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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

On solving split best proximity point and equilibrium problems in Hilbert spaces

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ABSTRACT. In this paper, we introduce a split best proximity point and equilibrium problem, and find a solution of the best proximity point problem such that its image under a given bounded linear operator is a solution of the equilibrium problem. We construct an iterative algorithm to solve such problem in real Hilbert spaces and obtain a weak convergence theorem. Finally, we also give an example to illustrate our result.

1. INTRODUCTION

In this paper, we introduce a new split problem, which is called a split best proximity point and equilibrium problems (SBPEP). Let H_1 and H_2 be two real Banach spaces. Let C and D be two subsets of H_1 with $d(C, D) = \inf\{\|c - d\| : c \in C \text{ and } d \in D\}$, K a closed convex subset of H_2, $A : H_1 \to H_2$ a bounded linear operator. Let $S : C \to D$ be a mapping and $f : K \times K \to \mathbb{R}$ be a bi-function. The SBPEP is

$$\text{to find a element } p \in C \text{ such that } \|p - Sp\| = d(C, D), \tag{1.1}$$

and

$$\text{such that } u := Ap \in K \text{ solves } f(u, v) \geq 0, \forall v \in K. \tag{1.2}$$

If we consider only (1.1), then (1.1) is a classical best proximity point problem. The best proximity point problem for nonlinear mappings is an interesting topic in the optimization theory (see [2, 3, 9]). It is well known that the concept of a best proximity point includes that of a fixed point as a special case.

On the other hand, if we consider only (1.2), then (1.2) is a classical equilibrium point problem. Various problems arising in physics, optimization and economics can be modeled as equilibirum problems. So equilibrium problem plays very important role in solving existence of solution of these problems (see [4, 11]). Some authors have proposed some methods to find the solution of the best proximity point problems (see [13, 5]) and equilibrirum problem (see [4, 11, 6, 7, 8, 10]).

In this paper, we construct some iterative algorithm for solving the SBPEP when the nonlinear mapping is best proximally nonexpansive in Hilbert spaces. Some weak convergence theorems are established. The results obtained in this paper can be established as the common solution of best proximity point problem and equilibrium problem. Finally, an example are given to illustrate our result.

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2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Recall that a mapping $T : H \to H$ is said to be

(1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} x, y \in H;$$

(2) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - q\| \leq \|x - q\| \quad \text{for all} \quad x \in H, q \in F(T),$$

where $F(T) = \{x \in C : Tx = x\}$. Observe that a nonexpansive mapping with at least one fixed point is quasi-nonexpansive.

Let A and B be two nonempty closed convex subsets of H. We define A_0 and B_0 by the following sets:

$$A_0 = \{x \in A : \|x - y\| = D(A, B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = D(A, B), \text{ for some } x \in A\}.$$

We recall some useful definitions and lemmas, which will be used in the later sections.

Let C be a nonempty closed convex subset of Hilbert space H. For any $x \in H$, its projection onto C is defined as

$$P_C(x) = \text{argmin}\{\|y - x\| | y \in C\}$$

The mapping $P_C : H \to C$ is called a *projection operator*, which has the well-known properties in the following lemma.

Lemma 2.1 ([1]). *Let C be a nonempty closed convex subset of Hilbert space H. Then for all $x, y \in H$ and $z \in C$,*

(1) $\langle P_Cx - x, z - P_Cx\rangle \geq 0$;
(2) $\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle$;
(3) $\|P_Cx - z\|^2 \leq \|x - z\|^2 - \|P_Cx - x\|^2$;
(4) $\|z - P_Cx\|^2 + \|x - P_Cx\|^2 \leq \|x - z\|^2$

A Banach space $(X, \|\cdot\|)$ said to satisfy *Opial's condition* if, for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well-known that each Hilbert space satisfies Opial's condition.

Lemma 2.2 ([13]). *Let A, B be two nonempty subsets of a uniformly convex Banach spaces X such that A is closed and convex. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$. Then $F(P_AT|_{A_0}) = Best_A(T)$.*

Definition 2.1 ([13]). Let A and B be two nonempty subsets of a real Hilbert space H and C a subset of A. A mapping $T : A \to B$ is said to be *C-nonexpansive* if

$$\|Tx - Tz\| \leq \|x - z\|$$

for all $x \in A$ and $z \in C$. If $C = Best_AT$, we say that T is a *best proximally nonexpansive mapping*.

Definition 2.2 ([12]). Let A and B be closed subsets of a metric space (X, d). Then, A and B are said to satisfy the *P-property* if, for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:

$$d(x_1, y_1) = d(x_2, y_2) = D(A, B) \to d(x_1, x_2) = d(y_1, y_2).$$

Notic that, for any pair (A, B) of nonempty closed and convex subsets of a real Hilbert space, H has the P-property.

Lemma 2.3 ([5]). *Let A, B be two nonempty subsets of a uniformly convex Banach space X such that A is closed and convex. Suppose that $T : A \to B$ is mapping such that $T(A_0) \subseteq B_0$. Then, $T|_{A_0}$ satisfies the proximal property if and only if $I - P_A T|_{A_0}$ is demiclosed at zero.*

Lemma 2.4 ([4]). *Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions:*

- $(A1)$ *$F(x, x) = 0$ for all $x \in K$;*
- $(A2)$ *is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;*
- $(A3)$ *for each $x, y \in K$,*

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

- $(A4)$ *for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } x, y \in K.$$

Lemma 2.5 ([8]). *Let K be a nonempty closed convex subset of H and let F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to K$ as follows:*

$$T_r^F(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in K \right\} \tag{2.3}$$

for all $x \in H$. Then the following hold:

(1) T_r^F is single-valued;

(2) T_r^F is firmly-nonexpansive, that is, for any $x, y \in H$,

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

(3) $F(T_r^F) = EP(F)$ for all $r > 0$;

(4) $EP(F)$ is closed and convex.

Lemma 2.6 ([10]). *Let K be a nonempty closed convex subset of H. For $x \in H$, let the mapping T_r^F be the same as in Lemma 2.5. Then for $r, s > 0$ and $x, y \in H$,*

$$\|T_r^F(x) - T_r^F(y)\| \leq \|y - x\| + \frac{|s - r|}{s}\|T_s^F(y) - y\|.$$

3. Main results

In this section, we prove some weak convergence theorem for SBPEP in Hilbert spaces.

Theorem 3.1 (Weak convergence theorem). *Let H_1 and H_2 be two real Hilbert spaces and $C, D \subset H_1, K \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $Best_C S \neq \emptyset$ and $f : K \times K \to \mathbb{R}$ a bi-function satisfying $(A1) - (A4)$ with $EP(f) \neq \emptyset$. Suppose that S satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0, \\ u_n = (1 - \alpha_n)x_n + \alpha_n P_C S x_n, \quad \forall n \geq 1, \\ x_{n+1} = P_C\left[u_n + \gamma A^*(T_{r_n}^f - I)Au_n\right], \quad n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_n\} \subset (0,1]$ *with* $\limsup_{n\to\infty} \alpha_n < 1$*,* $r_n \subset (0,\infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* $\gamma \in \left(0, \frac{1}{\|A^*\|^2}\right)$ *is a constant. Suppose that* $\Omega = \{p \in Best_C S : Ap \in EP(f)\} \neq \emptyset$*, then the sequence* $\{x_n\}$ *converges waekly to an element* $x^* \in \Omega$*.*

Proof. Let $p \in \Omega$. So $p \in Best_C S$ and $Ap \in EP(f)$. Since S is a best proximally nonexpansive mapping we have

$$\begin{aligned} \|u_n - p\|^2 &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|P_C S x_n - p\|^2 - \alpha_n(1-\alpha_n)\|P_C S x_n - x_n\|^2 \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|P_C S x_n - x_n\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|P_C S x_n - x_n\|^2. \end{aligned} \tag{3.5}$$

By Lemma 2.5 and $p \in EP(f)$, we have

$$\begin{aligned} \|T^f_{r_n} Au_n - Ap\|^2 &= \|T^f_{r_n} Au_n - T^f_{rn} Ap\|^2 \\ &\le \langle T^f_{r_n} Au_n - T^f_{rn} Ap, Au_n - Ap\rangle \\ &= \frac{1}{2}\left[\|T^f_{r_n} Au_n - T^f_{r_n} Ap\|^2 + \|Au_n - Ap\|^2 - \|T^f_{r_n} Au_n - Au_n\|^2\right], \end{aligned}$$

which implies that

$$\|T^f_{r_n} Au_n - Ap\|^2 \le \|Au_n - Ap\|^2 - \|T^f_{r_n} Au_n - Au_n\|^2. \tag{3.6}$$

Consider, by (3.6), we obtain

$$\begin{aligned} &2\gamma\langle u_n - p, A^*(T^f_{r_n} - I)Au_n\rangle \\ &= 2\gamma\langle A(u_n - p, (T^f_{r_n} - I)Au_n)\rangle \\ &= \langle A(u_n - p) + (T^f_{r_n} - I)Au_n - (T^f_{r_n} - I)Au_n, (T^f_{r_n} - I)Au_n\rangle \\ &= 2\gamma\left[\langle T^f_{r_n} Au_n - Ap, (T^f_{r_n} - I)Au_n\rangle - \|(T^f_{r_n} - I)Au_n\|^2\right] \\ &= 2\gamma\left[\frac{1}{2}\|T^f_{r_n} Au_n - Ap\|^2 + \frac{1}{2}\|(T^f_{r_n} - I)Au_n\|^2 - \frac{1}{2}\|Au_n - Ap\|^2 - \|(T^f_{r_n} - I)Au_n\|^2\right] \\ &\le 2\gamma\left[\frac{1}{2}\|Au_n - Ap\|^2 + \frac{1}{2}\|(T^f_{r_n} - I)Au_n\|^2 - \frac{1}{2}\|Au_n - Ap\|^2 - \|(T^f_{r_n} - I)Au_n\|^2\right] \\ &= 2\gamma\left[-\frac{1}{2}\|(T^f_{r_n} - I)Au_n\|^2\right] \\ &= -\gamma\|(T^f_{r_n} - I)Au_n\|^2. \end{aligned} \tag{3.7}$$

From, we have

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \|P_C\left[u_n + \gamma A^*(T^f_{r_n} - I)Au_n\right] - p\|^2 \\ &= \|P_C\left[u_n + \gamma A^*(T^f_{r_n} - I)Au_n\right] - P_C p\|^2 \\ &\le \|u_n + \gamma A^*(T^f_{r_n} - I)Au_n - p\|^2 \\ &= \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T^f_{r_n} - I)Au_n\|^2 + 2\gamma\langle u_n - p, A^*(T^f_{r_n} - I)Au_n\rangle \\ &\le \|u_n - p\|^2 + \gamma^2\|A^*\|^2\|(T^f_{r_n} - I)Au_n\|^2 - \gamma\|(T^f_{r_n} - I)Au_n\|^2 \\ &= \|u_n - p\|^2 - \gamma(1 - \gamma\|A^*\|^2)\|(T^f_{r_n} - I)Au_n\|^2. \end{aligned} \tag{3.8}$$

Since $\gamma \in (0, \frac{1}{\|A^*\|^2})$, $\gamma(1-\gamma\|A^*\|^2) > 0$. It follows from (3.5) and (3.8) that

$$\|x_{n+1} - p\| \leq \|u_n - p\| \leq \|x_n - p\| \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Therefore $\lim_{n\to\infty} \|x_n - p\| = r \geq 0$. Again by (3.9), we have $\lim_{n\to\infty} \|u_n - p\| = r$. By (3.5), we have $\lim_{n\to\infty} \|P_C S x_n - x_n\| = 0$, which implies that

$$\|u_n - x_n\| = \alpha_n \|P_C S x_n - x_n\| \to 0. \tag{3.10}$$

Because $\lim_{n\to\infty} \|x_n - p\| = r$, then $\{x_n\}$ is bounded, hence $\{x_n\}$ has a weakly convergence subsequence $\{x_{n_j}\}$. Assume that $x_{n_j} \rightharpoonup x^*$ for some $x^* \in C$. Then $u_{n_j} \rightharpoonup x^*$, and $Au_{n_j} \rightharpoonup Ax^*$ by (3.10) and A is a bounded linear operator.

Now we prove $x^* \in \Omega$, that is $x^* \in Best_C S$ and $Ax^* \in EP(f)$. Since $lim_{n\to\infty}\|u_n - p\| = \lim_{n\to\infty} \|x_n - p\| = r$ and (3.5), we have

$$\lim_{n\to\infty} \|P_C S x_n - x_n\| = 0. \tag{3.11}$$

Since S satisfy the proximal property, by Lemma 2.3, we have $I - P_C S|_{C_0}$ is demiclosed at zero. It follows by (3.11) that $P_C S x^* = x^*$, i.e., $x^* \in Best_C S$. Since $lim_{n\to\infty}\|u_n - p\| = \lim_{n\to\infty} \|x_n - p\| = r$ and (3.8), we have

$$\lim_{n\to\infty} \|T^f_{r_n} Au_n - Au_n\| = 0. \tag{3.12}$$

Otherwise, if $T^f_{r_n} Ax^* \neq Ax^*$ for some $r > 0$, then by Opial's condition, Lemma 2.6 and (3.8), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|Au_{n_j} - Ax^*\| &< \liminf_{j\to\infty} \|Au_{n_j} - T^f_{r_{n_j}} Ax^*\| \\
&\leq \liminf_{j\to\infty} \left\{ \|Au_{n_j} - T^f_{r_{n_j}} Au_{n_j}\| + \|T^f_{r_{n_j}} Au_{n_j} - T^f_r Ax^*\| \right\} \\
&= \liminf_{j\to\infty} \|T^f_{r_{n_j}} Au_{n_j} - T^f_r Ax^*\| \\
&\leq \liminf_{j\to\infty} \left(\|Au_{n_j} - Ax^*\| + \frac{|r_{n_j} - r|}{r} \|T^f_r Au_{n_j} - Au_{n_j}\| \right) \\
&= \liminf_{j\to\infty} \|Au_{n_j} - Ax^*\|,
\end{aligned}$$

which is a contradiction. Therefore $T^f_r Ax^* = Ax^*$ for all $r > 0, i.e., Ax^* \in EP(f)$. The proof is completed. □

By setting $H = H_1 = H_2$ and $A := I$ (the identity mapping) in Theorem 3.1, we have immediately the following collary.

Corollary 3.1. *Let H be a real Hilbert spaces, and C, D be nonempty closed convex subsets of H. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $Best_C S \neq \emptyset$ and $f : C \times C \to \mathbb{R}$ a bi-function satisfying $(A1)-(A4)$ with $EP(f) \neq \emptyset$. Suppose that S satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0, \\ u_n = (1-\alpha_n)x_n + \alpha_n P_C S x_n, \\ x_{n+1} = (1-\gamma)u_n + \gamma T^f_{r_n} u_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$, $r_n \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in \left(0, \frac{1}{\|I\|^2}\right)$ is a constant. Suppose that $Best_C S \cap EP(f) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $x^* \in Best_C S \cap EP(f)$.

By setting $f(x,y) = 0$ for all $x, y \in C, r_n = 1$ for all $n \in \mathbb{N}$ in Corollary 3.1. Then we obtain the algorithm which converges weakly to the best proximity point for best proximally nonexpansive operators, which is difference from general Mann algorithm defined by Suparatulatorn *et al* [14].

Corollary 3.2. *Let H be a real Hilbert spaces, and C, D be nonempty closed convex subsets of H. Let $S : C \to D$ be best proximally nonexpansive mapping such that $S(C_0) \subset D_0$ with $Best_C S \neq \emptyset$. Suppose that S satisfies the proximal property. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C_0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C S x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\gamma \in \left(0, \frac{1}{\|I\|^2}\right)$ is a constant. Then the sequence $\{x_n\}$ converges weakly to an element $x^* \in Best_C S$.

By setting $S := I$ (the identity mapping) in Corollary 3.1, we have immediately the following collary

Corollary 3.3. *Let H be a real Hilbert spaces, and C be nonempty closed convex subsets of H. Let $f : C \times C \to \mathbb{R}$ a bi-function satisfying $(A1) - (A4)$ with $EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\gamma)x_n + \gamma T_{r_n}^f x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $r_n \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\gamma \in \left(0, \frac{1}{\|I*\|^2}\right)$ is a constant. Then the sequence $\{x_n\}$ converges weakly to an element $x^* \in EP(f)$.

4. Numerical Example

We give an example and numerical result for supporting our main theorem.

Example 4.1. *Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}$, $C = [-1,0] \times [0,1], D = [3,7] \times [0,1]$ and $K = [-3,0]$. Define two mappings $A : \mathbb{R}^2 \to \mathbb{R}$ and $S : C \to D$ by $A(x^{(1)}, x^{(2)}) = 3x^{(1)}$ for all $(x^{(1)}, x^{(2)}) \in \mathbb{R}^2$ and $S(x^{(1)}, x^{(2)}) = (3 - x^{(1)}, \dfrac{x^{(2)}}{2})$ for all $(x^{(1)}, x^{(2)}) \in C$. Then $C_0 = \{(0,z) : 0 \leq z \leq 1\}$. Let $f(u,v) = (u-1)(v-u)$ for all $u, v \in K$. Choose $\alpha_n = \dfrac{n}{2n+1}, r_n = \dfrac{n}{n+1}$ and $\gamma = \dfrac{1}{20}$. It is easy to check that f satisfies all conditions in Theorem 3.1 such that $EP(f) = \{0\}$ and S is a best proximally nonexpansive mappings such that $S(C_0) \subseteq D_0$ with $Best_C S = \{(0,0)\}$*

Then Algorithm (3.4) can be simplified as

$$\begin{cases} x_0 \in \{(0,z) : 0 \leq z \leq 1\} \\ u_n = \left(\dfrac{(n+1)x_n^{(1)}}{2n+1}, \dfrac{(3n+2)x_n^{(2)}}{4n+2} \right), \\ x_{n+1} = \left(1, u_n^{(2)}\right), \ \ \forall n \geq 1, \end{cases}$$

Next, choosing the initial point $x_0 = (0,1)$ and the stopping criterion for our testing method is $E_n = \|x_{n+1} - x_n\| \leq 1 \times 10^{-9}$. The following table shows the numerical experiment of the proposed algorithm. From Table 1, we observe that the sequence $\{x_n\}$ converges to $(0,0)$ which is a best proximity point of S and $A(0,0) = 0$ is an equilibrium point of f. Figure 1 shows the errors $E_n = \|x_{n+1} - x_n\|$ in each iteration.

n	x_n	E_n
0	*(0, 1)*	-
1	*(0, 0.8333)*	*0.1667*
2	*(0, 0.667)*	*0.1667*
3	*(0, 0.5238)*	*0.1429*
⋮	⋮	⋮
70	*(0, 1.4320e-08)*	*1.5758e-09*
71	*(0, 1.0765e-09)*	*1.1848e-09*
72	*(0, 4.4745e-09)*	*8.9082e-10*

TABLE 1. Numerical results for Algorithm 3.4

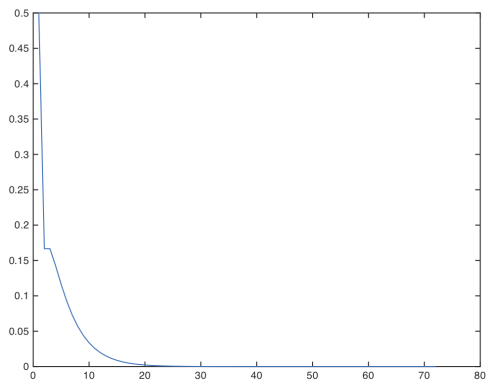

FIGURE 1. The error ploting of $E_n = \|x_{n+1} - x_n\|$

5. CONCLUSION

This paper introduced and discussed the split best proximity point and equilibrium problems in Hilbert spaces. We proposed an algorithms for solving such problems. We also considered its convergence results and gives a numerical example.

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