtained, Algorithm (3.22) would, perhaps, be preferred to either Algorithm (1.3) or Algorithm (1.2) in any possible application.

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