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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

Strong convergence of inertial subgradient extragradient method for solving variational inequality in Banach space

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ABSTRACT. In this paper, we introduce a modified inertial subgradient extragradient algorithm in a 2-uniformly convex and uniformly smooth real Banach space and prove a strong convergence theorem for approximating a common solution of fixed point equation with a demigeneralized mapping and a variational inequality problem of a monotone and Lipschitz mapping. We present an example to validate our new findings. This work substantially improves and generalizes some well-known results in the literature.

1. INTRODUCTION

Let E be a Banach space and $\phi : E \times E \to [0, \infty)$ denotes the Lyapunov functional defined as

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \forall\, x, y \in E.$$

The functional ϕ satisfies the following properties (see Nilsrakoo and Saejung [26]):

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \ \forall\, x, y, z \in E, \text{ and} \tag{1.1}$$

$$\phi(z, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \le \alpha\phi(z, x) + (1-\alpha)\phi(z, y), \tag{1.2}$$

where $\alpha \in (0, 1)$ and $x, y \in E$.

Remark 1.1. If E is strictly convex and smooth Banach space, then for $x, y \in E,\ \phi(x, y) = 0$ if and only if $x = y$ (See Remark 2.1 in [24]).

Let E be a reflexive, strictly convex and smooth Banach space and C a nonempty closed and convex subset of E. Following Alber [1], for each $x \in E$, there exists a unique element $u \in C$ (denoted by $\Pi_C x$) such that $\phi(u, x) = \min_{y\in C} \phi(y, x)$. The mapping $\Pi_C : E \to C$, defined by $\Pi_C x = u$ is called the generalized projection operator (see [2]). Let $T : C \to E$ be a mapping. Then T is said to be demiclosed at $y \in C$, if a sequence $\{x_n\}$ converges weakly to x and the sequence $\{Tx_n\}$ converges strongly to y, then $T(x) = y$. In particular, if $y = 0$, then T is demiclosed at 0. A point $p \in C$ is said to be a fixed point of T if $Tp = p$,(see [17, 32]). We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C which converges weakly to p and $||x_n - Tx_n|| \to 0$ as $n \to \infty$. The set of all asymptotic fixed points of T is denoted by $\widehat{F(T)}$.

Definition 1.1. A mapping $T : C \to E$ is said to be:

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(i) relative nonexapnsive (see [24]) if $F(T) \neq \emptyset,\ F(T) \neq \widehat{F(T)}$ and

$$\phi(p, Tx) \leq \phi(p, x),\ \forall\, x \in C,\ p \in F(T),$$

(ii) quasi-ϕ-strictly psudocontractive (see [34]) if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that

$$\phi(p, Tx) \leq \phi(p, x) + k\phi(x, Tx),\ \forall\, x \in C \text{ and } p \in F(T). \tag{1.3}$$

Recently, Takahashi et al. [30] introduced a new mapping as follows: Let E be a smooth Banch space and let J be the duality mapping on E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called $(\eta, s)-$demigeneralized if, for any $x \in C$ and $q \in F(U)$, we have

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux) + s\phi(Ux, x); \tag{1.4}$$

in particular, an $(\eta, 0)-$demigeneralized mapping satisfies

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux). \tag{1.5}$$

We notice from (1.1) that

$$\phi(p, Tx) = \phi(p, x) + \phi(x, Tx) + 2\langle p - x, Jx - JTx\rangle. \tag{1.6}$$

Combining (1.3) and (1.6), we obtain $\phi(p, x) + \phi(x, Tx) + 2\langle p - x, Jx - JTx\rangle \leq \phi(p, x) + k\phi(x, Tx)$, which implies $2\langle x-p, Jx-JTx\rangle \geq (1-k)\phi(x, Tx)$ holds $\forall k \in (-\infty, 1)$. Hence, T in Definition 1.1(ii) is a $(k, 0)-$demigeneralized mapping in the sense of Takahashi et al. [30]. Also by (1.5) and (1.6), every $(0, 0)$-demigeneralized mapping is relative nonexapnsive mapping in the sense of Definition 1.1(i). For more examples, see [30].
An operator A from C into E^* (the dual of E) is said to be: (i) $monotone$ if for all $x, y \in C$, we have

$$\langle x - y, Ax - Ay\rangle \geq 0, \tag{1.7}$$

(ii) maximal monotone if A is monotone and the graph of A i.e. $G(A) := \{(x, y) \in E \times E^* : y \in A(x)\}$ is not properly contained in the graph of any other monotone operator. (iii) α-inverse strongly monotone if there exists a positive real number α such that $\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2,\ \forall\, x, y \in C$.
(iv) L-Lipschitz if $||Ax - Ay|| \leq L||x - y||$, holds for all $x, y \in C$ and some $L > 0$.
The variational inequality problem with respect to the operator A is to find an element $u \in C$ such that:

$$\langle v - u, Au\rangle \geq 0,\ \forall v \in C. \tag{1.8}$$

As in [29], we denote the set of solutions of variational inequality problem by $VI(C, A)$. The problem of solving a variational inequality of the form (1.8) has been intensively studied by many authors [3, 5, 6, 16, 19, 31]. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proving to be productive and innovative. Iterative methods for solving these problems have been proposed and analyzed by many authors when A is monotone and Lipschitz (see, [11, 12] and references therein).
In the case of real Hilbert space, Korpelevič [20] introduced extragradient method as follows: For $x_0 \in H$;

$$\begin{cases} y_n = P_C(x_n - \tau A(x_n)), \\ x_{n+1} = P_C(x_n - \tau A(y_n)), \end{cases} \tag{1.9}$$

where A is monotone and Lipschitz. Censor et al. [13] modified the method proposed by Korpelevič [20] by replacing one of the projections with a projection onto a half-space.

This subgradient extragradient method has received great attention in Hilbert spaces by many authors (see [15, 21]). Now, in order to speed up rate of convergence of the iterative algorithm, Polyak [27] studied the heavy ball method, an inertial extrapolation for minimizing a smooth convex function as follows: For $x_0, x_1 \in H$;

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = w_n + \lambda_n \nabla h(x_n),\ n \geq 1 \end{cases} \tag{1.10}$$

where $\theta_n \in [0, 1)$ and λ_n is a step-size parameter to be chosen sufficiently small. Here, the inertial is represented by the term $\theta_n(x_n - x_{n-1})$ which serves as a remarkable tool for improving the performance of the algorithms and has some nice convergence properties. As a result of this, the study of inertial-type algorithms has attracted the attention of several researchers (see for example [4, 8]). Recently, Thong and Hieu [31] combined the inertial technique with the subgradient extragradient method and proposed an algorithm, called inertial subgradient extragradient method. Under several appropriate conditions imposed on parameters, they proved that the iterative scheme converges weakly to a solution of a variational inequality in a Hilbert space.

In 2015, Nakajo [25] introduced the so called CQ method in a 2-uniformly convex and uniformly smooth real Banach space. He proved that the CQ algorithm converges strongly to the nearest common solution of fixed point of relatively nonexpansive mapping and a variational inequality problem.

Motivated by the work of Censor et al. [13], recently, Chidume and Nnakwe [18] proposed Krasnoselkii-type subgradient extragradient algorithm and proved a weak convergence theorem for obtaining a common element of solutions of variational inequality problem and common fixed points of a countable family of relatively nonexpansive mappings in a uniformly smooth and 2-uniformly convex real Banach space. For similar strong convergence results in Banach spaces, we refer the reader to Ansari and Rehan [7] and Ceng at al. [9].

Motivated and inspired by the work of Thong and Hiew [31] and Chidume and Nnakwe [18] , in this paper, we introduce a modified inertial subgradient extragradient algorithm in a 2-uniformly convex and uniformly smooth real Banach space and prove a strong convergence theorem for approximating a common solution of fixed point of a $(k, 0)$-demigeneralized mapping and solution of variational inequality problem of a monotone and Lipschitz mapping. We also present some important special cases of our main result.

2. Preliminaries

Let E be a reflexive, strictly convex and smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} (the duality mapping from E^* into E) is single-valued, one to one and onto. The following mapping is studied by Alber [1].

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2 \tag{2.11}$$

for all $x \in E$ and $x^* \in E^*$. From the definition of ϕ, we get $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$is a continuous, convex function from E^* into $\mathbb{R}$.

Lemma 2.1. [1] *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.11). *Then*

$$V(x, x^*) + 2\langle J^{-1} - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.2. (see [23, 24]) *Let C be a nonempty closed and convex subset of a smooth Banach space E. Then $u = \Pi_C x$ if and only if $\langle u - y, Jx - Ju \rangle \geq 0, \forall y \in C$ and $x \in E$.*

Lemma 2.3. [24] *Let E be a uniformly convex and smooth Banach space and $\{u_n\}$ and $\{v_n\}$ be two sequences in E. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$ and either $\{u_n\}$ or $\{v_n\}$ is bounded, then $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Lemma 2.4. (Xu, [33]) *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant α such that $\alpha||x - y||^2 \leq \phi(x, y), \forall x, y \in E$.*

Lemma 2.5. (Rockafellar, [28]) *Let C be nonempty closed and convex subset of a reflexive Banach space E and A a monotone, hemicontiuous map of C into E^*. Let $T : E \to 2^{E^*}$ be an operator defined by:*

$$Tu = \begin{cases} Au + N_C(u), & u \in C \\ \emptyset, & u \notin C, \end{cases}$$

where $N_C(u)$ is the normal cone at $u \in C$ and is defined as follows:

$$N_C(u) = \{w^* \in E^* : \langle u - z, w^*\rangle \geq 0,\ \forall z \in C\}.$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.6. (Xu, [33]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let*

$$B_r(0) := \{x \in E : ||x|| \leq r\}.$$

Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous strictly increasing convex function

$$g : [0, 2r] \to \mathbb{R},$$

such that $g(0) = 0$ and for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big\| \sum_{n=1}^{\infty} \lambda_n x_n \Big\|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||). \tag{2.12}$$

Lemma 2.7. (Chidume and Nnakwe, [18]) *Let E be a uniformly smooth and 2-uniformly convex real Banach space and C be a nonempty closed and convex subset of E. Let $A : E \to E^*$ be monotone map and k-Lipschitz, τ be a positive number and suppose that $VI(C, A) \neq \emptyset$. Let J be the normalized duality map on E and let $\{x_n\}$ be a sequence in E defined by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \tau A(x_n)), \\ T_n = \{w \in E : \langle w - y_n, Jx_n - \tau A(x_n) - Jy_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{T_n} J^{-1}(Jx_n - \tau A(y_n)). \end{cases}$$

Then for all $u \in VI(C, A)$, we have

$$\phi(u, x_{n+1}) \leq \phi(u, x_n) - \Big(1 - \frac{\tau k}{\alpha}\Big)(\phi(y_n, x_n) + \phi(x_{n+1}, y_n)). \tag{2.13}$$

where $\tau < \frac{\alpha}{k}$, and $\alpha \in (0, 1)$ is the constant in Lemma 2.4.

Lemma 2.8. (Mainge, [22]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.9. (Chani and Riachi, [10]) *Let $\{a_n\},\ \{\gamma_n\},\ \{\delta_n\},$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n a_{n-1} + t_n s_n + \delta_n,$$

where $\sum_{n=n_0}^{\infty} t_n = +\infty,\ \sum_{n=n_0}^{\infty} \delta_n < +\infty$ for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}],\ \limsup_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges to zero.

Lemma 2.10. (Takahashi et al., [30]) *Let E be a smooth Banach space and C be a nonempty closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$ and s be a real number with $s \in [0, \infty)$. Let U be an $(\eta, s)-$demigeneralized mapping of C into E. Then*

(i) *$F(U)$ is closed and convex.*
(ii) *For any $\alpha \in [0, 1)$, let $T = J^{-1}(\alpha J + (1-\alpha)JU)$, where J is the duality mapping on E. If U is $(\eta, 0)-$demigeneralized mapping, then $T : C \to E$ is also $(\eta, 0)-$demigeneralized mapping.*

3. Main results

Lemma 3.11. *Let E be a smooth Banach space and C a nonempty closed and convex subset of E. Let $k \in (-\infty, 0]$ and $T : C \to E$ be a $(k, 0)-$demigeneralized mapping with $F(T) \neq \emptyset$. Let λ be a real number in $(0, 1]$ and define $S = J^{-1}((1-\lambda)J + \lambda JT)$, where J is the duality mapping on E. Then*

(i) *$F(T) = F(S)$,*
(ii) *S is relatively-nonexpansive mapping of C into E.*

Proof. It is easy to show that $F(T) = F(S)$. For (ii), since T is $(k, 0)$-demigeneralized, therefore for any $p \in F(T)$ and $x \in C$, from (1.5), we obtain

$$\langle x - p, Jx - JSx\rangle = \langle x - p, Jx - J(J^{-1}((1-\lambda)Jx + \lambda JTx))\rangle = \lambda\langle x - p, Jx - JTx\rangle$$

$$\geq \lambda\frac{1-k}{2}\phi(x, Tx) \geq \lambda\phi(x, Tx). \tag{3.14}$$

By (1.2), we obtain

$$\phi(x, Sx) = \phi(x, J^{-1}((1-\lambda)Jx + JTx)) \leq \lambda\phi(x, Tx).$$

From (3.14) we obtain

$$2\langle x - p, Jx - JSx\rangle \geq \phi(x, Sx),$$

so it follows from (1.1), that

$$\phi(p, x) + \phi(x, Sx) - \phi(p, Sx) \geq \phi(x, Sx)$$

which implies that

$$\phi(p, Sx) \leq \phi(p, x).$$

Therefore S is a relatively nonexpansive mapping from C into E. We notice that if $k \in (-\infty, 1)$, then from (3.15), it follows that

$$2\langle x - p, Jx - JSx\rangle \geq (1-k)\phi(x, Sx),$$

which shows that S is $(k, 0)-$demigeneralized mapping (as in Lemma 2.10 (ii)).

□

Theorem 3.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E and J be the normalized duality mapping on E. Let $T : C \to E$ be a $(k, 0)-$demigeneralized mapping and demiclosed at zero with $k \in (-\infty, 0]$. Let $A : C \to E^*$ be a monotone and $L-$Lipschits mapping with $L > 0$. Let τ be a positive real number such that $\tau \in \left(0, \frac{\alpha}{L}\right)$ and $\alpha \in (0, 1)$ is the constant in Lemma 2.4. Assume*

$F(T)\cap VI(C,A)\neq\emptyset$. For any fixed $u\in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by arbitrarily chosen $x_0, x_1\in E$:

$$\begin{cases} w_n = J^{-1}(Jx_n+\delta_n(Jx_{n-1}-Jx_n)),\\ y_n=\Pi_C J^{-1}(Jw_n-\tau Aw_n),\\ T_n=\{x\in E:\langle x-y_n, Jw_n-\tau Aw_n-Jy_n\rangle\leq 0\},\\ z_n=\Pi_{T_n}J^{-1}(Jw_n-\tau Ay_n),\\ v_n=J^{-1}((1-\lambda_n)Jz_n+\lambda_n JTz_n),\\ x_{n+1}=J^{-1}(\alpha_n Jx_n+\beta_n Jv_n+\gamma_n Ju),\ n\geq 1\end{cases}\tag{3.15}$$

where $\{\delta_n\}\subset[0,\frac{1}{2}]$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$, and $\{\lambda_n\}\subset(0,1)$ which satisfy the following conditions:

(C1) $0<a\leq\delta_n<\beta_n\leq\frac{1}{2}$, *for all* $n\geq 1$,
(C2) $\lim_{n\to\infty}\gamma_n=0$ *and* $\sum_{n=1}^{\infty}\gamma_n=+\infty$,
(C3) $0<\liminf_{n\to\infty}\alpha_n\leq\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1$.

Then $\{x_n\}$ converges strongly to $p:=\Pi_{F(T)\cap VI(C,A)}u$.

Proof. Let $p\in F(T)\cap VI(C,G)$. By (3.15) and (2.13), we obtain

$$\phi(p,z_n)\leq\phi(p,w_n)-\Big(1-\frac{\tau k}{\alpha}\Big)(\phi(y_n,w_n)+\phi(z_n,y_n)).\tag{3.16}$$

Also by (3.15) and (1.2), we obtain

$$\begin{aligned}\phi(p,w_n)&=\phi(p,J^{-1}(Jx_n+\delta_n(Jx_{n-1}-Jx_n)))\\ &\leq(1-\delta_n)\phi(p,x_n)+\delta_n\phi(p,x_{n-1}).\end{aligned}\tag{3.17}$$

Furthermore, as T is a $(k,0)-$demigeneralized mapping, so $S_n:=J^{-1}((1-\lambda_n)JI+\lambda_n JT)$ is relatively-nonexpansive mapping and $F(S_n)=F(T)$ by Lemma 3.11. Therefore from (3.15), we obtain

$$\phi(p,v_n)=\phi(p,S_nz_n)\leq\phi(p,z_n).\tag{3.18}$$

Hence, by (3.15), (2.11), (3.16), (3.17), (3.18) and Lemma 2.6, we obtain

$$\begin{aligned}\phi(p,x_{n+1})&=\phi(p,J^{-1}(\alpha_nJx_n+\beta_nJv_n+\gamma_nJu))=V(p,\alpha_nJx_n+\beta_nJv_n+\gamma_nJu)\\ &=||p||^2-2\langle p,\alpha_nJx_n+\beta_nJv_n+\gamma_nJu\rangle+||\alpha_nJx_n+\beta_nJv_n+\gamma_nJu||^2\\ &\leq(1-\gamma_n-\beta_n\delta_n)\phi(p,x_n)+\beta_n\delta_n\phi(p,x_{n-1})+\gamma_n\phi(p,u)-\alpha_n\beta_ng(||Jx_n-Jv_n||)\\ &\quad-\Big(1-\frac{\tau k}{\alpha}\Big)(\phi(y_n,w_n)+\phi(z_n,y_n))\leq(1-\gamma_n-\beta_n\delta_n)\phi(p,x_n)\\ &\quad+\beta_n\delta_n\phi(p,x_{n-1})+\gamma_n\phi(p,u)\leq\max\{\phi(p,x_n),\phi(p,x_{n-1}),\phi(p,u)\}.\end{aligned}\tag{3.19}$$

By induction, $\phi(p,x_n)\leq\max\{\phi(p,x_1),\phi(p,x_0),\phi(p,u)\}$. Hence, $\{x_n\}$ is bounded. Thus $\{v_n\}$, $\{z_n\}$, $\{y_n\}$, $\{Tz_n\}$ and $\{w_n\}$ are also bounded. Therefore, by (3.19), we obtain

$$\begin{aligned}&\beta_n\delta_ng(||Jx_n-Jv_n||)+\Big(1-\frac{\tau k}{\alpha}\Big)(\phi(y_n,w_n)+\phi(z_n,y_n))\\ &\leq\ (\phi(p,x_n)-\phi(p,x_{n-1}))+\beta_n\beta_n(\phi(p,x_{n-1})-\phi(p,x_n))+\gamma_n\phi(p,u).\end{aligned}\tag{3.20}$$

The remaining proof will be divided into two cases.

Case 1. Assume that $\{\phi(p,x_n)\}_{n=1}^{\infty}$ is non-increasing sequence of real numbers. As $\phi(p,x_n)\leq D$, for all $n\geq 1$, where $D:=\max\{\phi(p,x_1),\phi(p,x_0),\phi(p,u)\}$, so $\{\phi(p,x_n)\}_{n=1}^{\infty}$ is bounded;

thus its limit exists. Therefore $\lim_{n\to\infty}(\phi(p,x_n)-\phi(p,x_{n+1}))=\lim_{n\to\infty}(\phi(p,x_{n-1})-\phi(p,x_n))=0$. Since $\beta_n\delta_n>0$ and $\left(1-\frac{\tau k}{\alpha}\right)>0$, therefore by (3.20) we obtain

$$\lim_{n\to\infty}\phi(y_n,w_n)=\lim_{n\to\infty}\phi(z_n,y_n)=\lim_{n\to\infty}g(||Jx_n-Jv_n||)=0.$$

Thus, it follows by Lemma 2.3 and property of g in Lemma 2.6, respectively, that

$$\lim_{n\to\infty}||y_n-w_n||=\lim_{n\to\infty}||z_n-y_n||=0 \tag{3.21}$$

and

$$\lim_{n\to\infty}||Jx_n-Jv_n||=0. \tag{3.22}$$

As J is uniformly norm-to-norm continuous on bounded sets, so (3.21) becomes

$$\lim_{n\to\infty}||Jy_n-Jw_n||=\lim_{n\to\infty}||Jz_n-Jy_n||=0. \tag{3.23}$$

Also by (x_{n+1}) in (3.15), it follows by (C2) and (3.22) that

$$\lim_{n\to\infty}||Jx_{n+1}-Jx_n||=0. \tag{3.24}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, therefore we get

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0. \tag{3.25}$$

By (3.15) and (3.24), we obtain

$$\lim_{n\to\infty}||Jw_n-Jx_n||\le\lim_{n\to\infty}\delta_n||Jx_{n-1}-Jx_n||=0. \tag{3.26}$$

It now follows by (3.22), (3.23) and (3.26) that

$$\lim_{n\to\infty}||Jz_n-Jv_n||=0. \tag{3.27}$$

Also, by (3.22) and (3.27), we obtain

$$\lim_{n\to\infty}||Jx_n-Jz_n||\le\lim_{n\to\infty}||Jx_n-Jv_n||+\lim_{n\to\infty}||Jv_n-Jz_n||=0.$$

By the uniform continuity of J^{-1} on bounded sets, we get

$$\lim_{n\to\infty}||x_n-z_n||=0. \tag{3.28}$$

Also (3.21) and (3.28) imply

$$\lim_{n\to\infty}||x_n-y_n||\le\lim_{n\to\infty}||x_n-z_n||+\lim_{n\to\infty}||z_n-y_n||=0. \tag{3.29}$$

Since T is $(k,0)-$demigeneralized and $p\in F(T)$, therefore by (v_n) in (3.15) and (1.5), we obtain

$$\langle z_n-p,Jz_n-Jv_n\rangle=\lambda_n\langle z_n-p,Jz_n-JTz_n\ge\lambda_n\frac{1-k}{2}\phi(z_n,Tz_n).$$

Now $k\le 0$ and $\lambda_n>0$ for all $n\ge 1$ give $\lambda(1-k)>0$ and so we get

$$\phi(z_n,Tz_n)\le\frac{2}{\lambda(1-k)}||z_n-p||||Jz_n-Jv_n||,$$

using the boundedness of $\{z_n\}$ and (3.27), we obtain $\lim_{n\to\infty}\phi(z_n,Tz_n)=0$. Now by Lemma 2.3, we obtain

$$\lim_{n\to\infty}||z_n-Tz_n||=0. \tag{3.30}$$

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to z in E as $j\to\infty$. By (3.28), we get z_{n_j} converges weakly to z as

$j \to \infty$. By demiclosedness of T, we obtain from (3.30), that $z \in F(T)$.

Next, we show that $z \in VI(C, A)$. Let $U : E \to 2^{E^*}$ be defined by

$$Ux = \begin{cases} Ax + N_C(x), & x \in C \\ \emptyset, & x \notin C, \end{cases}$$

where $N_C x$ is the normal cone at $x \in C$ as defined in Lemma 2.5. The operator U is maximal monotone and $0 \in Ux$ if and only $x \in VI(C, A)$. It is known that if U is maximal monotone, then for given $(y, v^*) \in E \times E^*$ such that $\langle y - v, y^* - v^* \rangle \geq 0, \ \forall \ (v, v^*) \in G(U)$, one has $y^* \in Uy$. Let $(x, q) \in G(U)$. We want to show that $\langle x - z, q \rangle \geq 0$. Now $(x, q) \in G(U)$ implies that $q \in Ux = Ax + N_C x$ which means $q - Ax \in N_C x$. Thus $\langle x - v, q - Ax \geq 0, \ \forall \ v \in C$. Since $y_n = \Pi_C J^{-1}(Jw_n - \tau A w_n)$ and $x \in C$, we obtain by Lemma 2.2 that

$$\langle y_n - x, Jw_n - \tau A w_n - Jy_n \rangle \geq 0, \text{ that is, } \left\langle x - y_n, \frac{Jy_n - Jw_n}{\tau} + Aw_n \right\rangle \geq 0.$$

As $y_n \in C$ and $q - Ax \in N_C x$, so we get $\langle x - y_{n_j}, q - Ax \rangle \geq 0$ which implies that

$$\langle x - y_{n_j}, q \rangle \geq \langle x - y_{n_j}, Ax \rangle \geq \langle x - y_{n_j}, Ax - Ay_{n_j} \rangle + \langle x - y_{n_j}, Ay_{n_j} - Aw_{n_j} \rangle$$

$$-\left\langle x - y_{n_j} \frac{Jy_{n_j} - Jw_{n_j}}{\tau} \right\rangle \geq -kM||w_{n_j} - y_{n_j}|| - M||Jw_{n_j} - Jw_{n_j}||, \tag{3.31}$$

where $M = \max\{||x - y_{n_j}||, \frac{1}{\tau}\}$ (note $\{y_{n_j}\}$ is bounded). As x_{n_j} converges weakly to z, so by (3.29), we get y_{n_j} converges weakly to z as $j \to \infty$. Then by (3.31), (3.21) and (3.23), we obtain

$$\langle x - z, q \rangle \geq 0.$$

As U is maximal monotone, we have $z \in U^{-1}0$ and so $z \in VI(C, A)$. Therefore $z \in F(T) \cap VI(C, A)$.

Finally, we show that x_n converges strongly to $p := \Pi_{F(T)\cap VI(C,A)} u$. Now $\limsup_{n\to\infty} \langle x_n - p, Ju - Jp \rangle = \lim_{j\to\infty} \langle x_{n_j} - p, Ju - Jp \rangle = \langle z - p, Ju - Jp \rangle$ and by Lemma 2.2, we have $\langle z - p, Ju - Jp \rangle \leq 0$ and hence

$$\limsup_{n\to\infty} \langle x_n - p, Ju - Jp \rangle = \langle z - p, Ju - Jp \rangle \leq 0. \tag{3.32}$$

It follows from (3.24) and (3.32) that

$$\limsup_{n\to\infty} \langle x_{n+1} - p, Ju - Jp \rangle \leq 0. \tag{3.33}$$

Now, by (3.15), (3.16), (3.17), (3.18) and Lemma 2.1, we obtain

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}(\alpha_n Jx_n + \beta_n Jv_n + \gamma_n Ju)) = V(p, \alpha_n Jx_n + \beta_n Jv_n + \gamma_n Ju) \\ &\leq V(p, \alpha_n Jx_n + \beta_n Jv_n + \gamma_n Ju - \gamma_n(Ju - Jp)) + 2\gamma_n \langle x_{n+1} - p, Ju - Jp \rangle \\ &\leq \alpha_n \phi(p, x_n) + \beta_n[(1 - \delta_n)\phi(p, x_n) + \delta_n \phi(p, x_{n-1})] + 2\gamma_n \langle x_{n+1} - p, Ju - Jp \rangle \\ &= (1 - \gamma_n - \beta_n\delta_n)\phi(p, x_n) + \beta_n\delta_n\phi(p, x_{n-1}) + 2\gamma_n \langle x_{n+1} - p, Ju - Jp \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \phi(p, x_{n+1}) \quad &\leq \quad (1 - \gamma_n - \beta_n\delta_n)\phi(p, x_n) + \beta_n\delta_n\phi(p, x_{n-1}) \\ &\quad\quad + 2\gamma_n \langle x_{n+1} - p, Ju - Jp \rangle. \end{aligned} \tag{3.34}$$

By Lemma 2.9 and (3.34), we obtain $\phi(p, x_n) \to 0$ as $n \to \infty$ which implies by Lemma 2.3 that $||x_n - p|| \to 0$ as $n \to \infty$. Hence $x_n \to p := \Pi_{F(T)\cap VI(C,A)} u$.

Case 2. Assume that $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is non-decreasing sequence of real numbers. As in

Lemma 2.8, set $\Gamma_n := \phi(p, x_n)$ and let $r : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough), defined by

$$r(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, r is a non-decreasing sequence such that $r(n) \to \infty$ as $n \to \infty$. Thus

$$0 \leq \Gamma_{r(n)} \leq \Gamma_{r(n)+1}, \ \forall\, n \geq n_0$$

which means that $\phi(p, x_{r(n)}) \leq \phi(p, x_{r(n)+1})$, for all $n \geq n_0$. Since $\{\phi(p, x_{r(n)})\}$ is bounded, therefore $\lim_{n\to\infty} \phi(p, x_{r(n)})$ exists. Thus following the same line of action as in Case 1, we can show that the following hold:

$$\lim_{n\to\infty} ||y_{r(n)} - w_{r(n)}|| = \lim_{n\to\infty} ||z_{r(n)} - y_{r(n)}|| = \lim_{n\to\infty} ||v_{r(n)} - x_{r(n)}|| = 0$$

and

$$\lim_{n\to\infty} ||x_{r(n+1)} - x_{r(n)}|| = \lim_{n\to\infty} ||x_{r(n)} - y_{r(r)}|| = \lim_{n\to\infty} ||v_{r(n)} - x_{r(r)}|| = 0.$$

Also $\lim_{n\to\infty} ||z_{r(n)} - Tz_{r(n)}|| = 0$. Since $\{x_{r(n)}\}$ is bounded, there exists a subsequence of $\{x_{r(n)}\}$, still denoted by $\{x_{r(n)}\}$ such that $x_{r(n)}$ converges weakly to z as $n \to \infty$. By an argument similar to that in Case 1, we can show that $z \in F(T) \cap VI(C, A)$ and

$$\lim_{n\to\infty} \langle x_{r(n)+1} - p, , Ju - Jp\rangle \leq 0. \tag{3.35}$$

Also, by (3.34), and $\Gamma_{r(n)} \leq \Gamma_{r(n)+1}$, we get

$$\phi(p, x_{r(n)}) \leq (1 - \gamma_{r(n)} - \beta_{r(n)}\delta_{r(n)})\phi(p, x_{r(n)}) + \beta_{r(n)}\delta_{r(n)}\phi(p, x_{r(n)-1})$$

$$+2\gamma_{r(n)}\langle x_{r(n)+1} - p, Ju - Jp\rangle \leq (1 - \gamma_{r(n)})\phi(p, x_{r(n)+1}) + 2\gamma_{r(n)}\langle x_{r(n)+1} - p, Ju - Jp\rangle.$$

Therefore

$$\phi(p, x_{r(n)}) \leq \phi(p, x_{r(n)+1}) \leq 2\langle x_{r(n)+1} - p, Ju - Jp\rangle. \tag{3.36}$$

which implies by (3.35) $\limsup_{n\to\infty} \phi(p, x_{r(n)}) \leq 0$. Thus $\lim_{n\to\infty} \phi(p, x_{r(n)}) = 0$. Now by (3.36), we have $\lim_{n\to\infty} \phi(p, x_{r(n)+1}) = 0$ and therefore $\lim_{n\to\infty} \Gamma_{r(n)} = \lim_{n\to\infty} \Gamma_{r(n)+1} = 0$. For all $n \geq n_0$, we have that $\Gamma_{r(n)} \leq \Gamma_{r(n)+1}$ if $n \neq r(n)$ (that is, $r(n) < n$), because $\Gamma_{k+1} \leq \Gamma_k$ for $r(n) \leq k \leq n$. As a consequence, we get for all $n \geq n_0$

$$0 \leq \Gamma_n \leq \max\{\Gamma_{r(n)}, \Gamma_{r(n)+1}\} = \Gamma_{r(n)+1}.$$

So $\lim_{n\to\infty} \Gamma_n = 0$ gives that $\lim_{n\to\infty} \phi(p, x_n) = 0$, which implies that $\lim_{n\to\infty} ||p - x_n|| = 0$. Thus $x_n \to p := \Pi_{F(T)\cap VI(C,A)}u$. □

Since every relative nonexpansive mapping is $(0, 0)$-demigeneralized mapping, therefore the following result follows from Theorem 3.1.

Corollary 3.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E and J be the normalized duality mapping on E. Let $T : C \to E$ be a relative nonexapnsive mapping and demiclosed at zero. Let $A, \tau,\ \{\delta_n\},\ \{\alpha_n\},\ \{\beta_n\},\ \gamma_n, u$ and $\{x_n\}_{n=1}^{\infty}$ be as in Theorem 3.1 with $\lambda_n = 1, \forall n \geq 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a point $\Pi_{F(T)\cap VI(C,A)}u$.*

It is well known that generalized projection Π_C on a closed convex subset C of a Hilbert space H coincides with the metric projection P_C. A generalized hybrid mapping $T : C \to H$ is $(0, 0)$-generalized mapping.Hence from Theorem 3.1, we obtain:

Corollary 3.2. *Let H be a real Hilbert space and C a nonempty closed and convex subset of H. Let $T : C \to E$ be (α, β)-generalized hybrid mapping and demiclosed at zero where $\alpha, \beta \in \mathbb{R}$. Let $A : C \to H$, τ, $\{\delta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, γ_n, u, $\{x_n\}_{n=1}^{\infty}$ be as in Theorem 3.1 with $\lambda_n = 1, \forall n \geq 1$ and the duality mapping $J := I$ (the identity mapping of H). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a point $\Pi_{F(T)\cap VI(C,A)}u$.*

4. Numerical example

Let $E = \mathbb{R}^4$ be the four-dimensional Euclidean space with the usual inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

where $x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ and usual norm

$$||x||^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2,\ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

Given a half-space $C = \{z \in \mathbb{R}^4 : \langle u, z - w_0\rangle \leq 0\}$ of $\mathbb{R}^4$, where $u(\neq 0)$ and w_0 are two fixed element of $\mathbb{R}^4$. Then for any $x_0 \in \mathbb{R}^4$, we have

$$P_C x_0 = \begin{cases} x_0 - \frac{\langle u, x_0 - w_0\rangle}{||u||^2}u, & \langle u, x_0 - w_0\rangle > 0; \\ x_0, & \langle u, x_0 - w_0\rangle \leq 0. \end{cases}$$

Let $Tx = -\frac{2}{3}x$. Then T is $(-1/5, 0)$-demigeneralized mapping and $Ax = (-2x_3 + x_2, x_4 - x_1, 2x_1 - 2x_4, -x_2 + 2x_3)$ is monotone and $2\sqrt{2}$-Lipschitz operator. Let $\lambda_n = \frac{2n-1}{6n}$, $\alpha_n = \frac{3n+1}{6n}$, $\beta_n = \frac{2n-1}{4n}$, $\gamma_n = \frac{1}{12n}$, and $\delta_n = \frac{1}{4}$, $\forall n \geq 1$. Now $\tau \in (0, \frac{\alpha}{L}) = (0, \frac{\alpha}{2\sqrt{2}})$, for $\alpha \in (0, 1)$, put $\alpha = \frac{1}{2}$, so that we can take $\tau = \frac{1}{8}$. All the conditions of Theorem 3.1 are satisfied. So by its conclusion, we have

$$\begin{cases} w_n = \frac{1}{4}(3x_n - x_{n-1}), \\ y_n = P_C(w_n - \frac{1}{8}Aw_n), \\ T_n = \{x \in \mathbb{R}^4 : \langle x - y_n, w_n - \frac{1}{8}Aw_n - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(w_n - \frac{1}{8}Ay_n), \\ v_n = \frac{8n+5}{18n}z_n, \\ x_{n+1} = \frac{3n+1}{6n}x_n + \frac{2n-1}{4n}v_n + \frac{1}{12n},\ n \geq 1. \end{cases}$$

(Case 1.) Take $x_0 = [-2, -3, -2, 3]^T$, $x_1 = [-1, 2, -1, 2]^T$ and $u = [-2, 1, 0.5, -0.7]$;
(Case 2.) Take $x_0 = [-1, 2, -1, 2]^T$, $x_1 = [-2, -3, -2, 3]^T$ and $u = [-2, 1, 0.5, -0.7]^T$;
(Case 3.) Take $x_0 = [-2.5, -0.3, -2, 3]^T$, $x_1 = [1, 3, 0.10, 0.5]^T$ and $u = [-2.4, 0.3, 0.5, -0.7]^T$.

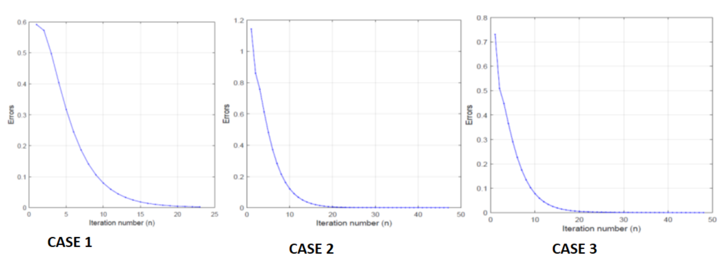

FIGURE 1. Errors vs number of iterations.

Remark 4.2.

(i) Corollary 3.1 improves the main result of Nakajo [25] in the sense that in his algorithm at each iteration three subsets of C, namely $C_n,\ Q_n$ and $C_n \cap Q_n$ need to be computed while in our algorithm (3.15), these subsets have been replaced with the half space T_n.

(ii) Corollary 3.1 generalizes, Theorem 3.1 of Thong and Hieu [31] from Hilbert space setting to uniformly smooth and 2-uniformly convex real Banach space.

(iii) Corollary 3.2 improves and extends the results of Korpelevič [20] and Kraikaew and Saejung [21] in the following aspects:
  (a) it improves the result of Korpelevič [20] from weak convergence to strong convergence and replaces one of the projections in the algorithm (1.9) by a half-space.
  (b) it extends the main theorem in [21] from quasi-nonexpansive mappings to (α, β)-generalized hybrid mappings.

(iv) w_n in our algorithm being a convex combination, substantially speeds up its rate of convergence and reduces the computational cost. This makes a lot of difference in comparison with the previously known subgradient extragradient or inertial subgradient extragradient methods given in [13, 18, 31].

(v) We provided example to illustrate the convergence of our algorithm, with different initial values of x_0, x_1 and u in **Cases(1-3)**. Figure 1 shows that the algorithm (3.15) converges to 0 quickly with less number of iterations.

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