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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} *anniversary*

Efficient solution and value function for non-convex variational problems

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ABSTRACT. In this paper, we want to investigate a wide range of non-convex variational problems and obtain some sufficient and necessary conditions for existence of a feasible solution for these problems. Hence, we define optimal value function corresponding to these problems and obtain a relationship between subdifferential of the optimal value function and the set of Lagrange multipliers.

1. INTRODUCTION

Non-convex variational problems have played a crucial role in mathematical economics. One of the most important subjects in economic observations is to analyze the relationship between traders and productions. In this paper, our purpose is to find the total cost of a material that is purchased by a trader. The outline of this paper is as follows: In this Section, we define the non-convex variational problem and some preliminary definitions and results which will used in the sequel. In Section 2, we obtain some sufficient and necessary conditions for be a feasible solution for the non-convex variational problem.
In Section 3, we define optimal value function corresponding to the non-convex variational problem and obtain a relationship between subdifferential of the optimal value function and the set of Lagrange multipliers.
Let X, Y and Z be Banach spaces, K and W be closed, convex and pointed cones in Y and Z, respectively. Cone K induces a partial ordering on Y, defined so that $x \leq_K y$ if and only if $y - x \in K$. We denote the positive polar cone of K by K^{++} as

$$K^{++} = \{y^* \in Y^* : \langle y^*, y\rangle > 0 \ \forall y \in K\}.$$

Definition 1.1 (Definition A.10 [8])**.** Let J be an interval in X, then a function $f : J \longrightarrow Y$ is Bochner integrable if there exists a sequence of simple functions $f_n : J \longrightarrow Y$ such that the following two conditions are satisfied

$$\lim_{n\longrightarrow\infty} f_n(t) = f(t), \text{ almost everywhere,}$$

and

$$\lim_{n\longrightarrow\infty} \int_J \|f_n(t) - f(t)\| = 0.$$

Definition 1.2 (Definition 2.4. [11])**.** Function $f : X \longrightarrow Y$ is called K-convex, if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1-\lambda)f(y) \in f(\lambda x + (1-\lambda)y) + K.$$

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It is called K-concave, if $-f$ is K-convex.

Suppose $\Gamma : [0,1] \longrightarrow 2^X$ is a set-valued mapping such that for all $t \in [0,1]$, $\Gamma(t)$ is a nonempty and closed set and

$$\mathcal{Z} = \{\gamma \in L^1([0,1], X) : \ \gamma(t) \in \Gamma(t) \ a.e. \ t \in [0,1]\}.$$

Suppose that $f_0 : [0,1] \times X \longrightarrow Y$ and $g_0 : [0,1] \times X \longrightarrow Z$ are Bochner integrable functions. Let us define the maps $f : L^1([0,1], X) \longrightarrow Y$ and $g : L^1([0,1], X) \longrightarrow Z$ as

$$f(\gamma) = \int_0^1 f_0(t, \gamma(t))dt,$$

$$g(\gamma) = \int_0^1 g_0(t, \gamma(t))dt,$$

where g_0 is W-convex in the second argument. We consider the following optimization problem

$$\min_{\gamma \in K(a)} f(\gamma) \qquad\qquad (OP(a))$$

where for all $a \in Z$, we define $K(a) = \{\gamma \in \mathcal{Z} : \ g(\gamma) \in -W + a\}$, clearly if g_0 is a W-convex function in the second argument and by the convexity of W, we get the convexity of $K(a)$. We denote the solution set of Problem $(OP(a))$ by $S(a)$. The Problem $(OP(a))$ is a generalization of the Problem (1.1) in [2], the Problem (P) in [1], the Problem $(\mathcal{P})$ in [12] and the Problem (P) in [14] . Hence, the amount of the material j to be purchased by the trader t is denoted by $\gamma_j(t)$. Therefore, the Problem $(OP(a))$ means the total cost of set $K(a)$ that consists of materials γ. The existence result due to Bazan et al. [2] requires
(i) the lower semi-continuity of $f_0(t, .)$;
(ii) continuity for a.e. $t \in [0,1]$ of $g_0(t, .)$;
(iii) measurablity of $g_0(t, .)$;
(iv) f_0 is a Borel function;
(v) $f_0(., \gamma(.))$ and $g_0(., \gamma(.))$ are member of $L^1([0,1], \mathbb{R}^n)$.
Our goal is to obtain a solution for Problem $(OP(a))$ in Banach spaces where the functions $f_0(., \gamma(.))$ and $g_0(., \gamma(.))$ are Bochner integrable. For undefined notions we refer to [7].

Definition 1.3. A Set-valued operator $T : X \longrightarrow 2^Y$ is called:
(a) closed if $\mathrm{Gr}(T) = \{(x,y) \in X \times Y : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.
(b) intersectionally closed on $A \subseteq X$, if;

$$\bigcap_{x \in A} cl\big(T(x)\big) = cl\big(\bigcap_{x \in A} T(x)\big).$$

(c) topological pseudomonotone, if for all $a, b \in Y$,

$$cl\big(\bigcap_{u \in [a,b]} T(u)\big) \cap [a,b] = \bigcap_{u \in [a,b]} T(u) \cap [a,b],$$

where $[a,b] = \{y \in Y : \ a \leq_K y \leq_K b\}$.
(d) KKM map, if

$$\mathrm{conv} H \subseteq \bigcup_{x \in H} T(x), \ \text{for each } H \in \langle X \rangle,$$

where we denote by $\langle X \rangle$ the family of all nonempty finite subsets of the set X.

Definition 1.4 (Definition 3.6.1 in [6]). Let X be a Banach space and A be a nonempty subset of X. The Normal cone of A, $N(A,a) : X \longrightarrow 2^{X^*}$ is defined as

$$N(A,a) = \begin{cases} \{v \in X^* : \langle v, x-a\rangle \leq 0 \ \forall x \in A\} & \text{if } a \in A, \\ \emptyset & \text{o.w,} \end{cases}$$

Theorem 1.1 (Theorem 2 in [9]). *Let K be a nonempty and convex subset of a Hausdorff topological vector space X and $T : K \longrightarrow 2^K$. Suppose that the following conditions hold:*
$(A1)$ T is a KKM map;
$(A2)$ for each $H \in \langle K\rangle$, the set-valued map $T \cap \mathrm{conv}H$ is intersectionally closed on $\mathrm{conv}H$;
$(A3)$ T is topological pseudomonotone;
$(A4)$ there exist a nonempty subset B of K and a nonempty compact subset D of K such that $\mathrm{conv}(H \cup B)$ is compact for any $H \in \langle K\rangle$, and for each $y \in K\setminus D$, there exists $x \in \mathrm{conv}(B \cup \{y\})$ such that $y \notin T(x)$.
Then, $\bigcap_{x\in K} T(x) \neq \emptyset$.

2. Existence of solution for problem $(OP(a))$

In this section, we consider sufficient conditions for the existence of a solution of Problem $(OP(a))$. In the following, we obtain an existence result of Problem $(OP(a))$ by using a fixed point theorem.

Theorem 2.2. *Suppose the set-valued map $T : K(a) \cap (L^1([0,1], X)) \longrightarrow 2^{K(a)\cap(L^1([0,1],X))}$ defined as*

$$T(\gamma) = \{\alpha \in L^1([0,1], X) \cap K(a) : \ f(\gamma) - f(\alpha) \in K\}.$$

Also consider the following conditions hold:
(a) $K \subseteq Y$ is a pointed cone (not necessary convex) such that $Y = K \cup -K$;
(b) f_0 is a lower semi continuous map and K-convex in the second argument;
(c) g_0 is a lower semi continuous map and W-convex in the second argument;
(d) there exist a nonempty subset B of $(K(a) \cap L^1([0,1], X))$ and a nonempty compact subset D of $(K(a) \cap L^1([0,1], X))$ such that $\mathrm{conv}(H \cup B)$ is compact for any $H \in \langle K\rangle$, and for each $\alpha \in K \setminus D$ there exists $\gamma \in \mathrm{conv}(B \cup \{\alpha\})$ such that $\alpha \notin T(\gamma)$.
Then Problem $(OP(a))$ has a solution.

Proof. We will show that T is a KKM map. Suppose on the contrary that T is not a KKM map, then there exists $H = \{\gamma_1, \ \gamma_2, \ ..., \ \gamma_n\} \subseteq (K(a) \cap L^1([0,1], X))$, $t_i \geq 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that $\gamma = \sum_{i=1}^n t_i\gamma_i \notin \bigcup_{i=1}^n T(\gamma_i)$. Thus for each $i = 1, 2, ..., n$, we have $\gamma \notin T(\gamma_i)$, therefore $f(\gamma_i) - f(\gamma) \notin K$. Since $Y = K \cup -K$, then $f(\gamma) - f(\gamma_i) \in K$. For all ${y_0}^* \in K^{++}$ and for all i

$$\int_0^1 {y_0}^*(f_0(t, \gamma(t)) - f_0(t, \gamma_i(t)))dt = {y_0}^*(\int_0^1 f_0(t, \gamma(t))dt - \int_0^1 f_0(t, \gamma_i(t))dt) > 0. \tag{2.1}$$

Therefore

$${y_0}^*(f_0(t, \gamma(t)) - f_0(t, \gamma_i(t))) > 0 \qquad \text{a.e. } \ t \in [0,1]. \tag{2.2}$$

Since for all $i \in \{1, 2, ..., n\}$, $\ t_i \geq 0$, then for almost everywhere $t \in [0,1]$ we have

$${y_0}^*(f_0(t, \gamma(t)) - \Sigma_{i=1}^n t_i f_0(t, \gamma_i(t))) = \Sigma_{i=1}^n t_i({y_0}^*(f_0(t, \gamma(t)) - f_0(t, \gamma_i(t)))) \geq 0, \tag{2.3}$$

on the other hand, f_0 is K-convex in the second argument,

$$\Sigma_{i=1}^n t_i f_0(t, \gamma_i(t)) - f_0(t, \gamma(t)) = \Sigma_{i=1}^n t_i f_0(t, \gamma_i(t)) - f_0(t, \Sigma_{i=1}^n t_i\gamma_i(t)) \in K, \tag{2.4}$$

then $y_0{}^*(\Sigma_{i=1}^n t_i f_0(t,\gamma_i(t)) - f_0(t,\gamma(t))) > 0$ so,

$$y_0{}^*(f_0(t,\gamma(t)) - \Sigma_{i=1}^n t_i f_0(t,\gamma_i(t))) < 0, \tag{2.5}$$

which contradicts (2.3), thus T is a KKM map and therefore,

$$f_0(t,\gamma_i(t)) - f_0(t,\gamma(t)) \in K, \qquad a.e. \quad t \in [0,1].$$

Now we show that for each $H \in \langle A \rangle$, the set-valued map $T \cap \mathrm{conv}(H)$ is intersectionally closed on $\mathrm{conv}(H)$; i.e.,

$$\bigcap_{\gamma \in \mathrm{conv}H} cl(T(\gamma) \cap \mathrm{conv}H) = cl(\bigcap_{\gamma \in \mathrm{conv}H} (T(\gamma) \cap \mathrm{conv}H)). \tag{2.6}$$

The inclusion $\supset$ in (2.6), is always true. Conversely; let $\alpha \in \bigcap_{\gamma \in \mathrm{conv}H} cl(T(\gamma) \cap \mathrm{conv}H)$, then for all $\gamma \in \mathrm{conv}H$, we have $\alpha \in cl(T(\gamma) \cap \mathrm{conv}H)$ and there exists a sequence $(\alpha_n) \subseteq T(\gamma) \cap \mathrm{conv}H$ such that $\alpha_n \longrightarrow \alpha$. Since $\alpha_n \in T(\gamma)$ and f_0 is lower semi continuous, then $\alpha \in T(\gamma)$. So for all $\gamma \in \mathrm{conv}H$, we have $\alpha \in T(\gamma) \cap \mathrm{conv}H$ and

$$\alpha \in \bigcap_{\gamma \in \mathrm{conv}H} (T(\gamma) \cap \mathrm{conv}H) \subseteq cl(\bigcap_{\gamma \in \mathrm{conv}H} (T(\gamma) \cap \mathrm{conv}H)).$$

Here we show that T is topological pseudomonotone. For fixed $a, b \in X$, obviously:

$$\bigcap_{\gamma \in [a,b]} (T(\gamma) \cap [a,b]) = (\bigcap_{\gamma \in [a,b]} T(\gamma)) \cap [a,b] \subseteq cl(\bigcap_{\gamma \in [a,b]} T(\gamma)) \cap [a,b].$$

Conversely, let $\alpha \in cl(\bigcap_{\gamma \in [a,b]} T(\gamma)) \cap [a,b]$, then there exists a sequence $(\alpha_n) \subseteq (\bigcap_{\gamma \in [a,b]} T(\gamma)) \cap [a,b]$ such that $\alpha_n \longrightarrow \alpha$. Since $\alpha_n \in T(\gamma)$ and f_0 is lower semi continuous in the second argument, then for all $\gamma \in [a,b]$ we have $\alpha \in T(\gamma)$. So for all $\gamma \in [a,b]$, we have $\alpha \in T(\gamma) \cap [a,b]$ and therefore,

$$\alpha \in \bigcap_{\gamma \in [a,b]} (T(\gamma) \cap [a,b]).$$

By Theorem 1.1 we have $\bigcap_{\gamma \in A} T(\gamma) \neq \emptyset$, let γ_0 be in this intersection, then γ_0 is a solution of Problem $(OP(a))$. □

Let $\gamma_0 \in L^1([0,1], X)$. As a reminder, if there exists a neighbourhood U of γ_0 such that

$$f(\gamma) - f(\gamma_0) \in K \;\forall \gamma \in U \cap K(a),$$

then γ_0 is a local efficient solution of Problem $(OP(a))$. Hence every solution of Problem $(OP(a))$ is said to be a global efficient solution.
Bazan et al. [2] showed that if f_0 and g_0 are measurable, lower semi-continuous and continuous, respectively and W is a closed convex cone, then $0 \in int[g(C_0) + W]$ and each local efficient solution for $(OP(a))$ is a global efficient solution. Here, we shall obtain by different assumptions that each local efficient solution of $(OP(a))$ is a global efficient solution.

Theorem 2.3. *Let f_0 be a lower semi continuous and K-convex function in the second argument and g_0 be W-convex function in the second argument. Then each local efficient solution of $(OP(a))$ is a global efficient solution.*

Proof. Let $\gamma_0 \in L^1([0,1], X) \cap K(a)$ be a local efficient solution, then there exists a neighbourhood U of γ_0 such that

$$f(\gamma) - f(\gamma_0) \in K \qquad \forall \gamma \in U \cap K(a),$$

therefore,

$$\int_0^1 f_0(t,\gamma(t))dt - \int_0^1 f_0(t,\gamma_0(t))dt \in K\ \forall \gamma \in U \cap K(a).$$

So for all $y^* \in K^+$

$$y^*(\int_0^1 (f_0(t,\gamma(t)) - f_0(t,\gamma_0(t)))dt) = \int_0^1 y^*(f_0(t,\gamma(t)) - f_0(t,\gamma_0(t)))dt \geq 0\ \forall \gamma \in U \cap K(a).$$

Therefore, $y^*(f_0(t,\gamma(t)) - f_0(t,\gamma_0(t))) \geq 0$ almost everywhere $t \in [0,1]$ and

(2.7) $$f_0(t,\gamma(t)) - f_0(t,\gamma_0(t)) \in K\ a.e.\ t \in [0,1] \qquad \forall \gamma \in U \cap K(a).$$

Let $\alpha \in L^1([0,1],X) \cap K(a)$ be given. By W-convexity of g_0 for all $r \in]0,1[$, we have $\gamma_0 + r(\alpha - \gamma_0) \in K(a)$. On the other hand, there exists $r_0 \in (0,1)$ such that for all $r \in [0,r_0]$, we have $\gamma_0 + r(\alpha - \gamma_0) \in U \cap K(a)$ and from (2.7), we obtain

(2.8) $$f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)) - f_0(t,\gamma_0(t)) \in K \qquad a.e.\ t \in [0,1].$$

On the other hand, f_0 is K-convex in the second argument, then

$$-f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)) + (1-r)f_0(t,\gamma_0(t)) + rf_0(t,\alpha(t)) \in K,$$

and

$$-rf_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)) + (r-1)f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t))$$
$$+(1-r)f_0(t,\gamma_0(t)) + rf_0(t,\alpha(t)) \in K,$$

therefore

$$r(f_0(t,\alpha(t)) - f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)))$$
$$+(r-1)(f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)) - f_0(t,\gamma_0(t))) \in K.$$

From (2.8) and the above relation, we have

(2.9) $$f_0(t,\alpha(t)) - f_0(t,(\gamma_0 + r(\alpha - \gamma_0))(t)) \in K\ a.e.\ t \in [0,1]\ .$$

by summing (2.8) and (2.9), we obtain

$$f_0(t,\alpha(t)) - f_0(t,\gamma_0(t)) \in K\ a.e.\ t \in [0,1],$$

and for all $y^* \in K^+$, we obtain $y^*(f_0(t,\alpha(t)) - f_0(t,\gamma_0(t))) \geq 0$, therefore
$\int_0^1 y^*(f_0(t,\alpha(t)) - f_0(t,\gamma_0(t)))dt =$

$$y^*(\int_0^1 f_0(t,\alpha(t)) - f_0(t,\gamma_0(t))dt) = y^*(f(\alpha) - f(\gamma_0)) \geq 0.$$

So $f(\alpha) - f(\gamma_0) \in K$ for almost everywhere $t \in [0,1]$. □

Here, we consider sufficient conditions for the existence of a feasible solution of Problem $(OP(a))$.

Definition 2.5. Let $\eta : X \times X \longrightarrow X$.

- [15] A subset Ω of X is said to be invex with respect to η if for any $x, y \in \Omega$ and $\lambda \in [0,1]$, $y + \lambda\eta(x,y) \in \Omega$.
- [10] Let $\Omega \subset X$ be an invex set with respect to η and $F : \Omega \longrightarrow 2^Y$. F is said to be K-preinvex with respect to η on Ω if for any $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$, one has

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(x_2 + \lambda\eta(x_1,x_2)) + K.$$

Definition 2.6. [5] Given a subset A of a Banach space Y which is ordered by a closed and convex cone $K \subset Y$, we say that $\bar{a} \in A$ is a super minimal point of A $(\bar{a} \in SE(A;K))$ if there is a number $M > 0$ such that

$$cl[cone(A - \bar{a})] \cap (B_Y - K) \subset MB_Y,$$

where B_Y denotes the closed unit ball of Y. Let $(\bar{x}, \bar{y}) \in grF$ with $\bar{x} \in \Omega$, then $(\bar{x}, \bar{y})$ is a local super minimmizer of problem

$$\text{minimize } F(x), \text{ subject to } x \in \Omega,$$

if there is a neighbourhood U of $\bar{x}$ such that $\bar{y} \in SE(F(\Omega \cap U); K)$.
Obviously every local super efficient solution of the above problem is a local efficient solution.

Theorem 2.4. [13] *Let Ω be a closed and invex set, K be a closed convex pointed ordering cone in Y and $F : \Omega \longrightarrow 2^Y$ be a K-preinvex map with respect to η which is continuous with respect to the second argument. Suppose that $(\bar{x}, \bar{y}) \in GrF$ and there is a $y^* \in intK^+$ such that*

$$0 \in \partial F(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega).$$

Then $(\bar{x}, \bar{y})$ is a local super minimizer of Problem

$$\text{minimize } F(x), \text{ subject to } x \in \Omega.$$

Theorem 2.5. *Let K be a closed convex pointed ordering cone in Y and $f : L^1([0,1], \mathbb{R}^n) \longrightarrow Y$ be a W-convex map. Suppose that there is a $y^* \in intK^+$ such that*

$$0 \in \partial f(\bar{\gamma})(y^*) + N(\bar{\gamma}; L^1([0,1], \mathbb{R}^n)).$$

Then $\bar{\gamma}$ is a local super minimizer of Problem $(OP(a))$.

Proof. Obviously $L^1([0,1], \mathbb{R}^n)$ is a closed and convex set and therefore it is invex with respect to η which $\eta(x, y) = x - y$. Now an appeal to Theorem 2.4 completes the proof. $\square$

3. Existence of solution for $(OP(0))$ and optimal value function

For Problem $(OP(a))$ and fixed $y_0{}^* \in K^+$, the optimal value function $\psi : Z \longrightarrow \mathbb{R}$ is defined as:

$$\psi(a) = \begin{cases} inf\{y_0{}^*(f(\gamma)) : \ g(\gamma) \in -W + a\} & K(a) \neq \emptyset, \\ \\ +\infty & \text{o.w}, \end{cases}$$

Obviously, if there exists $\gamma_0 \in K(a)$ such that $\psi(a) = y_0{}^*(f(\gamma_0))$, then $\gamma_0 \in S(a)$ and the converse holds as well. In the following results, we focus on the set of Lagrange multipliers $f(\gamma) + \langle \lambda, g(\gamma) \rangle$, where $\lambda \in Z^*$ and $\gamma \in L^1([0,1], X)$. We shall show a relation between subdifferential of the optimal value function for Problem $(OP(0))$ and the set of Lagrange multipliers which refines the Corollary 3.7 in [2]. Motivated by an idea in [5], we consider the following result.

Theorem 3.6. $-\lambda \in \partial\psi(0)$ *if and only if*
(i) $\lambda \in W^+$;
(ii) for all $\gamma \in K(a)$,

$$\psi(0) \leq y_0{}^*(f(\gamma)) + \langle \lambda, g(\gamma) \rangle.$$

Proof. Let $-\lambda \in \partial\psi(0) = \{a^* \in Z^* : \ \langle a^*, a \rangle \leq \psi(a) - \psi(0) \ \forall a \in Z\}$. Since $\psi(a) = \inf\{y_0{}^*(f(\gamma)) : \ g(\gamma) \in -W + a\}$ and $\psi(0) = \inf\{y_0{}^*(f(\gamma)) : \ g(\gamma) \in -W\}$ for all $a \in W$, we have

$$\{y_0{}^*(f(\gamma)) : \ g(\gamma) \in -W\} \subseteq \{y_0{}^*(f(\gamma)) : \ g(\gamma) \in -W + a\}.$$

Therefore, $\psi(a) \leq \psi(0)$. Since $-\lambda \in \partial\psi(0)$, then

$$\langle -\lambda, a - 0\rangle \leq \psi(a) - \psi(0) \leq 0.$$

Therefore we get for all $a \in W$, $\langle \lambda, a\rangle \geq 0$ and $\lambda \in W^+$ which completes the proof of part (i).
For part (ii), suppose that $-\lambda \in \partial\psi(0)$ and $\langle -\lambda, g(\gamma)\rangle \leq \psi(g(\gamma)) - \psi(0)$, therefore,

$$\psi(0) \leq \psi(g(\gamma)) + \langle \lambda, g(\gamma)\rangle \leq {y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma)\rangle.$$

Conversely, let $\lambda \in W^+$ and $\gamma \in K(a)$, then $g(\gamma) \in -W + a$ and $\langle \lambda, g(\gamma) - a\rangle \leq 0$. On the other hand, from our assumption (ii), we have

$$\begin{aligned}\psi(0) &\leq {y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma)\rangle \\ &= {y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma) - a\rangle + \langle \lambda, a\rangle \\ &\leq {y_0}^*(f(\gamma)) + \langle \lambda, a\rangle.\end{aligned}$$

By taking the infimum on the left hand side over $K(a)$, we obtain

$$\psi(0) \leq {y_0}^*(f(\gamma)) + \langle \lambda, a\rangle,$$

and

$$\psi(0) \leq \psi(a) + \langle \lambda, a\rangle.$$

Thus, $-\langle \lambda, a\rangle \leq \psi(a) - \psi(0)$ and $-\lambda \in \partial\psi(0)$. □

In the following theorem, we obtain a relationship between solutions of Problem $(OP(0))$ and the set of Lagrange multipliers.

Theorem 3.7. *Let $(\bar{x}, \lambda) \in X \times Z^*$ then $-\lambda \in \partial\psi(0)$ and $\bar{x} \in S(0)$, if and only if*
(i) $\lambda \in W^+$;
(ii) $\bar{x}$ is a solution of the following problem (I)

$$\min_{\gamma \in K(a)} ({y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma)\rangle)\ (I).$$

(iii) $\langle \lambda, g(\bar{x})\rangle = 0$.

Proof. Let $-\lambda \in \partial\psi(0)$ and $\bar{x} \in S(0)$, then by applying Theorem 3.6, we know $\lambda \in W^+$. From definition of the subdifferential of ψ and by our assumptions, we have

$$\begin{aligned}{y_0}^*(f(\bar{x})) &= \inf_{\gamma \in K(0)} {y_0}^*(f(\gamma)) \\ &= \inf\{{y_0}^*(f(\gamma)) :\ g(\gamma) \in -W\} = \psi(o).\end{aligned}$$

On the other hand,

$$0 \leq \langle -\lambda, g(\gamma)\rangle \leq \psi(g(\gamma)) - \psi(0). \tag{3.10}$$

Since $\bar{x} \in K(0)$, then $g(\bar{x}) - g(\bar{x}) = 0 \in -W$ and $g(\bar{x}) \in g(\bar{x}) - W$, so $\psi(g(\bar{x})) \leq {y_0}^*(f(\bar{x})) = \psi(0)$. From 3.10, $\langle \lambda, g(\bar{x})\rangle = 0$ and from Theorem 3.6, for all $\gamma \in K(a)$

$$\begin{aligned}{y_0}^*(f(\bar{x})) + \langle \lambda, g(\bar{x})\rangle &= {y_0}^*(f(\bar{x})) \\ &= \psi(o) \\ &\leq {y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma)\rangle.\end{aligned}$$

Conversely, let $\lambda \in W^+$ and $\bar{x}$ be a solution of Problem (I). If $\gamma \in K(0)$, then $g(\gamma) \in -W$, and since $\lambda \in W^+$, we have $\langle \lambda, g(\gamma)\rangle \leq 0$. Therefore, by assumption (ii) we have

$$\begin{aligned}{y_0}^*(f(\gamma)) &\geq {y_0}^*(f(\gamma)) + \langle \lambda, g(\gamma)\rangle \\ &\geq {y_0}^*(f(\bar{x})) + \langle \lambda, g(\bar{x})\rangle.\end{aligned}$$

So $y_0{}^*(f(\gamma)) \geq y_0{}^*(f(\bar{x})) + \langle \lambda, g(\bar{x}) \rangle$, and by using our assumption $\langle \lambda, g(\bar{x}) \rangle = 0$ and

$$y_0{}^*(f(\gamma)) \geq y_0{}^*(f(\bar{x})).$$

Taking the infimum on the right hand side over $K(0)$, we obtain

$$\psi(0) = \inf_{\gamma \in K(0)} y_0{}^*(f(\gamma)) = y_0{}^*(f(\bar{x})).$$

Thus $\bar{x} \in S(0)$. Now we show that $-\lambda \in \partial\psi(0)$. If $\gamma \in K(a)$, and by having $\lambda \in W^+$ and $g(\gamma) \in -W + a$, we get $\langle \lambda, g(\gamma) \rangle \leq \langle \lambda, a \rangle$. On the other hand, for all $\gamma \in K(0)$, we have $g(\gamma) \in -W$ and by using assumption (ii), we deduce that

$$\begin{aligned} \psi(0) \leq y_0{}^*(f(\bar{x})) &= y_0{}^*(f(\bar{x})) + \langle \lambda, g(\bar{x}) \rangle \\ &\leq y_0{}^*(f(\gamma)) + \langle \lambda, g(\gamma) \rangle \\ &\leq y_0{}^*(f(\gamma)) + \langle \lambda, a \rangle. \end{aligned}$$

Then,

$$\psi(0) \leq y_0{}^*(f(\gamma)) + \langle \lambda, a \rangle.$$

Taking the infimum over $K(a)$, we obtain

$$\psi(0) \leq \psi(a) + \langle \lambda, a \rangle.$$

Therefore, $-\lambda \in \partial\psi(0)$. □

Now we consider the Problem $(OP(a))$. For this idea, we define the Hamiltonian function $H : [0, 1] \times Z^* \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ corresponding to Problem $(OP(a))$ as

$$H(t, \lambda) = \sup_{\gamma \in 3} \{\langle \lambda, g(\gamma) \rangle - y_0{}^*(f(\gamma))\}.$$

In the following theorem, we will show the existence of a solution of the Problem $(P(a))$ is equivalent to

$$H(t, \lambda) = \sup_{\gamma \in 3} \{\langle \lambda, g(\gamma) \rangle - y_0{}^*(f(\gamma))\},$$

and we will obtain a relationship between the subdifferential of the optimal value function of Problem $(P(a))$ and the set of Lagrange multipliers. In fact,we obtain a necessary and sufficient condition for the existence of solution for Problem $(P(a))$.

Theorem 3.8. *If $\lambda \in \partial\psi(a)$ and $\bar{x} \in S(a)$, then*
(i) $-\lambda \in W^+$;
(ii) $\lambda \in N(W, -g(\bar{x}) - a)$ and

$$H(t, \lambda) = y_0{}^*(f(\bar{x})) - \langle \lambda, g(\bar{x}) \rangle.$$

Conversely, if $-\lambda \in W^+$, $\lambda \in N(W, -g(\bar{x}) + a)$ and

$$H(t, \lambda) = y_0{}^*(f(\bar{x})) - \langle \lambda, g(\bar{x}) \rangle,$$

then $\lambda \in \partial\psi(a)$ and $\bar{x} \in S(a)$.

Proof. Suppose that $\lambda \in \partial\psi(a)$ and $\bar{x} \in S(a)$, select $b \in W$, if $g(\gamma) \in -W + a$ then $g(\gamma) \in -W + a + b$. Therefore,

$$\{y_0{}^*(f(\gamma)) : \; g(\gamma) \in -W + a\} \subseteq \{y_0{}^*(f(\gamma)) : \; g(\gamma) \in -W + a + b\},$$

and having $\psi(a + b) \leq \psi(a)$, then

$$\langle \lambda, b \rangle = \langle \lambda, b + a - a \rangle \leq \psi(b + a) - \psi(a) \leq 0,$$

so $-\lambda \in W^+$.
Since $\lambda \in \partial\psi(a)$, then

$$\langle \lambda, b - a \rangle \leq \psi(b) - \psi(a) \; b \in Z.$$

On the other hand, $\bar{x} \in S(a)$ i.e. $\psi(a) = {y_0}^*(f(\bar{x}))$. Then,

$$\begin{aligned}\langle \lambda, g(\gamma) - g(\bar{x})\rangle &= \langle \lambda, g(\gamma) - g(\bar{x}) - a + a\rangle \\ &\leq \psi(g(\gamma) - g(\bar{x}) + a) - \psi(a) \\ &\leq {y_0}^*(f(\gamma)) - {y_0}^*(f(\bar{x})).\end{aligned}$$

Therefore,

$${y_0}^*(f(\bar{x})) - \langle \lambda, g(\bar{x})\rangle \leq {y_0}^*(f(\gamma)) - \langle \lambda, g(\gamma)\rangle,$$

then $H(t, \lambda) = \langle \lambda, g(\bar{x})\rangle - {y_0}^*(f(\bar{x}))$.
Since $g(\bar{x}) \in -W + a$, then $\langle \lambda, g(\bar{x}) - a\rangle \geq 0$. Furthermore, for all $b \in W$ we have

$$\langle \lambda, b + g(\bar{x}) + a\rangle \leq \psi(b + a) - \psi(g(\bar{x})) \leq 0.$$

Therefore, $\lambda \in N(W, -g(\bar{x}) - a)$.
Conversely, since $\lambda \in N(W, -g(\bar{x}) + a)$ and for all $\gamma \in K(a)$, $g(\gamma) \in -W + a$ then $a - g(\gamma) \in W$ and

$$\langle \lambda, a - g(\gamma) + g(\bar{x}) - a\rangle \leq 0. \tag{3.11}$$

Thus, $\langle \lambda, g(\bar{x}) - g(\gamma)\rangle \leq 0$. On the other hand, $H(t, \lambda) = \langle \lambda, g(\bar{x})\rangle - {y_0}^*(f(\bar{x}))$, then

$$\begin{aligned}{y_0}^*(f(\bar{x})) &\leq {y_0}^*(f(\gamma)) - \langle \lambda, g(\gamma)\rangle + \langle \lambda, g(\bar{x})\rangle \\ &= {y_0}^*(f(\gamma)) + \langle \lambda, g(\bar{x}) - g(\gamma)\rangle \\ &\leq {y_0}^*(f(\gamma)).\end{aligned}$$

Hence, $\bar{x} \in S(a)$ and $\psi(a) = {y_0}^*(f(\bar{x}))$.
Now we will show $\lambda \in \partial\psi(a)$. Since

$$H(t, \lambda) = \langle \lambda, g(\bar{x})\rangle - {y_0}^*(f(\bar{x})).$$

Then for all $\gamma \in Z$, we have

$$\begin{aligned}{y_0}^*(f(\bar{x})) = \psi(a) &\leq {y_0}^*(f(\gamma)) + \langle \lambda, g(\bar{x}) - g(\gamma)\rangle \\ &= {y_0}^*(f(\gamma)) + \langle \lambda, g(\bar{x}) - g(\gamma) - b + b - a + a\rangle \\ &= {y_0}^*(f(\gamma)) + \langle \lambda, a - b\rangle + \langle \lambda, g(\bar{x}) - g(\gamma) + b - a\rangle.\end{aligned} \tag{3.12}$$

Since $\lambda \in N(W, -g(\bar{x}) + a)$, if $\gamma \in K(b)$, then $b - g(\gamma) \in W$, and

$$\langle \lambda, g(\bar{x}) - g(\gamma) + b - a\rangle \leq 0,$$

from 3.12, we obtain

$$\psi(a) = {y_0}^*(f(\bar{x})) \leq {y_0}^*(f(\gamma)) + \langle \lambda, a - b\rangle,$$

then

$$\langle \lambda, b - a\rangle \leq {y_0}^*(f(\gamma)) - \psi(a).$$

By taking the infimum over $K(b)$, we obtain

$$\langle \lambda, b - a\rangle \leq \psi(b) - \psi(a),$$

then $\lambda \in \partial\psi(a)$. □

The following result is a generalization of Theorem 4.1 in [2].

Corollary 3.1. *$\lambda \in \partial\psi(0)$ and $\bar{x} \in S(0)$, if and only if*
(i) $-\lambda \in W^+$;
(ii) $\lambda \in N(W, -g(\bar{x}))$ and

$$H(t, \lambda) = \langle \lambda, g(\bar{x})\rangle - {y_0}^*(f(\bar{x})).$$

Proof. It suffies to set $a = 0$ in the Theorem 3.8. □

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