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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

Characterizing the Lagrange multiplier rule in nonconvex set-valued optimization

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ABSTRACT. In this article, by using the notions of contingent derivative, contingent epiderivative and generalized contingent epiderivative, we obtain some characterizations of the Lagrange multiplier rule at points which are not necessarily local minima.

1. INTRODUCTION AND PRELIMINARIES

Set-valued optimization is very known in optimization theory and Economics (see for instance [1, 2, 3, 10, 13]). In recent years, many scholars have a great attention to the set-valued optimization problems; see [3, 4, 8, 11, 13, 17] and the references therein. In set-valued optimization, derivative of a set-valued map plays an important role in different aspects. These derivatives have been formulated in different ways. Aubin [1] introduced the concept of the contingent derivative. While, Chen and Jahn [3] introduced a generalized contingent epiderivative of set-valued map, which exists under standard assumptions.

Lagrange [14] published his multiplier rule, which is a vital tool in constrained optimization. The Lagrange rule remains valid in the vector optimization problem. Jahn and Khan [11] extended the Lagrange rule to set-valued optimization for the notion of generalized contingent epiderivative. However, it is important to find some conditions ensuring that the multiplier of the objective function is nonzero. In scalar and vector optimization theory of such conditions are referred to as constraint qualification.

We use the notions of contingent derivative, contingent epiderivative and generalized contingent epiderivative and obtain some necessary and sufficient conditions for establishing the Lagrange multiplier rule. Also similar results by weak contingent epiderivative and the second order composed adjacent contingent derivatives are obtained.

Here, we give some notation and definitions, which will be used in the sequel. Let X be a normed space and let A be a nonempty subset of X. The symbols $\mathrm{int}(A)$, $\overline{A}$, $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ denote the interior, closure, convex hull and closed convex hull of A, respectively. A nonempty subset C of X is called a convex cone if C is convex and $tC \subseteq C$ for any $t \geq 0$. We say that the cone C is solid whenever $\mathrm{int}(C) \neq \emptyset$ and it is called pointed if $C \cap \{-C\} = \{0\}$. Also the smallest cone containing of a nonempty subset B of X is denoted by $\mathrm{cone}(B)$. It is easy to see that

$$\mathrm{cone}(B) = \cup_{t \geq 0} tB.$$

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Additionally, we set

$$\text{cone}_+(B) = \cup_{t>0} tB.$$

Let X and Y be normed spaces and $F : X \rightrightarrows Y$ be a set-valued map. Let $C \subset Y$ be a closed convex cone inducing a partial ordering in Y. The effective domain, the graph and the epigraph of F are defined as follows:

$$\mathrm{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\},$$

$$\mathrm{graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\},$$

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y \mid x \in X, y \in F(x) + C\}.$$

Let A be a subset of Y, an element $\bar{y} \in A$ is said to be a minimal point of A if $A \cap (\bar{y} - C) = \{\bar{y}\}$. The set of all minimal points of A with respect to the cone C will be denoted by $\mathrm{Min}(A, C)$.

If $\mathrm{int}(C) \neq \emptyset$, then $\bar{y} \in A$ is said to be a weakly minimal point of A if $A \cap (\bar{y} - \mathrm{int}(C)) = \emptyset$. The set of all weakly minimal points of A with respect to cone C, will be denoted by $\mathrm{WMin}(A, C)$.

Throughout this paper, let X, Y, Z be real normed spaces, C and D be pointed, closed convex cone in Y and Z, respectively. Suppose S is a nonempty subset of X and $F : S \rightrightarrows Y$ and $G : S \rightrightarrows Z$ are two set-valued maps. We consider the following set-valued optimization problems:

$$\text{(P)} \ \min F(x) \text{ subject to } x \in S$$

and

$$\text{(CP)} \ \min F(x) \text{ subject to } x \in S_1 := \{x \in S \mid G(x) \cap (-D) \neq \emptyset\}.$$

Let $\bar{x} \in S_1$. The point $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$ is a (CP)- minimizer, if $\bar{y} \in \mathrm{Min}(F(S_1), C)$; that is $F(S_1) \cap (\bar{y} - C) = \{\bar{y}\}$, where $F(S_1) = \cup_{x \in S_1} F(x)$.

Suppose that $\bar{x} \in \overline{S}$. The contingent cone (resp. the adjacent cone) of S at $\bar{x}$, $T(S, \bar{x})$ is defined as follows:

$$T(S, \bar{x}) := \{x \in X : \exists t_k \downarrow 0, \exists x_k \in X, x_k \to x, \bar{x} + t_k x_k \in S, \forall k\}.$$

$$(\text{resp. } A(S, \bar{x}) := \{x \in X : \forall \{\lambda_n\} \subset \mathbb{R}_+ \backslash \{0\}, \exists \{x_n\}_{n \in \mathbb{N}} \subset X, x_n \to x, \bar{x} + \lambda_n x_n \in S, \forall n \in \mathbb{N}\}.)$$

If $T(S, \bar{x}) = A(S, \bar{x})$, we say that the set S is derivable at $\bar{x}$.

Now, we give the notions of contingent derivative, contingent epiderivative and generalized contingent epiderivative of a set-valued map.

Definition 1.1. [1, 3, 12] Assume that $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$.

(1) A set-valued map $DF(\bar{x}, \bar{y}) : X \rightrightarrows Y$ which is defined by

$$DF(\bar{x}, \bar{y})(x) := \{y \in Y \mid (x, y) \in T(\mathrm{graph}(F), (\bar{x}, \bar{y}))\}$$

is called the contingent derivative of F at $(\bar{x}, \bar{y})$.

(2) A single valued map $D_E F(\bar{x}, \bar{y}) : X \to Y$ is called the contingent epiderivative of F at $(\bar{x}, \bar{y})$, if the epigraph of $D_E F(\bar{x}, \bar{y})$ coincides with the contingent cone to the epigraph of F at $(\bar{x}, \bar{y})$, that is $\mathrm{epi}(D_E F(\bar{x}, \bar{y})) = T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$.

(3) A set-valued map $D_g F(\bar{x}, \bar{y}) : X \rightrightarrows Y$, is called generalized contingent epiderivative of F at $(\bar{x}, \bar{y})$, if the following identity holds

$$D_g F(\bar{x}, \bar{y})(x) = \mathrm{Min}(D(F + C)(\bar{x}, \bar{y})(x), C), x \in \mathrm{dom}(D(F + C)(\bar{x}, \bar{y})).$$

Remark 1.1. If $\mathrm{int}(C) \neq \emptyset$, then we can replace the "Min" in the Definition 1.1 (3) by "WMin" and define similarly $D_w F(\bar{x}, \bar{y})(.)$, the weak contingent epiderivative of F at $(\bar{x}, \bar{y})$.

The map F is called Lipschitz-like around $(\bar{x}, \bar{y}) \in \text{graph}(F)$, if there exist a constant $L \geqslant 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that for every $x_1, x_2 \in U \cap \text{dom}(F)$, one has

$$F(x_1) \cap V \subseteq F(x_2) + L||x_1 - x_2||B_Y,$$

where B_Y is the closed unit ball of Y.

As a necessary optimality condition for set-valued optimization problem, we refer to the following Lagrange multiplier rule along with a constraint qualification.

Theorem 1.1 (Theorem 12.1.11, [13]). *Let C, D be solid cones. For $\bar{x} \in S$, let $(\bar{x}, \bar{y}) \in \text{graph}(F)$, be a local weak minimizer of (CP), and let $\bar{z} \in G(\bar{x}) \cap (-D)$. Assume that either $(F + C)$ is Lipschitz-like around $(\bar{x}, \bar{y})$ or $(G + D)$ is Lipschitz-like around $(\bar{x}, \bar{z})$. Suppose that either $\text{epi}(F)$ is derivable at $(\bar{x}, \bar{y})$ or $\text{epi}(G)$ is derivable at $(\bar{x}, \bar{z})$. Let $(D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(\Omega)$ be convex, where $\Omega := \text{dom}(D(F + C)(\bar{x}, \bar{y})) \cap \text{dom}(D(G + D)(\bar{x}, \bar{z}))$. Then there exists $(\xi, \zeta) \in Y^* \times Z^* \setminus \{(0, 0)\}, \xi \in C^+, \zeta \in D^+$:*

$$\xi(y) + \zeta(z) \geqslant 0, \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(\Omega).$$

Furthermore, if additionally we consider the following constraint qualification

$$D(G + D)(\bar{x}, \bar{y})(\Omega) + \text{cone}(D + \bar{z}) = Z,$$

then $\xi \neq 0$.

2. Main results

In this section, we give some characterizations of the Lagrange multiplier rule at points which are not necessarily local minima. Theorems of alternative type, play an important role in optimization. In the following we give an alternative theorem for the contingent derivative.

Theorem 2.2. (Alternative type) *Suppose that C, D are solid, $\bar{x} \in S_1$, $(\bar{x}, \bar{y}) \in \text{graph}(F)$ and $\bar{z} \in G(\bar{x}) \cap -D$. Let B be a nonempty cone subset of $\text{dom}(D(F + C)(\bar{x}, \bar{y})) \cap \text{dom}(D(G + D)(\bar{x}, \bar{z}))$, and let $(D(F+C)(\bar{x}, \bar{y}), D(G+D)(\bar{x}, \bar{z}))(B)$ be convex, then only one of the following statements holds.*

(i) There exists $v \in B$ such that

$$D(F + C)(\bar{x}, \bar{y})(v) \cap (-\text{int}(C)) \neq \emptyset \text{ and } D(G + D)(\bar{x}, \bar{z})(v) \cap (-\text{int}(D)) \neq \emptyset.$$

(ii) There exists $(\xi, \zeta) \in Y^* \times Z^* \setminus \{(0, 0)\}$ such that $\xi \in C^+, \zeta \in D^+$, $\zeta(\bar{z} = 0$ and

$$\xi(y) + \zeta(z) \geqslant 0,\ \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B).$$

Proof. Consider $A := \Phi(B) + (0, \bar{z})$, where $\Phi := (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))$, then A is convex. From standard separation theorem, exactly one of the following two assertions holds.

(i) $A \cap (-\text{int}(C \times D)) \neq \emptyset$.

(ii) There exists a continuous and linear functional $s : Y \times Z \to \mathbb{R}$ such that $s = (\xi, \zeta) \in (C^+ \times D^+) \setminus \{(0, 0)\}$ and $s(y, z) \geqslant 0$ for every $(y, z) \in A$, and so

$$\xi(y) + \zeta(z + \bar{z}) \geqslant 0, \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B).$$

Since $(0, 0) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B)$, we get $\zeta(\bar{z}) \geqslant 0$. Since $\zeta \in D^+$ and $\bar{z} \in -D$, we have $\zeta(\bar{z}) \leq 0$. Therefore, $\zeta(\bar{z}) = 0$ and

$$\xi(y) + \zeta(z) \geqslant 0,\ \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B).$$

□

Notice that the functionals ξ and ζ in condition (ii) of the above theorem are called the Lagrange multiplier and this condition is a necessary optimal condition, see [13, Theorem 12.1. 10]. Also, if $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $Z = \mathbb{R}^m$, F, G are differentiable single-valued functions and B is convex cone, then the above alternative theorem reduce to Theorem 3.1 of [6]. However, an important question is to fined conditions ensuring that $\xi \neq 0$. In what follows we give an answer to this question.

Let us consider the CP-weak minimizers:

$$V_0 := \mathrm{WMin}(D(F + C)(\bar{x}, \bar{y})(G_0(\bar{x}, \bar{z})), C),$$

where $\bar{z} \in G(\bar{x}) \cap (-D)$, B is a cone and

$$G_0(\bar{x}, \bar{z}) := \{v \in \overline{\mathrm{co}}B : D(G + D)(\bar{x}, \bar{z})(v) \cap (-\mathrm{int}(D)) \neq \emptyset\}.$$

By Remark 1.1, we have $V_0 = D_w F(\bar{x}, \bar{y})(G_0(\bar{x}, \bar{z}))$.

In the following result, we obtain a necessary and sufficient condition of the Lagrange multiplier rule along with $\xi \neq 0$, by using the alternative theorem at points which are not necessarily local weak minimizer.

Corollary 2.1. *If $G_0(\bar{x}, \bar{z}) \neq \emptyset$, then in Theorem 2.2 $\xi \neq 0$, and $[(ii) \Leftrightarrow 0 \in V_0]$.*

Now, we obtain a characterization of the Lagrange multiplier rule along with $\xi \neq 0$, without using the alternative theorem and constraint qualification.

Theorem 2.3. *Let C be solid, $\Omega = \mathrm{dom}(D(F + C)(\bar{x}, \bar{y})) \cap \mathrm{dom}(D(G + D)(\bar{x}, \bar{z}))$ and B be a cone in Ω. If $\bar{x} \in S$ and $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$, then the following assertions are equivalent.*

(a) $\overline{\mathrm{co}}(\Phi(B)) \cap -(\mathrm{int}(C) \times \{0\}) = \emptyset$, where $\Phi(B) := (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B)$.

(b) There exist $(\xi, \zeta) \in Y^* \setminus \{0\} \times Z^*$ such that $\xi \in C^+, \zeta \in D^+$ and

$$\xi(y) + \zeta(z) \geqslant 0,\ \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(B).$$

Proof. $(a) \Rightarrow (b)$ Let $c \in \mathrm{int}(C)$ and V be an open convex neighborhood of 0 in Y such that $-c + V \subset -\mathrm{int}(C)$. Assume that W is an open convex neighborhood of 0 in Z and

$$U := (-c, 0) + (V(0_Y) \times W(0_Z)).$$

Then, $\mathrm{cone}_+(U)$ is an open convex set and condition (b) implies that

$$\mathrm{co}(\Phi(B)) \cap \mathrm{cone}_+(U) = \emptyset.$$

By the Hahn-Banach separation theorem, there exists $(\xi, \zeta) \in Y^* \times Z^* \setminus \{(0, 0)\}$ such that

$$\xi(y) + \zeta(z) \geqslant 0,\ \forall (y, z) \in \Phi(B),$$

and

$$\xi(y) + \zeta(z) \leqslant 0,\ \forall (y, z) \in \mathrm{cone}(U).$$

Now we show that $\xi \neq 0$. Suppose on the contrary that $\xi = 0$, then $\zeta(z) \leqslant 0$, for any $z \in W$ and so $\zeta = 0$, which contradicts $(\xi, \zeta) \neq (0, 0)$. Let us prove that $\xi \in C^+$ and $\zeta \in D^+$. Since $D(F+C)(\bar{x}, \bar{y})(.) = D_E F(\bar{x}, \bar{y})(.)+C$, $D(G+D)(\bar{x}, \bar{z})(.) = D_E G(\bar{x}, \bar{z})(.)+D$ and $(0, 0) \in (D_E F(\bar{x}, \bar{y}), D_E G(\bar{x}, \bar{z}))(B)$, we put $y = v + c$, $z = w + d$ and by taking $d = 0$, $v = 0$ and $w = 0_Z$, we obtain $\xi(c) \geqslant 0$ for every $c \in C$. Hence, $\xi \in C^+$. Also by taking $c = 0$, $v = 0$ and $w = 0$, we obtain $\zeta \in D^+$.

$(b) \Rightarrow (a)$ Let $\xi(y) + \zeta(z) \geqslant 0$, for any $(y, z) \in \Phi(B), \xi \in C^+, \zeta \in D^+, \xi \neq 0$. Therefore, $\xi(y) + \zeta(z) \geqslant 0$, for every $(y, z) \in \overline{\mathrm{co}}(\Phi(B))$.

Since $\xi(y) + \zeta(z) = \xi(y) < 0$, for each $(y, z) \in -(\mathrm{int}(C) \times \{0\})$, then (b) holds. □

Theorem 2.4. *Let C be a solid cone, and let B be a cone in Ω. For $\bar{x} \in S$, let $(\bar{x}, \bar{y}) \in \mathrm{graph}(F)$, and let $\bar{z} \in G(\bar{x}) \cap (-D)$. Consider the following conditions*

(a) $\overline{\Phi(\overline{\mathrm{co}B})} \cap -(\mathrm{int}(C) \times \{0\}) = \emptyset$.

(b) There exists $(\xi, \zeta) \in Y^* \setminus \{0\} \times Z^*$ *such that* $\xi \in C^+$, $\zeta \in D^+$ *and* $\xi(y) + \zeta(z) \geqslant 0$, *for every* $(y, z) \in \Phi(B)$.

If F *is* $C-$*convex and* G *is* $D-$*convex, then condition* (a) *implies* (b).

Proof. First we show that $\Phi(\overline{\text{co}B})$ is convex. If $(y_1, z_1), (y_2, z_2) \in \Phi(\overline{\text{co}B})$, then there exist $x_1, x_2 \in \overline{\text{co}B}$ such that

$$(y_i, z_i) \in (D_E F(\bar{x}, \bar{y})(x_i), D_E G(\bar{x}, \bar{z})(x_i)) + (C \times D).$$

Therefore,

$$(x_i, y_i) \in \text{epi}(D_E F(\bar{x}, \bar{y})) = T(\text{epi}(F), (\bar{x}, \bar{y})),$$

and

$$(x_i, z_i) \in \text{epi}(D_E G(\bar{x}, \bar{z})) = T(\text{epi}(G), (\bar{x}, \bar{z})).$$

Since $T(\text{epi}(F), (\bar{x}, \bar{y}))$ and $T(\text{epi}(G), (\bar{x}, \bar{z}))$ are convex, then for all $0 \leqslant \lambda \leqslant 1$ we have

$$\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in T(\text{epi}(F), (\bar{x}, \bar{y})),$$

and

$$\lambda(x_1, z_1) + (1 - \lambda)(x_2, z_2) \in T(\text{epi}(G), (\bar{x}, \bar{z})).$$

Hence,

$$\lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \in (D_E F(\bar{x}, \bar{y}), D_E G(\bar{x}, \bar{z}))(\lambda x_1 + (1 - \lambda)x_2) + (C \times D),$$

and so $\lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \in \Phi(\overline{coB})$. Now, suppose (a) holds, $c \in \text{int}(C)$ and define

$$U := (-c, 0) + V \times W,$$

where $-c + V \subset -\text{int}(C)$, V and W are open convex neighborhoods of 0 in Y and Z, respectively. Then, $\text{cone}_+(U)$ is an open convex set and by (a), we have

$$\Phi(\overline{\text{co}}B) \cap \text{cone}_+(U) = \emptyset.$$

By the Hahn-Banach separation theorem, there exists $(\xi, \zeta) \in Y^* \times Z^* \setminus \{(0, 0)\}$ such that

$$\xi(y) + \zeta(z) \geqslant 0, \ \forall (y, z) \in \Phi(\overline{\text{co}B}) \supset \Phi(B),$$

and

$$\xi(y) + \zeta(z) \leqslant 0, \ \forall (y, z) \in \text{cone}(U).$$

Now, we prove that $\xi \neq 0$. Suppose on the contrary that $\xi = 0$. Then $\zeta(z) \leqslant 0$, for each $z \in W$ which implies $\zeta = 0$. This contradicts $(\xi, \zeta) \neq (0, 0)$. Therefore, $\xi \neq 0$. Also, by a similar proof as that of the proof of Theorem 2.3, we have $\xi \in C^+$ and $\zeta \in D^+$. □

Remark 2.2. (I) Suppose that $X = \mathbb{R}^n, Y = Z = \mathbb{R}, C = D = \mathbb{R}_+$, $F = f$ and $G = g$ are single-valued and differentiable at point $\bar{x}$. By [9, Remark 15.2] $D(f + C)(\bar{x}, f(\bar{x}))(x) = \langle \nabla f(\bar{x}), x \rangle + C, D(g + D)(\bar{x}, g(\bar{x}))(x) = \langle \nabla g(\bar{x}), x \rangle + D$ and we have

- (i) condition (b) in Theorem 2.3, is the same condition (d) in [6, Theorem 4.1] and condition (a) is the same condition (b) (KKT condition) in [6, Theorem 4.1], in the case where $|I| = 1$;
- (ii) condition (a) in Theorem 2.4 turns into the condition (c) in [6, Theorem 4.1], with $|I| = 1$, and condition (b) in Theorem 2.4 is the same condition (b) in [6, Theorem 4.1].

(II) Let $f : X \to Y$ be Hadamard directionally differentiable at $\bar{x} \in X$ in the direction $x \in X$. Then by [13, Proposition 11.1.3]

$$Df(\bar{x}) := Df(\bar{x}, f(\bar{x}))(x) = \{f'(\bar{x}, x)\},$$

where $f'(\bar{x}, x)$ is a Hadamard directional derivative of f at $\bar{x} \in X$ in the direction $x \in X$. Also, if $f : X \to \mathbb{R}$ is a convex function and f is continuous at $\bar{x} \in X$, then by [9,

Theorem 15.15], the contingent epiderivative is equal to the directional derivative. Hence, in these cases from Theorem 2.3 we obtain an equivalent condition to KKT condition in the nonsmooth case.
(III) When the directional derivative or the Fréchet derivative of single-valued map does not exist, the contingent derivative can be very useful, see [16].

Condition (b) in Theorems 2.3 and 2.4 is a necessary optimality condition but it is not a sufficient optimality condition in the general case. Moreover, by considering convexity assumptions on F and G, it is proved that condition (b) in Theorems 2.3 and 2.4 is a sufficient optimality condition, see [3, 8, 11, 12, 13].

In order to show that condition (b) in the above theorems can be a sufficient optimality condition, we need a similar notion of convexity to Definition 12.3.3 of [13] as follows.

Definition 2.2. Let $S \subset X$, and $\Xi \subset Y \times Z$. Let $F : S \subseteq X \rightrightarrows Y$, $G : S \subseteq X \rightrightarrows Z$ be two set-valued maps, and let $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$, $(\bar{x}, \bar{z}) \in \operatorname{graph}(G)$. The pair (F, G) is said to be Ξ-composite-contingently-quasi-convex at $(\bar{x}, \bar{y}, \bar{z})$ if for every $x \in S$, the condition $[(F, G)(x) - (\bar{y}, \bar{z})] \cap \Xi \neq \emptyset$ ensures that $(DF(\bar{x}, \bar{y}), DG(\bar{x}, \bar{z}))(x - \bar{x}) \cap \Xi \neq \emptyset$.

The following result is similar to [13, Theorem 12.3.6] and its proof is similar to that of Theorem 12.3.4 in [13] and so it is omitted.

Theorem 2.5. *Let C be solid, S be convex, $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{graph}(F, G)$ and let $S - \bar{x} \subseteq \Omega$, where $\Omega = \operatorname{dom}(D(F + C)(\bar{x}, \bar{y})) \cap \operatorname{dom}(D(G + D)(\bar{x}, \bar{z}))$. Assume that there exist $\xi \in C^{+} \setminus \{0\}$ and $\zeta \in D^{+}$ with $\zeta(\bar{z}) = 0$ such that for all $x \in S$,*

$$\xi(y) + \zeta(z) \geqslant 0, \ \forall (y, z) \in (D(F + C)(\bar{x}, \bar{y}), D(G + D)(\bar{x}, \bar{z}))(x - \bar{x}).$$

Let $\hat{S} := \{x \in S \mid G(x) \cap (-D + cone(\bar{z}) - cone(\bar{z})) \neq \emptyset\}$. If $(F + C, G + D) : \hat{S} \rightrightarrows Y \times Z$ is Ξ-composite-contingently-quasi-convex at $(\bar{x}, \bar{y}, \bar{z})$ with $\Xi = (-intC) \times (-D + cone(\bar{z}) - cone(\bar{z}))$, then $(\bar{x}, \bar{y}))$ is a weak minimizer of F on $\hat{S}$.

Since the inclusion $(DF(\bar{x}, \bar{y}), DG(\bar{x}, \bar{z}))(.) \supseteq D(F, G)(\bar{x}, \bar{y})(.)$ always holds, while the reverse inclusion in general not true. Thus, the above theorem is an improvement of [13, Theorem 12.3.6]. Also, since $S_1 \subseteq \hat{S}$, from Theorem 2.5 one can obtain that $(\bar{x}, \bar{y})$ is a weak minimizer of (CP).

In the following result by using generalized contingent epiderivative, we establish another characterization of the Lagrange multiplier rule along with $\xi \neq 0$, without constraint qualification.

Theorem 2.6. *Let C be solid, $(\bar{x}, \bar{y}) \in \operatorname{graph}(F)$ and let $\bar{z} \in G(\bar{x}) \cap (-D)$. Let*

$$\Omega := \operatorname{dom}(D_g F(\bar{x}, \bar{y})) \cap \operatorname{dom}(D_g G(\bar{x}, \bar{z})) \neq \emptyset,$$

and $(0, 0) \in \overline{(D_g F(\bar{x}, \bar{y}), D_g G(\bar{x}, \bar{z}))(B)}$, where B be a cone in Ω. Then the following conditions are equivalent.

(a) $\overline{\operatorname{co}}[(D_g F(\bar{x}, \bar{y}), D_g G(\bar{x}, \bar{z}))(B) + (C \times D)] \cap -(\operatorname{int}(C) \times \{0_Z\}) = \emptyset.$

(b) There exists $(\xi, \zeta) \in Y^{*} \times Z^{*} \setminus \{(0, 0)\}$ with $\xi \in C^{+}$ and $\zeta \in D^{+}$ such that $\xi \neq 0$ and $\xi(y) + \zeta(z) \geqslant 0$, for every $(y, z) \in (D_g F(\bar{x}, \bar{y}), D_g G(\bar{x}, \bar{z}))(B)$.

Proof. $(a) \Rightarrow (b)$ Let

$$\overline{\operatorname{co}}[\Psi(B) + C \times D] \cap -(\operatorname{int}(C) \times \{0_Z\}) = \emptyset,$$

where $\Psi(B) := (D_g F(\bar{x}, \bar{y}), D_g G(\bar{x}, \bar{z}))(B)$. Set

$$U := (-c, 0) + V \times W,$$

where $c \in \mathrm{int}(C)$, $-c + V \subset -\mathrm{int}(C)$ and V, W are open convex neighborhood of 0 in Y and Z, respectively. Then, $\mathrm{cone}_+(U)$ is an open convex set and by (b), we have

$$\mathrm{co}(\Psi(B) + (C \times D)) \cap \mathrm{cone}_+(U) = \emptyset.$$

By the Hahn-Banach separation theorem there exists $(\xi, \zeta) \in Y^* \times Z^* \setminus \{(0,0)\}$, such that

$$\xi(u) + \zeta(v) \geqslant 0, \forall (u,v) \in (\Psi(B) + C \times D),$$

and

$$\xi(u) + \zeta(v) \leqslant 0, \forall (u,v) \in \mathrm{cone}(U).$$

We prove that $\xi \neq 0$. Suppose on the contrary that $\xi = 0$; then $\zeta(v) \leqslant 0$, for all $v \in W$ which implies $\zeta = 0$, and so we have a contradiction. Therefore, $\xi \neq 0$. Now we take $u = y + c$ and $v = z + d$,, therefore

$$\xi(y + c) + \zeta(z + d) \geqslant 0, \forall (y,z) \in \Psi(B),\ \forall (c,d) \in C \times D.$$

If we set $c = d = 0$, then we obtain $\xi(y) + \zeta(z) \geqslant 0$, for every $(y,z) \in \Psi(B)$. Since $\xi(u)+\zeta(v) \geqslant 0$, for each $(u,v) \in \Psi(B)+(C\times D)$, ξ and ζ are continuous, then the inequality holds for all $(u,v) \in \overline{\Psi(B)} + (C \times D)$. We now show that $\xi \in C^+$ and $\zeta \in D^+$. Since $(0,0) \in \overline{\Psi(B)}$ and $(0,0) \in C \times D$, by taking $c = 0, (y,z) = (0,0)$ and $u = y + c, v = z + d$ in the above inequality, we have $\zeta(d) \geqslant 0, \forall d \in D$, and by putting $d = 0$ and $(y,z) = (0,0)$, we have $\xi(c) \geqslant 0$, for any $c \in C$.
$(b) \Rightarrow (a)$ Since $\xi \in C^+$ and $\zeta \in D^+$, then

$$\xi(y + c) + \zeta(z + d) \geqslant 0,\ \forall (y,z) \in \Psi(B),\ \forall (c,d) \in C \times D.$$

Thus, $\xi(u) + \zeta(v) \geqslant 0,\ \forall (u,v) \in \overline{co}[\Psi(B) + C \times D]$. Also,

$$\xi(u) + \zeta(v) < 0,\ \forall (u,v) \in -(\mathrm{int}(C) \times \{0\}).$$

Therefore,

$$\overline{\mathrm{co}}[\Psi(B) + C \times D] \cap -(\mathrm{int}(C) \times \{0\}) = \emptyset.$$

$\square$

Remark 2.3. Note that the previous theorem is more involved than Theorem 2.3, because the set $\overline{D_g F(\bar{x}, \bar{y})(.) + C}$ could be convex, but the contingent cone $T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ can be non convex. However, $D(F + C)(\bar{x}, \bar{y})(.)$ is convex if and only if the contingent cone $T(\mathrm{epi}(F), (\bar{x}, \bar{y}))$ is convex. The following example [11] justifies our claim.

Example 2.1. Consider the set valued map $F : [0,1] \rightrightarrows \mathbb{R}^2$ defined in [Example 2.2, [11]] by:

$$F(x) = \{(y, -\sqrt{y}) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant x\} \cup \{(y, -y) \in \mathbb{R}^2 \mid y \leqslant 0\},$$

and $C = \mathbb{R}^2_+$. The generalized contingent epiderivative $D_g F(0, (0,0))$ of F at $(0, (0,0)) \in \mathrm{graph}(F)$ is given by:

$$D_g F(0, (0,0))(x) := \{(y,z) \in \mathbb{R}^2 \mid z = -y, y < 0\}, x \in \mathbb{R}_+;$$

Note that $\overline{D_g F(0, (0,0))(\mathbb{R}_+) + \mathbb{R}^2_+}$ is convex but the contingent cone $T(\mathrm{epi}(F), (0, (0,0)))$ is nonconvex.
Also $\overline{D_g F(0, (0,0))(\mathbb{R}_+) + \mathbb{R}^2_+} \cap -\mathrm{int}(\mathbb{R}^2_+) = \emptyset$, and condition (a) of Theorem 2.6 holds, for $G = 0$. Then the Lagrange multiplier rule holds with $\xi \neq 0$.

In the following example, condition (a) of Theorem 2.6 is fulfilled, while the regularity condition in the [11, Theorem 2.4] does not hold.

Example 2.2. Consider the (CP) with the set valued map $F : [0,1] \rightrightarrows \mathbb{R}^2$ given in example 2.1, and the set valued map $G : [0,1] \rightrightarrows \mathbb{R}$ such that $G(x) = [-x^3, +\infty)$ for every $x \in [0,1]$. If $D = \mathbb{R}_+$, then for every $x \in [0,1]$, we have $D(G+D)(0,0)(x) = \mathbb{R}_+$ and $D_gG(0,0)(x) = \{0\}$. Clearly, $S_1 = [0,1]$ and the point $(\bar{x},\bar{y}) = (0,(0,0)) \in \mathrm{graph}(F)$ is a minimizer of (CP). The point

$$(0,0) = ((0,0),0) \in \overline{(D_gF(\bar{x},\bar{y}), D_gG(\bar{x},\bar{z}))(S)}$$

and

$$\overline{co}[(D_gF(0,(0,0)), D_gG(0,0))(\mathbb{R}_+) + (\mathbb{R}^2_+ \times \mathbb{R}_+)] \cap -(int(\mathbb{R}^2_+) \times \{0\}) = \emptyset.$$

Therefore, by Theorem 2.6, there exist $(\xi,\zeta) \in Y^* \times Z^* \setminus \{(0,0)\}$ with $\xi \in C^+$ and $\zeta \in D^+$ such that $\xi \neq 0$ and

$$\xi(y) + \zeta(z) \geqslant 0, \ \forall (y,z) \in (D_gF(\bar{x},\bar{y}), D_gG(\bar{x},\bar{z}))(B),$$

while the regularity condition in [11, Theorem 2.4] does not hold, since

$$D_gG(0,0)(\mathbb{R}_+) + \mathrm{cone}(D + \bar{z}) = \mathbb{R}_+ \neq Z.$$

Definition 2.3. [11] The generalized contingent epiderivative $D_gF(\bar{x},\bar{y})(.)$ of the map F at point $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$, is said to have a domination at $x \in \mathrm{dom}(D_gF(\bar{x},\bar{y}))$ if $D(F+C)(\bar{x},\bar{y})(x) \subseteq D_gF(\bar{x},\bar{y})(x) + C$.

The following lemma in [11] ensured that the set $D_gF(\bar{x},\bar{y})(\Omega) + C$ is a convex cone.

Lemma 2.1. [11] *Let $F : X \rightrightarrows Y$ be a set valued map. Let the map F be locally C-convex at $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$, let $\Omega = \mathrm{dom}(D_gF(\bar{x},\bar{y}))$, and let $D_gF(\bar{x},\bar{y})(.)$ have a domination for all $x \in \Omega$. Then, the set $D_gF(\bar{x},\bar{y})(\Omega) + C$ is a convex cone.*

Remark 2.4. [11] If the set $D_gF(\bar{x},\bar{y})(0) + C$ is convex and $D_gF(\bar{x},\bar{y})$ have a domination at $0 \in \Omega$, then $0 \in \overline{D_gF(\bar{x},\bar{y})(\Omega)}$.

By using Lemma 2.1 and Remark 2.4, the following result deduces from Theorem 2.6, while $G = 0$.

Corollary 2.2. *Let $F : X \rightrightarrows Y$ be locally C-convex at $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$, let $\Omega = \mathrm{dom}(D_gF(\bar{x},\bar{y})$, and let $D_gF(\bar{x},\bar{y})(.)$ have a domination for all $x \in \Omega$. Suppose that the set $D_gF(\bar{x},\bar{y})(0) + C$ is convex and C is solid. Then the following conditions are equivalent.*

(a) There exists $\xi \in Y^* \setminus \{0\}$ with $\xi \in C^+$ such that $\xi(y) \geqslant 0$, for every $y \in D_gF(\bar{x},\bar{y})(\Omega)$.
(b) $\overline{[(D_gF(\bar{x},\bar{y}))(\Omega) + C]} \cap -(\mathrm{int}(C)) = \emptyset$.

In the following result, we give another equivalent condition with respect to weak contingent epiderivative $D_wF(\bar{x},\bar{y})$. Its proof is similar to the proof of Theorem 2.6, so it is omitted.

Theorem 2.7. *Let C be solid, $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ and let $\bar{z} \in G(\bar{x}) \cap (-D)$. Let*

$$\Omega := \mathrm{dom}(D_wF(\bar{x},\bar{y})) \cap \mathrm{dom}(D_wG(\bar{x},\bar{z})) \neq \emptyset,$$

and $(0,0) \in \overline{(D_wF(\bar{x},\bar{y}), D_wG(\bar{x},\bar{z})(B))}$, where B be a cone in Ω. Then the following conditions are equivalent.

(a) There exists $(\xi,\zeta) \in Y^* \times Z^* \setminus \{(0,0)\}$ with $\xi \in C^+$, $\zeta \in D^+$ such that $\xi \neq 0$ and $\xi(y) + \zeta(z) \geqslant 0$, for each $(y,z) \in (D_wF(\bar{x},\bar{y}), D_wG(\bar{x},\bar{z}))(B)$,
(b) $\overline{\mathrm{co}}[(D_wF(\bar{x},\bar{y}), D_wG(\bar{x},\bar{z}))(B) + (C \times D)] \cap -(\mathrm{int}(C) \times \{0_Z\}) = \emptyset$.

Example 2.3. Consider the set valued map $F : [0,1] \rightrightarrows \mathbb{R}^2$ defined in [Example 2.3, [11]] by:

$$F(x) := \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 \leqslant x^2\}.$$

Let $\mathbb{R}^2_+$ be ordering cone. Note that the point $(1,(0,-1)) \in \mathrm{graph}(F)$ and weak contingent epiderivative $D_w F(1,(0,-1)) : \mathbb{R}_- \rightrightarrows \mathbb{R}^2_+$ is given by:

$$D_w F(1,(0,-1))(x) := \{(y,z) \in \mathbb{R}^2 \mid z \geqslant |x|, y \in \mathbb{R}\}, \forall x \in \mathbb{R}_-.$$

Clearly condition (b) of the above theorem with $G = 0$ holds, for every cone B in $\mathbb{R}_-$, and so the Lagrange multiplier rule holds with $\xi \neq 0$.

Definition 2.4. [15] Let $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$ and $(\bar{u},\bar{v}) \in X \times Y$. The second order composed adjacent (resp. contingent) derivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ is the set valued map $D^{b''}F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ (resp. $D^{''}F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$) defined by

$$\mathrm{graph}\, D^{b''}F(\bar{x},\bar{y},\bar{u},\bar{v}) = A(A(\mathrm{graph}\, F,(\bar{x},\bar{y})),(\bar{u},\bar{v})).$$

$$(\text{resp. graph}\, D^{''}F(\bar{x},\bar{y},\bar{u},\bar{v}) = T(T(\mathrm{graph}\, F,(\bar{x},\bar{y})),(\bar{u},\bar{v}))).$$

Remark 2.5. [15] The second-order composed contingent derivative and the second order composed adjacent derivative is strictly positive homogeneous.

In the following by using the second order composed adjacent derivative, we give a characterization of the Lagrange multiplier rule. The proof of the following result is similar to that given in Theorem 2.6, so it is omitted.

Theorem 2.8. *Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be a set valued maps. Let $(\bar{x},\bar{y}) \in \mathrm{graph}(F)$, $\bar{z} \in G(\bar{x}) \cap (-D), (\bar{u},\bar{v},\bar{w}) \in X \times (-C) \times (-D)$, $\mathrm{int}(C) \neq \emptyset$ and $\Omega = \mathrm{dom}(D^{b''}(F + C)(\bar{x},\bar{y},\bar{u},\bar{v})) \cap \mathrm{dom}(D^{b''}(G + D)(\bar{x},\bar{z},\bar{u},\bar{w})) \neq \emptyset$, and B be a cone in Ω . Suppose $\Phi(B)$ is define by*

$$\Phi(B) := (D^{b''}(F + C)(\bar{x},\bar{y},\bar{u},\bar{v}), D^{b''}(G + D)(\bar{x},\bar{z},\bar{u},\bar{w}))(B).$$

If $(0_Y, 0_Z) \in \overline{\mathrm{co}}[\Phi(B)]$, then the following conditions are equivalent.

(i) There exists $(\xi,\zeta) \in Y^* \times Z^* \setminus \{(0,0)\}$ with $\xi \in C^+$, $\zeta \in D^+$ such that $\xi \neq 0$ and $\xi(y) + \zeta(z) \geqslant 0$, for every $(y,z) \in \Phi(B)$.

(ii) $\overline{\mathrm{co}}[\Phi(B) + (C \times D)] \cap -(\mathrm{int}(C) \times \{0_Z\}) = \emptyset$.

Notice that the above result also holds for the second-order composed contingent derivative.

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