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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} *anniversary*

Limiting proper minimal points of nonconvex sets in finite-dimensional spaces

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ABSTRACT. In this paper, limiting proper minimal points of nonconvex sets in Euclidean finite-dimensional spaces are investigated. The relationships between these minimal points and Borwein, Benson, and Henig proper minimal points, under appropriate assumptions, are established. Furthermore, a density property is derived and a linear characterization of limiting proper minimal points is provided.

1. INTRODUCTION

One of the most important solution concepts in modern vector optimization and set optimization is proper minimality. This notion has been defined and investigated by various scholars; see e.g. Kuhn and Tucker [10], Geoffrion [6], Benson [2], Borwein [3], and Henig [7] among others. See also, [1, 9, 13, 15].

In a recently published paper [14], we have introduced a new proper minimality notion, leading to defining a solution concept called limiting proper minimal point (LPM). These points have been defined from a dual space standpoint, invoking limiting (Mordukhovich) normal cone [11], a strong tool in variational analysis. In [14], we have characterized and established the properties of LPMs in (infinite-dimensional) Banach spaces. In the current work, we investigate LPMs more in a finite-dimensional setting, because of the following reasons: 1) Some results are derived which are not valid in an infinite-dimensional setting; 2) Some important restrictive assumptions imposed in [14] can be relaxed in finite-dimensional spaces; and 3) The practical-oriented vector optimization problems are usually modeled in a finite-dimensional framework and many scholars in applied optimization are not familiar with infinite-dimensional case.

In the current work, the relationships between LPMs from one side and Borwein, Benson, and Henig proper minimal points from other side are established. These connections highlight the density of LPMs in minimals. Moreover, a linear characterization of LPMs is provided. The required preliminaries are addressed in Section 2, and the main results are given in Section 3.

2. PRELIMINARIES

In this section, some standard notations and definitions are addressed which are used in the sequel. Given $\Omega \subseteq \mathbb{R}^n$, notations $int\Omega$, $conv\Omega$, and $cl\Omega$ stand for the interior, convex hull, and the closure of Ω, respectively.

A nonempty set $C \subseteq \mathbb{R}^n$ is said to be a cone if $\lambda C \subseteq C$ for each $\lambda \geq 0$. A cone C is convex if and only if $C + C \subseteq C$. A cone C is called pointed if $C \cap (-C) = \{0\}$.

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Furthermore, it is called nontrivial if $C \neq \mathbb{R}^n$ and $C \neq \{0\}$. A cone C is called an ordering cone if it is nontrival, convex, closed and pointed.

The standard inner product is denoted by $\langle \cdot, \cdot \rangle$. The nonnegative and positive dual cones of a cone $C \subseteq \mathbb{R}^n$ are respectively defined as

$$C^* := \{d \in \mathbb{R}^n : \langle d, c\rangle \geq 0,\ \forall c \in C\},$$

$$C^{*\circ} := \{d \in \mathbb{R}^n : \langle d, c\rangle > 0,\ \forall c \in C \setminus \{0\}\}.$$

The cone generated by $\Omega \subseteq \mathbb{R}^n$, denoted by $cone(\Omega)$, is defined as $cone(\Omega) := \bigcup_{\lambda \geq 0} \lambda\Omega$. The asymptotic cone to Ω is the set

$$As(\Omega) = \Big\{d \in \mathbb{R}^n : \exists\Big(x_\nu \in \Omega,\ t_\nu \downarrow 0\Big)\ s.t.\ t_\nu x_\nu \to d\Big\}.$$

Hereafter, the notation $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ while $x \in \Omega$. The Bouligand and Clarke tangent cones to Ω at $\bar{x}$ are expressed as follows

$$T(\bar{x}; \Omega) := \Big\{d \in \mathbb{R}^n : \exists\ \Big(t_\nu \downarrow 0, x_\nu \xrightarrow{\Omega} \bar{x}\Big)\ s.t.\ t_\nu(x_\nu - \bar{x}) \to d\Big\},$$

$$T_C(\bar{x}; \Omega) := \Big\{d \in \mathbb{R}^n : \forall\Big(t_\nu \downarrow 0,\ x_\nu \xrightarrow{\Omega} \bar{x}\Big),\ \exists\, d_\nu \to d\ s.t.\ \bar{x} + t_\nu d_\nu \in \Omega,\ \nu \in \mathbb{N}\Big\}.$$

It follows from the definitions that $T_C(\bar{x}; \Omega) \subseteq T(\bar{x}; \Omega)$. In contrast to $T(\bar{x}; \Omega)$, the Clarke tangent cone $T_C(\bar{x}; \Omega)$ is always convex. The contingent and Clarke normal cone to Ω at $\bar{x}$ are respectively defined as

$$N(\bar{x}; \Omega) = -T(\bar{x}; \Omega)^*,\ N_C(\bar{x}; \Omega) = -T_C(\bar{x}, \Omega)^*.$$

Given $\bar{x} \in \Omega$, the Fréchet normal cone to Ω at $\bar{x}$, is defined as

$$\hat{N}(\bar{x}; \Omega) := \left\{d \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle d, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0\right\}.$$

The limiting/Mordukhovich normal cone [11] to Ω at $\bar{x}$ is defined as

$$N_L(\bar{x}; \Omega) = \Big\{d \in \mathbb{R}^\nu : \exists\Big(x_\nu \xrightarrow{\Omega} \bar{x},\ d_\nu \to d\Big) \text{ with } d_\nu \in \hat{N}(x_\nu; \Omega),\ \nu \in \mathbb{N}\Big\}.$$

It is not difficult to show that $\hat{N}(\bar{x}; \Omega) \subseteq N_L(\bar{x}; \Omega)$.

Definition 2.1. [11] (normal regularity of sets) A set $\Omega \subseteq \mathbb{R}^n$ is called limiting normally regular at $\bar{x} \in \Omega$ if $N_L(\bar{x}; \Omega) = \hat{N}(\bar{x}; \Omega)$.

Let a nonempty set $\Omega \subseteq \mathbb{R}^n$ and an ordering cone $C \subseteq \mathbb{R}^n$ be given. A vector $\bar{x} \in \Omega$ is called a minimal point of Ω w.r.t C if $(\Omega - \bar{x}) \cap (-C) = \{0\}$. The set of minimal points of Ω w.r.t C is denoted by $E[\Omega, C]$.

Definition 2.2. [2, 3, 7] A point $\bar{x} \in \Omega$ is said to be

i) a Borwein proper minimal of Ω w.r.t C, written as $\bar{x} \in Bor[\Omega, C]$, if $T(\bar{x}; \Omega + C) \cap (-C) = \{0\}$.
ii) a Benson proper minimal of Ω w.r.t C, written as $\bar{x} \in Ben[\Omega, C]$, if $cl\ cone(\Omega + C - \bar{x}) \cap (-C) = \{0\}$.
iii) a Henig proper minimal of Ω w.r.t C, written as $\bar{x} \in He[\Omega, C]$, if there is an ordering cone C' such that $C \setminus \{0\} \subseteq int(C')$ and $\bar{x} \in E[\Omega, C']$.

It is known that

$$He[\Omega, C] = Ben[\Omega, C] \subseteq Bor[\Omega, C] \subseteq E[\Omega, C]. \tag{2.1}$$

The first equality in (2.1) comes from [7, Theorem 2.1] and [12, Proposition 2.1.4]. On the other hand, as $T(\bar{x}; \Omega) \subseteq clcone(\Omega - \bar{x})$, then $Ben[\Omega, C] \subseteq Bor[\Omega, C]$.

In a very recently-published paper [14], we have defined a new proper minimality notion, called limiting proper minimality. These points have been defined in terms of the limiting (Mordukhovich) normal cone.

Definition 2.3. [14] A point $\bar{x} \in E[\Omega, C]$ is called a limiting proper minimal of Ω w.r.t C, written as $\bar{x} \in L[\Omega, C]$, if $N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \neq \emptyset$.

The condition $\bar{x} \in E[\Omega, C]$ in Definition 2.3 cannot be dropped [14].

3. Main results

One of the main aims of this short paper is establishing the relationships between limiting proper minimality and other proper minimality notions in finite-dimensional spaces. Theorem 3.1 investigates the connection with Borwein proper minimality.

Theorem 3.1. *Let $\Omega + C$ be limiting normally regular at $\bar{x} \in \Omega$. Then, $\bar{x} \in L[\Omega, C]$ implies $\bar{x} \in Bor[\Omega, C]$. This converse holds if furthermore $\Omega + C$ is closed.*

Proof. Let $\bar{x} \in L[\Omega, C]$. Taking limiting normal regularity into account, there exists some $0 \neq d \in \hat{N}(\bar{x}; \Omega + C) \cap (-C^{*\circ})$.

Considering $0 \neq v \in T(\bar{x}; \Omega + C)$, there are sequences $x_\nu \in \Omega + C$ and $t_\nu > 0$ such that $t_\nu(x_\nu - \bar{x}) \to v$. Since $d \in \hat{N}(\bar{x}; \Omega + C)$, we get

$$\limsup_{x_\nu \longrightarrow \bar{x}} \frac{\langle d, x_\nu - \bar{x} \rangle}{\|x_\nu - \bar{x}\|} \leq 0,$$

which implies $\langle d, v \rangle \leq 0$. So,

$$\langle d, v \rangle \leq 0, \; v \in T(\bar{x}; \Omega + C). \tag{3.2}$$

If $\bar{x} \notin Bor[\Omega, C]$, then there exists some $\bar{v} \neq 0$ such that $\bar{v} \in T(\bar{x}; \Omega + C) \cap (-C)$. Since $\bar{v} \in -C \setminus \{0\}$ and $d \in -C^{*\circ}$, we have $\langle d, \bar{v} \rangle > 0$. This contradicts (3.2).

To prove the converse, assume that $\bar{x} \in Bor[\Omega, C]$ while $\bar{x} \notin L[\Omega, C]$. Then

$$\emptyset = \Big(N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \Big) \supseteq \Big(\hat{N}(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \Big).$$

By applying a standard separation theorem [8, Theorem 3.16], there exists $0 \neq p \in \mathbb{R}^n$ such that

$$\langle p, d \rangle \geq 0, \quad \forall d \in \hat{N}(\bar{x}; \Omega + C), \tag{3.3}$$

$$\langle p, c \rangle \leq 0, \quad \forall c \in -C^{*\circ}. \tag{3.4}$$

By (3.3) and limiting normal regularity, invoking [4, Theorem 11.36], we get

$$p \in (\hat{N}(\bar{x}; \Omega + C))^* = \Big(clconv(\hat{N}(\bar{x}; \Omega + C)) \Big)^* = \Big(cl\, conv(N_L(\bar{x}; \Omega + C)) \Big)^*$$

$$= (N_C(\bar{x}; \Omega + C))^* = -T_C(\bar{x}; \Omega + C) \subseteq -T(\bar{x}; \Omega + C).$$

On the other hand, by (3.4), $p \in -(-C^{*\circ})^* \subseteq C$. Therefore, $0 \neq -p \in T(\bar{x}; \Omega + C) \cap (-C)$. This contradicts $\bar{x} \in Bor[\Omega, C]$, and the proof is completed. □

Example 3.1 highlights the importance of limiting normal regularity assumption in Theorem 3.1.

Example 3.1. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \geq -x_1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > x_1\},$$
$$C = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 = x_2 \geq 0\}, \ \bar{x} = (0, 0).$$

Then, $N_L(\bar{x}; \Omega + C) = \{(x_1, x_2) : \ x_1 = x_2 \leq 0\}$ while $\hat{N}(\bar{x}; \Omega + C) = \{(0, 0)\}$. So, $\Omega + C$ is not limiting normally regular at $\bar{x}$. Furthermore,

$$T(\bar{x}; \Omega + C) \cap (-C) = cl(\Omega) \cap (-C) = \{(x_1, x_2) \ : \ x_1 = x_2 \leq 0\},$$
$$N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \neq \emptyset.$$

These imply $\bar{x} \in L[\Omega, C]$ while $\bar{x} \notin Bor[\Omega, C]$.

The closedness of $\Omega + C$ is essential for converse part in Theorem 3.1:

Example 3.2. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 > -|x_1|\},$$
$$C = \{(0, x_2) \in \mathbb{R}^2 : \ x_2 \geq 0\}, \ \bar{x} = (0, 0).$$

Here, $N_L(\bar{x}; \Omega + C) = N(\bar{x}; \Omega + C) = \{(0, 0)\}$, and so $\Omega + C$ is limiting normally regular at $\bar{x}$. Furthermore

$$T(\bar{x}; \Omega + C) \cap (-C) = cl(\Omega) \cap (-C) = \{(0, 0)\}, N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) = \emptyset.$$

These imply $\bar{x} \in Bor[\Omega, C]$ while $\bar{x} \notin L[\Omega, C]$. Notice that $\Omega + C = \Omega$ is not closed.

Theorem 3.2 addresses the relationship between limiting and Henig proper minimality.

Theorem 3.2. *Let $\Omega + C$ be limiting normally regular at $\bar{x} \in \Omega$.*

i) *If $\Omega + C$ is closed, then $\bar{x} \in He[\Omega, C]$ implies $\bar{x} \in L[\Omega, C]$.*
ii) *If $As(\Omega) \cap (-C) = \{0\}$, then $\bar{x} \in L[\Omega, C]$ implies $\bar{x} \in He[\Omega, C]$.*

Proof. i) Apply Theorem 3.1 and relation (2.1). ii) Apply Theorem 3.1 and [7, Theorems 2.1 and 2.2]. □

Corollary 3.1 is a direct consequence of Theorem 3.2 due to (2.1).

Corollary 3.1. *Let $\Omega + C$ be limiting normally regular at $\bar{x} \in \Omega$.*

i) *If $\Omega + C$ is closed, then $\bar{x} \in Ben[\Omega, C]$ implies $\bar{x} \in L[\Omega, C]$.*
ii) *If $As(\Omega) \cap (-C) = \{0\}$, then $\bar{x} \in L[\Omega, C]$ implies $\bar{x} \in Ben[\Omega, C]$.*

In Example 3.2, $\Omega + C$ is not closed and $\bar{x} \in Ben[\Omega, C]$, while $\bar{x} \notin L[\Omega, C]$. This highlights the necessity to closedness of $\Omega + C$ in Corollary 3.1(i).

Remark 3.1. One of the most powerful tools for working with limiting normal cones is "exact extremal principle"; see [11]. In the current work, for avoiding complicated proofs, we did not use this principle. As can bee seen from [14, Theorem 3.5], if one applies this principle in the proof of Theorem 3.2, then limiting normal regularity assumption will be redundant in Theorem 3.2(i), and then in Corollary 3.1(i).

Example 3.3 gives a situation to show that if condition $As(\Omega) \cap (-C) = \{0\}$ does not hold, then a limiting proper minimal may not be a Benson proper minimal.

Example 3.3. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq -x_2\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 1 \ \text{or} \ x_2 \geq 1\},$$

$\bar{x} = (0, 0)$, and $C = \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Here, Ω is limiting normally regular at $\bar{x}$. Furthermore,

$$cl\ cone(\Omega + C - \bar{x}) = \mathbb{R}^2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0, \ x_2 < 0\},$$
$$N_L(\bar{x}; \Omega + C) = \{(x_1, x_2) \ : \ x_1 = x_2 \leq 0\}.$$

Thus

$$cl\, cone(\Omega + C - \bar{x}) \cap (-C) = \{(x_1, 0) : \ x_1 \leq 0\} \cup \{(0, x_2) : \ x_2 \leq 0\} \neq \{(0,0)\},$$

and $N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \neq \emptyset$. These imply $\bar{x} \in L[\Omega, C]$ while $\bar{x} \notin Ben[\Omega, C]$. Notice the $As(\Omega) \cap (-C) \neq \{(0,0)\}$.

By Example 3.4, it is seen that if limiting normal regularity is not fulfilled, then a limiting proper minimal may not be a Benson proper minimal.

Example 3.4. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \geq -|x_1|, \ x_2 \geq -1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 1, \ x_2 \geq -2\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \leq -1, \ x_2 \geq -2, \ (x_1, x_2) \neq (-1, -2)\}, C := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 2x_1 \geq 0\}, \ \bar{x} = (0,0)$. We have, $N_L(\bar{x}; \Omega + C) = \{(x_1, x_2) : \ x_2 = x_1 \leq 0\}$ while $\hat{N}(\bar{x}; \Omega + C) = \{(0,0)\}$. So, $\Omega + C$ is not limiting normally regular at $\bar{x}$. Furthermore,

$$As(\Omega) \cap (-C) = \{(0,0)\}, cl\, cone(\Omega + C - \bar{x}) \cap (-C) \neq \{(0,0)\}, N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \neq \emptyset.$$

These imply $\bar{x} \in L[\Omega, C]$ while $\bar{x} \notin Ben[\Omega, C]$.

Corollary 3.2 results from Theorems 3.1 and 3.2 invoking [7, Theorems 2.1 and 2.2] and (2.1).

Corollary 3.2. *Let $\Omega + C$ be limiting normally regular at $\bar{x} \in \Omega$.*

i) *If $\Omega + C$ is closed, then*

$$\bar{x} \in Ben[\Omega, C] \Longleftrightarrow \bar{x} \in He[\Omega, C] \Longrightarrow \bar{x} \in Bor[\Omega, C] \Longrightarrow \bar{x} \in L[\Omega, C].$$

ii) *If $As(\Omega) \cap (-C) = \{0\}$, then*

$$\bar{x} \in L[\Omega, C] \Longrightarrow \bar{x} \in Bor[\Omega, C] \Longleftrightarrow \bar{x} \in He[\Omega, C] \Longleftrightarrow \bar{x} \in Ben[\Omega, C].$$

iii) *If Ω is compact, then*

$$\bar{x} \in L[\Omega, C] \Longleftrightarrow \bar{x} \in Bor[\Omega, C] \Longleftrightarrow \bar{x} \in He[\Omega, C] \Longleftrightarrow \bar{x} \in Ben[\Omega, C].$$

Corollary 3.3 presents an important result showing that the set of limiting proper minimals is dense in that of minimals (under appropriate assumptions).

Corollary 3.3. *Let $\Omega + C$ be limiting normally regular. Under either (i) or (ii), $L[\Omega, C]$ is dense in $E[\Omega, C]$.*

(i)] $\Omega + C$ is closed and $As(\Omega + C) \cap (-C) = \{0\}$; (ii)] Ω is compact.

Proof. Apply [7, Theorem 5.1] and Theorem 3.2(i). □

The next result provides a characterization of limiting proper minimal points utilizing a linear scalarization technique.

Theorem 3.3. *Let $\bar{x} \in \Omega$ be given.*

a) *If there exists some $\lambda \in C^{*\circ}$ such that $\langle \lambda, \bar{x} \rangle \leq \langle \lambda, x \rangle$ for any $x \in \Omega$, then $\bar{x} \in L[\Omega, C]$.*

b) *If $\Omega + C$ is convex and $\bar{x} \in L[\Omega, C]$, then there exists some $\lambda \in C^{*\circ}$ such that $\langle \lambda, \bar{x} \rangle \leq \langle \lambda, x \rangle$ for any $x \in \Omega$.*

Proof. (a) It is not difficult to see that $\bar{x} \in E[\Omega, C]$. As $\lambda \in C^{*\circ}$ and $\hat{N}(\bar{x}; \Omega + C) \subseteq N_L(\bar{x}; \Omega + C)$, it is sufficient to prove $-\lambda \in \hat{N}(\bar{x}; \Omega + C)$. Since $\lambda \in C^{*\circ}$ and $\langle \lambda, \bar{x} \rangle \leq \langle \lambda, x \rangle$ for any $x \in \Omega$, we get

$$\langle -\lambda, x + c - \bar{x} \rangle \leq 0, \ \forall \, x \in \Omega \Rightarrow -\lambda \in \hat{N}(\bar{x}; \Omega + C).$$

(b)] To prove the converse, let $\bar{x} \in L[\Omega, C]$. Then $\bar{x} \in E[\Omega, C]$ and there is some $d \in N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ})$. As $\Omega + C$ is convex, $\langle d, x - \bar{x} \rangle \leq 0$ for each $x \in \Omega$. By setting $\lambda = -d$, the proof is completed.

□

Example 3.3 shows the second part of the above theorem may not hold in the absence of the convexity of $\Omega + C$. We close the paper by two remarks.

Remark 3.2. The relationships between proximal proper minimal points and robust solutions from one side and limiting proper minimal points from other side in Euclidean finite-dimensional spaces have been proved in [14].

Remark 3.3. As discussed in [14], one cannot derive desired properties for limiting proper minimals if replaces the condition $N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \neq \emptyset$ in defining these points with $N_L(\bar{x}; \Omega) \cap (-C^{*\circ}) \neq \emptyset$. Provided that $0 \in E[clcone(\Omega), C]$; See [14, Theorem 3.24],

$$\left(N_L(\bar{x}; \Omega + C) \cap (-C^{*\circ}) \right) \subseteq \left(N_L(\bar{x}; \Omega) \cap (-C^{*\circ}) \right).$$

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