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Dedicated to Prof. Qamrul Hasan Ansari on the occasion of his 60^{th} anniversary

On quasi approximate solutions for nonsmooth robust semi-infinite optimization problems

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ABSTRACT. This paper devotes to the quasi ε-solution for robust semi-infinite optimization problems (RSIP) involving a locally Lipschitz objective function and infinitely many locally Lipschitz constraint functions with data uncertainty. Under the fulfillment of robust type Guignard constraint qualification and robust type Kuhn-Tucker constraint qualification, a necessary condition for a quasi ε-solution to problem (RSIP). After introducing the generalized convexity, we give a sufficient optimality for such a quasi ε-solution to problem (RSIP). Finally, we also establish approximate duality theorems in term of Wolfe type which is formulated in approximate form.

1. INTRODUCTION

In recent years, the study of a semi-infinite programming problem (SIP in brief), which is an optimization problem on a feasible set described by an infinite number of inequality constraints, has received a great deal of attention from scholars since most of several engineering problems are SIP, e.g., optimal control, transportation problems, etc; see [1, 2, 3, 4]. Recently, semi-infinite optimization problems without the convexity and differentiability assumptions have attracted many authors to study actively [3, 5, 6, 7, 8, 9]. Let us recall here some remarkable theoretical results from those mentioned earlier. For example, by using the separation-type theorem for nonconvex closed sets, a general Lagrange multiplier rule in terms of Clarke subdifferentials is obtained by Zheng and Yang [5]. Kanzi and Nobakhtian [7] derived a Fritz John type necessary optimality condition for optimal solution by using Mordukhovich and Clarke subdifferential. Moreover, Kanzi [9] investigated Karush-Kuhn-Tucker type necessary optimality conditions for optimal solution to nonsmooth semi-infinite programming problems where the objective and constraint functions are locally Lipschitz.

Taken from another viewpoint, the majority of practical optimization problems are often affected by data uncertainty due to prediction errors or lack of information, see [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein. Robust optimization [12, 13, 21, 22] has grown rapidly over the past two decades as a remarkable deterministic approach to treat mathematical optimization problems in the face of data uncertainty.

Besides, because the exact solutions do not exist while the approximate ones do even in the convex case, see [23, 24] and other references therein, the results on optimality conditions as well as duality for approximate solutions to semi-infinite programming

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problems under uncertainty have been studied [22, 25]. Motivated by this, some properties of approximate solutions of robust nonsmooth/nonconvex semi-infinite optimization problems seem to be developed. It is remarkable that unlike related works in [22, 25] we cannot apply the tool of ε-subdifferential to study ε-solution since the involved functions are not convex. This being a reason, we will focus on the quasi ε-solutions to nonsmooth/nonconvex semi-infinite optimization problems under a locally Lipschitz objective function and infinitely many locally Lipschitz constraint functions with data uncertainty. More precisely, let us consider the following semi-infinite optimization problem in the absence of data uncertainty

$$\text{(SIP)} \qquad \min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } g_t(x) \le 0,\ \forall t \in T,$$

where $f, g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$ are locally Lipschitz functions, and T is an index set with coordinately possibly infinite. The semi-infinite optimization problem (SIP) in the face of data uncertainty in the constraints can be captured by the problem

$$\text{(USIP)} \qquad \min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } g_t(x, v_t) \le 0,\ \forall t \in T,$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, t \in T$, are locally Lipschitz functions, and for each $t \in T, v_t \in \mathbb{R}^q$ is an uncertain parameter, which belongs to a convex compact set $\mathcal{V}_t \subset \mathbb{R}^q$. The uncertainty set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. The notation $v \in \mathcal{V}$ means that v is a *selection* of $\mathcal{V}$, i.e., $v : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$. The robust counterpart of (USIP) is as follows:

$$\text{(RSIP)} \qquad \min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } g_t(x, v_t) \le 0,\ \forall v_t \in \mathcal{V}_t,\ \forall t \in T.$$

The robust feasible set of (RSIP) is defined by

$$F := \{x \in \mathbb{R}^n : g_t(x, v_t) \le 0,\ \forall v_t \in \mathcal{V}_t,\ \forall t \in T\}.$$

Throughout this paper, we alway suppose that F is a nonempty set.

Below, let us recall the notion of a quasi ε-solution to problem (RSIP).

Definition 1.1. Let $\varepsilon \ge 0$ be given. A point $\bar{x} \in F$ is said to be a *quasi ε-solution* to problem (RSIP) if $f(x) \ge f(\bar{x}) - \varepsilon\|x - \bar{x}\|,\ \forall x \in F$.

Example 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g_t : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by

$$f(x) := \frac{1}{2}|x| + \frac{1}{2}x^3 \text{ and } g_t(x, v_t) := tx^3 - v_t x,$$

where $x \in \mathbb{R},\ t \in T := [0, 1],\ v_t \in \mathcal{V}_t := [-t + 2, t + 2]$. We can see that the robust feasible set $F = [0, 1]$. By taking $x_n := \frac{1}{n} \in F$ and $\varepsilon_n := \frac{1}{2} + \frac{3}{2}x_n$ where $n \in \mathbb{N}$, we obtain that for each $x \in F,\ f(x) \ge f(x_n) - \left(\frac{1}{2} + \frac{3}{2}x_n\right)|x - x_n|$. Hence, for each $n \in \mathbb{N}$, x_n is a quasi ε_n-solution of (RSIP).

In this article, we aim to establish necessary optimality conditions for a quasi ε-solution to problem (RSIP) under the fulfillment of new robust type constraint qualifications, robust type Guignard CQ and robust type Kuhn-Tucker CQ. With the help of generalized convex functions defined in terms of the Clarke subdifferentials, the obtained necessary conditions for quasi approximate solutions of the considered problem becomes sufficient. Afterworld, we also propose the weak and strong duality theorems, stated in approximate form, in the sense of Wolfe.

The layout of the paper is as follows. Section 2 collects definitions, notations and preliminary results that will be used later in the paper. Section 3 establishes necessary and sufficient of quasi ε-solution to problem (RSIP). Finally, duality results between the primal problem and its dual one in the sense of Wolfe are given in Section 5.

2. Preliminaries

Let us first recall some notation and preliminary results which will be used throughout this paper. $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. $\mathbb{R}^{(T)}_+$ denotes the set of mapping $\lambda : T \to \mathbb{R}_+$ (also denoted by $(\lambda_t)_{t\in T}$) such that $\lambda_t = 0$ except for finitely many indexes. For a given set $M \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the convex cone generated by M, by $\overline{M}$, conv (M), and cone (M), respectively. The polar cone and strict polar cone of M are respectively defined by $M^0 := \{d \in \mathbb{R}^n : \langle x, d\rangle \leq 0,\ \forall x \in M\}$ and $M^s := \{d \in \mathbb{R}^n : \langle x, d\rangle < 0,\ \forall x \in M\}$ where $\langle \cdot, \cdot\rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Also, $\overline{\text{cone}}(M)$ denotes the closed convex cone of M.

Definition 2.2. Let $M \subseteq \mathbb{R}^n$ and $\hat{x} \in \overline{M}$.

(i) *The contingent cone* to M at $\hat{x}$ is defined by

$$T(M,\hat{x}) := \left\{ d \in \mathbb{R}^n : \begin{array}{l} \exists \{t_k\} \subset \mathbb{R}_+;\ t_k \to 0,\ \exists \{d^k\} \subset \mathbb{R}^n;\ d^k \to d \\ \text{s.t. } \hat{x} + t_k d^k \in M,\ \forall k \in \mathbb{N} \end{array} \right\}.$$

(ii) *The cone of attainable directions* to M is defined by

$$A(M,\hat{x}) := \left\{ d \in \mathbb{R}^n : \begin{array}{l} \forall \{t_k\} \subset \mathbb{R}_+;\ t_k \to 0,\ \exists \{d^k\} \subset \mathbb{R}^n;\ d^k \to d \\ \text{s.t. } \hat{x} + t_k d^k \in M,\ \forall k \in \mathbb{N} \end{array} \right\}.$$

Notice that $T(M,\hat{x})$ and $A(M,\hat{x})$ are closed cones (generally nonconvex) in $\mathbb{R}^n$, and we always have $A(M,\hat{x}) \subseteq T(M,\hat{x})$.

Let $M \subseteq \mathbb{R}^n$ be a nonempty closed convex subset. The normal cone to M at $x \in M$ is defined by $N_M(x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle \leq 0, \forall y \in M\}$. The indicator function $\delta_M : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $\delta_M(x) := 0$ if $x \in M$; otherwise, $\delta_M(x) := +\infty$.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by dom f, that is, dom $f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. f is said to be convex if for all $\mu \in [0,1]$, $f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$. The function f is said to be concave whenever $-f$ is convex. In addition, if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then the one-sided or rather right-sided directional derivative always exists and is finite. The right-sided directional derivative of f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $f'(x; d)$, is defined as

$$f'(x; d) := \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}.$$

The subdifferential of convex function f at $x \in$ dom f is defined by

$$\partial f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle \leq f(y) - f(x), \forall y \in \mathbb{R}^n\}.$$

For $x \notin$ dom $f, \partial f(x)$ is empty.

Definition 2.3. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz* at $x \in \mathbb{R}^n$, if there exists a positive scalar L and a neighborhood N of x such that, for all $y, z \in N$, one has

$$|\varphi(y) - \varphi(z)| \leq L\|y - z\|.$$

Definition 2.4. [26] Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized directional derivative* of φ at x in the direction $d \in \mathbb{R}^n$, denoted $\varphi^o(x; d)$, is defined as

$$\varphi^o(x; d) := \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

Definition 2.5. [26] Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized subdifferential* of φ at x, denoted by $\partial_c \varphi(x)$, is defined as

$$\partial_c \varphi(x) := \{\xi \in \mathbb{R}^n : \varphi^o(x; d) \geq \langle \xi, d \rangle, \forall d \in \mathbb{R}^n\}.$$

Definition 2.6. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The function φ is said to be *regular* at $x \in \mathbb{R}^n$ if, for each $d \in \mathbb{R}^n$, the directional derivative $\varphi'(x; d)$ exists and coincides with $\varphi^o(x; d)$, i.e., $\varphi^o(x; d) = \varphi'(x; d), \ \forall d \in \mathbb{R}^n$.

Theorem 2.1. [26] *Let φ and ψ be locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and $\hat{x} \in$ dom $(\varphi) \cap$ dom (ψ). Then, the following properties hold:*

(i) $\varphi^o(\hat{x}; d) = \max \{\langle \xi, d \rangle : \xi \in \partial_c \varphi(\hat{x})\}, \ \forall d \in \mathbb{R}^n$.
(ii) $d \longmapsto \varphi^o(\hat{x}; d)$ *is a convex functon, and* $\partial_c \varphi(\hat{x}) = \partial \varphi^o(\hat{x}; \cdot)(0)$, *where* $\partial \varphi^o(\hat{x}; \cdot)$ *denotes the subdifferential of convex function* $\varphi^o(\hat{x}; \cdot)$.
(iii) $x \longmapsto \partial_c \varphi(x)$ *is an upper semicontinuous set-valued function.*
(iv) $\partial_c(\varphi + \psi)(\bar{x}) \subseteq \partial_c(\varphi)(\bar{x}) + \partial_c(\psi)(\bar{x})$.
If φ and ψ are regular at $\bar{x}$, then $\varphi + \psi$ is also regular at $\bar{x}$, and equality holds above.
(v) *If $\hat{x}$ is a minimum point of φ over $\mathbb{R}^n$, then $0 \in \partial_c \varphi(\hat{x})$.*

We assume here that each function $g_t, t \in T$ satisfying the following assumption.

Assumption For a given compact subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$, the following conditions will be considered in this paper.

(C1) $g(x, v)$ is upper semicontinuous in (x, v);
(C2) g is a locally Lipschitz in x, uniformly for v in $\mathcal{V}$, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in \mathcal{V}$, one has $|g(y, v) - g(z, v)| \leq L\|y - z\|$;
(C3) for each $(x, v) \in \mathbb{R}^n \times \mathcal{V}$, the function $g(\cdot, v)$ is regular at x, that is, $g^o(x, v; \cdot) = g'(x, v; \cdot)$, the derivatives being with respect to x;
(C4) set-valued map $\mathbb{R}^n \times \mathcal{V} \ni (x, v) \longmapsto \partial_c g(\cdot, v)(x)$ is upper semicontinuous where $\partial_c g(\cdot, v)(x)$ denotes the Clarke subdifferential of g with respect to x.

Remark 2.1. In a suitable setting, if the function g is convex in x and continuous in v, the conditions (C2), (C3), and (C4) are then automatically satisfied. These conditions also hold whenever the derivative $\nabla_x g(x, v)$ with respect to x exists and is continuous in (x, v).

Remark 2.2. [18] Under the assumptions (C1) and (C2) the function $\psi : \mathbb{R}^n \to \mathbb{R}$,

$$\psi(x) := \max\{g(x, v) : v \in \mathcal{V}\},$$

is defined and finite. Further, ψ is locally Lipschitz on $\mathbb{R}^n$, and hence for each $x \in \mathbb{R}^n$ the set $\mathcal{V}(x)$ defined as

$$\mathcal{V}(x) := \{v \in \mathcal{V} : g(x, v) = \psi(x)\},$$

is a nonempty closed subset of $\mathbb{R}^q$.

We conclude this section by the following lemmas which useful in our later analysis.

Lemma 2.1. [26] *Let the function ψ be defined in Remark 2.2. Suppose that the conditions (C1) - (C4) are fulfilled. Then the usual one-sided directional derivative $\psi'(x; d)$ exists, and satisfies the following : for each $x, d \in \mathbb{R}^n$,*

$$\psi'(x; d) = \psi^o(x; d) = \max_{v \in \mathcal{V}(x)} g^o_x(x, v; d) = \max\{\langle \xi, d \rangle : \xi \in \partial_c g(\cdot, v)(x), \ v \in \mathcal{V}(x)\}.$$

Lemma 2.2. [27] *For a given compact convex subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, suppose that the basic conditions (C1) - (C4) are fulfilled. Further, suppose that $g(x, \cdot)$ is concave on $\mathcal{V}$, for each $x \in \mathbb{R}^n$. Then*

$$\partial_c \psi(x) = \{\xi \in \mathbb{R}^n : \exists v \in \mathcal{V}(x) \text{ such that } \xi \in \partial_c g(\cdot, v)(x)\}.$$

3. Robust approximate optimality conditions

In this section, we establish the necessary and sufficient optimality conditions for approximate solutions to problem (RSIP).

Let T be an arbitrary (but nonempty) index set, and $\{g_t : \mathbb{R}^n \times \mathcal{V}_t \to \mathbb{R}\}$ be a family of functions satisfying (C1) and (C2). In the sequel, for a given $x \in F$, we turn our attention to define for simplicity,

$$T(x) := \{t \in T : \exists v_t \in \mathcal{V}_t \text{ s.t. } g_t(x, v_t) = 0\} \text{ and}$$

$$Z(x) := \bigcup_{\substack{t \in T(x) \\ v_t \in \mathcal{V}_t(x)}} \partial_c g_t(\cdot, v_t)(x),$$

with the convention $\bigcup_{t\in\emptyset} X_t = \emptyset$, for any set $X_t,\ t \in T(x)$.

Remark 3.3. As a Clarke subdifferential of a locally Lipschitz function is always nonempty, compact convex cone, we actually have $Z(x) \neq \emptyset$ and $Z(x) = \overline{\text{cone}}\, Z(x)$.

Similar to the notions of Guignard CQ and Kuhn-Tucker CQ stated in [9, Definition 3.1], we now present the following robust type constraint qualification for F.

Definition 3.7. Let $\bar{x} \in F$. We say that $\sigma := \{g_t(x, v_t) \leq 0,\ v_t \in \mathcal{V}_t,\ t \in T\}$ satisfies

(i) The *robust type Guignard CQ* (RGCQ) at $\bar{x}$, if $(Z(\bar{x}))^0 \subseteq \overline{\text{conv}}\ (T(F, \bar{x}))$.

(ii) The *robust type Kuhn-Tucker CQ* (RKTCQ) at $\bar{x}$, if $(Z(\bar{x}))^0 \subseteq A(F, \bar{x})$.

Remark 3.4. We point out that when considering the system σ as the form $\{\psi_t(x),\ t \in T\}$, by virtue of [9, Remark 3.8], the RKTCQ guarantees the fulfillment of RGCQ which can be seen as the most generalization of the robust type constraint qualifications for nonconvex semi-infinite system. Furthermore, we refer the reader to [9] for some related constraint qualifications and their connections.

Now, with the aid of [28, Theorem 2.22, p.33], we turn our attention to give the necessary robust approximate optimality condition theorem for a quasi ε-solution (RSIP).

Theorem 3.2. *Suppose that the RKTCQ is satisfied at $\bar{x} \in F$, and each $t \in T$, g_t satisfies (C1) - (C4) and $g_t(\bar{x}, \cdot)$ is concave on $\mathcal{V}_t$. If $\bar{x}$ is a quasi ε-solution of* (RSIP), *then*

(i) $0 \in \partial_c f(\bar{x}) + \overline{\mathit{cone}}(Z(\bar{x})) + \varepsilon\mathbb{B}$;

(ii) *If, in addition, the set cone$(Z(\bar{x}))$ is closed, then there exist $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t(\bar{x}),\ t \in T(\bar{x})$, such that*

$$0 \in \partial_c f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \partial_c g_t(\cdot, v_t)(\bar{x}) + \varepsilon\mathbb{B}. \tag{3.1}$$

Proof. (i) A quasi ε-solution $\bar{x}$ of (RSIP) can be viewed as an optimal solution of the following semi-infinite programming problem:

$$\min_{x\in\mathbb{R}^n} f(x) + \varepsilon\|x - \bar{x}\| \text{ s.t. } \psi_t(x) \leq 0,\ \forall t \in T,$$

where $\psi_t(x) := \max_{v_t\in\mathcal{V}_t} g_t(x, v_t)$. Note that in view of Lemma 2.1 that for each $t \in T(\bar{x})$, ψ_t is regular at $\bar{x}$. Moreover, it follows from Lemma 2.2 that

$$\partial_c\psi(\bar{x}) = \{\xi \in \mathbb{R}^n : \exists v_t \in \mathcal{V}_t(\bar{x}) \text{ s.t. } \xi \in \partial_c g_t(\cdot, v_t)(\bar{x})\},$$

which results in $Z(\bar{x}) = \cup_{t\in T(\bar{x})}\partial_c\psi_t(\bar{x})$. Set $\phi(x) := f(x) + \varepsilon\|x - \bar{x}\|$, observe that ϕ is a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$, inasmuch as $\|\cdot - \bar{x}\|$ is convex along with [28, Theorem 2.22, p.33]. So, we apply [9, Theorem 4.3(a)] to assert that

$$0 \in \partial_c(f + \varepsilon\|\cdot - \bar{x}\|)(\bar{x}) + \overline{\text{cone}}(Z(\bar{x})).$$

Then, there exists $\xi \in \partial_c(f + \varepsilon\| \cdot - \bar{x}\|)(\bar{x})$ such that $-\xi \in \overline{\text{cone}}(Z(\bar{x}))$. It then follows from [28, Theorem 3.13, p. 97] that for any $d \in \mathbb{R}^n$,

$$\begin{aligned} f^o(\bar{x}; d) + \varepsilon\|d\| &\geq f^o(\bar{x}; d) + \varepsilon(\| \cdot - \bar{x}\|)'(\bar{x}; d) \\ &= f^o(\bar{x}; d) + \varepsilon(\| \cdot - \bar{x}\|)^o(\bar{x}; d) \\ &\geq (f + \varepsilon\| \cdot - \bar{x}\|)^o(\bar{x}; d) \\ &\geq \langle \xi, d \rangle. \end{aligned}$$

This means that 0 is an optimal solution of the following unconstrained convex optimization problem:

$$\min_{d \in \mathbb{R}^n} f^o(\bar{x}; d) + \varepsilon\|d\| - \langle \xi, d \rangle,$$

which results in $0 \in \partial(f^o(\bar{x}; \cdot) + \varepsilon\| \cdot \| + \{-\xi\})(0)$. Invoking the classical Sum Rule for convex functions, together with the fact that $\partial\| \cdot \|(0) = \mathbb{B}$ and Theorem 2.1(ii), one has $\xi \in \partial_c f(\bar{x}) + \varepsilon\mathbb{B}$, and so, (i) has been justified.

(ii) The proof is done by (i) together with [29, Example 1.3.5, p. 77]

□

Now, let us provide an example illustrating our necessary optimality condition for (RSIP).

Example 3.2. Let f, g_t, T and $\mathcal{V}_t$ be defined as in Example 1.1. We now consider $\bar{x} := 0$, $\varepsilon := \frac{1}{2}$. It can be verified that $\bar{x}$ is a quasi ε-solution of (RSIP). On the other hand, we have $\partial_c f(\bar{x}) = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\partial_c g_t(\bar{x}, v_t) = \{-v_t\}$, $T(\bar{x}) = [0, 1]$ and $\mathcal{V}_t(\bar{x}) = \mathcal{V}_t = [-t+2, t+2]$. It is easy to see that $Z(\bar{x}) = [-3, -1]$ and $A(F, \bar{x}) = [0, +\infty)$. So, RKTCQ is satisfied at $\bar{x}$. In addition, letting $v_t := 2$ for all $t \in T(\bar{x})$ and

$$\lambda_t = \begin{cases} 0, & \text{if } t \in [0, 1), \\ \frac{1}{2}, & \text{if } t = 1, \end{cases}$$

one has

$$0 \in [-2, 0] = \left[-\frac{1}{2}, \frac{1}{2}\right] + \frac{1}{2}\{-2\} + \left[-\frac{1}{2}, \frac{1}{2}\right] = \partial_c f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \partial_c g_t(\cdot, v_t)(\bar{x}) + \varepsilon\mathbb{B}.$$

As an immediate consequence of [9, Theorem 4.4], we can obtain another necessary robust approximate optimality condition theorem for a quasi ε-solution (RSIP).

Theorem 3.3. *Let $\bar{x}$ be a quasi ε-solution of* (RSIP). *Assume that the RGCQ satisfies at $\bar{x}$ and additionally $(f + \varepsilon\| \cdot - \bar{x}\|)^o(\bar{x}; .)$ is a concave function. Then, (i) and (ii) in Theorem 3.2 are satisfied.*

It is worth noting that the relation obtained in (3.1) hints us to state a robust type approximate Karush-Kuhn-Tucker (KKT) type condition when treating approximate solutions of (RSIP) as follows:

Definition 3.8. Let F be the robust feasible set of (RSIP). A point $\bar{x} \in F$ is said to satisfy the *robust approximate (KKT) condition* on F if there exists there exist $(\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t(\bar{x}), t \in T(\bar{x})$, which $(\lambda_t)_{t \in T}$ are not all zero such that

$$0 \in \partial_c f(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t \partial_c g_t(\cdot, v_t)(\bar{x}) + \varepsilon\mathbb{B}.$$

Next, we give sufficient conditions for a feasible point of problem (RSIP) to be a quasi ε-solution. To this aim, we consider the following generalized convexity notion which states an analogous manner as in [30] and other references therein.

Definition 3.9. Let $g_T := (g_t)_{t\in T}$. We say that (f, g_T) is a *generalized convex* on F at $\bar{x} \in F$, if for any $x \in F,\ \xi \in \partial_c f(\bar{x}),\ \gamma_t \in \partial_c g_t(\cdot, v_t)(\bar{x}),$ and $v_t \in \mathcal{V}_t(\bar{x}),\ t \in T$, there exists $d \in \mathbb{R}^n$ such that

$$\begin{aligned}
&f(x) - f(\bar{x}) \geq \langle \xi, d\rangle,\\
&g_t(x, v_t) - g_t(\bar{x}, v_t) \geq \langle \gamma_t, d\rangle,\ \ \forall t \in T,\\
&\langle b, d\rangle \leq \|x - \bar{x}\|,\ \forall b \in \mathbb{B}.
\end{aligned}$$

Example 3.3. Let f, g_t, T and $\mathcal{V}_t$ be defined as in Example 1.1. Recall that for $\bar{x} = 0$, we have $\partial_c f(\bar{x}) = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\partial_c g_t(\bar{x}, v_t) = \{-v_t\},\ T(\bar{x}) = [0, 1]$ and $\mathcal{V}_t(\bar{x}) = \mathcal{V}_t = [-t+2, t+2]$. For any $x \in F = [0, 1],\ \xi \in \partial_c f(\bar{x}),\ \gamma_t \in \partial_c g_t(\bar{x}, v_t)$, by taking $d := x$, it follows that $d \in \mathbb{R}$,

$$\begin{aligned}
&x^3 + (1 - 2\xi)x \geq 0,\ \forall \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]\\
&tx^3 \geq 0,\ \forall t \in [0, 1]\\
&(1 - b)x \geq 0,\ \forall b \in [-1, 1].
\end{aligned}$$

and consequently,

$$\begin{aligned}
&f(x) - f(\bar{x}) = \frac{1}{2}x + \frac{1}{2}x^3 \geq \xi d,\\
&g_t(x, v_t) - g_t(\bar{x}, v_t) = tx^3 - v_t x \geq \gamma_t d \text{ and}\\
&|x - \bar{x}| = x \geq bd,\ \forall b \in [-1, 1].
\end{aligned}$$

This shows that (f, g_T) is generalized convex on F at $\bar{x}$.

Theorem 3.4. *Suppose that $\bar{x} \in F$ satisfies the robust approximate (KKT) condition. If (f, g_T) is generalized convex on F at $\bar{x}$, then $\bar{x}$ is a quasi ε-solution of* (RSIP).

Proof. Let $\bar{x} \in F$ be satisfied the robust ε-approximate (KKT) condition. Therefore, there exist $v_t \in \mathcal{V}_t,\ \lambda_t \geq 0,\ t \in T(\bar{x}),\ \xi \in \partial_c f(\bar{x}),\ \gamma_t \in \partial_c g_t(\cdot, v_t)(\bar{x})$ and $b \in \mathbb{B}$ such that

$$0 = \xi + \sum_{t\in T(\bar{x})} \lambda_t \gamma_t + \varepsilon b. \tag{3.2}$$

Suppose on contrary that $\bar{x}$ is not a quasi ε-solution of (RSIP). It then follows that there exists $\hat{x} \in F$ satisfying

$$f(\hat{x}) + \varepsilon\|\hat{x} - \bar{x}\| < f(\bar{x}). \tag{3.3}$$

By the generalized convexity of (f, g_T) at $\bar{x}$, there exists $d \in \mathbb{R}^n$ such that

$$f(\hat{x}) - f(\bar{x}) \geq \langle \xi, d\rangle, \tag{3.4}$$

$$g_t(\hat{x}, v_t) - g_t(\bar{x}, v_t) \geq \langle \gamma_t, d\rangle,\ \forall t \in T(\bar{x}), \tag{3.5}$$

$$\langle b, d\rangle \leq \|\hat{x} - \bar{x}\|,\ \forall b \in \mathbb{B}. \tag{3.6}$$

Combining (3.3) and (3.4), we obtain that

$$\langle \xi, d\rangle + \varepsilon\|\hat{x} - \bar{x}\| < 0. \tag{3.7}$$

Since $\hat{x} \in F,\ v_t \in \mathcal{V}_t$, and $t \in T(\bar{x})$, we have $g_t(\hat{x}, v_t) \leq 0 = g_t(\bar{x}, v_t)$. From (3.5), we can concluded that $\langle \gamma_t, d\rangle \leq 0$. Since $\lambda_t,\ t \in T(\bar{x})$ are not all zero and by (3.7), we obtain that

$$\langle \xi, d\rangle + \sum_{t\in T(\bar{x})} \lambda_t\langle \gamma_t, d\rangle + \varepsilon\|\hat{x} - \bar{x}\| < 0. \tag{3.8}$$

On the other hand, (3.6) together with (3.8) yields $\left\langle \xi + \sum_{t\in T(\bar{x})} \lambda_t \gamma_t + b, d \right\rangle < 0$, which contradicts to (3.2) that $\left\langle \xi + \sum_{t\in T(\bar{x})} \lambda_t \gamma_t + b, d \right\rangle = \langle 0, d\rangle = 0$. Consequently, $\bar{x}$ is a quasi ε-solution of (RSIP). $\square$

4. Approximate duality theorem

Now, we formulate a Wolfe-type dual problem (RSID) of (RSIP) as follows:

$$\text{(RSID)} \qquad \max_{(y,v,\lambda)} f(y) + \sum_{t\in T} \lambda_t g_t(y, v_t) \text{ s.t. } (y, v, \lambda) \in F_D,$$

where F_D, the feasible set of (RSID), is defined by

$$F_D := \left\{ (y, v, \lambda) : \begin{array}{l} 0 \in \partial_c f(y) + \sum_{t\in T} \lambda_t \partial_c g_t(y, v_t) + \varepsilon\mathbb{B} \\ (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)},\ v_t \in \mathcal{V}_t,\ t \in T \end{array} \right\}.$$

Definition 4.10. Let $\varepsilon \geq 0$. Then $(\bar{x}, \bar{v}, \bar{\lambda})$ is a quasi ε-solution of (RSID) if for any $(y, v, \lambda) \in F_D$,

$$f(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq f(y) + \sum_{t\in T} \lambda_t g_t(y, v_t) - \varepsilon\|y - \bar{x}\|.$$

Now, we establish the following approximate weak duality theorem, which holds between (RSIP) and (RSID).

Theorem 4.5. (Approximate weak duality theorem) *Let x and (y, v, λ) be feasible solution of* (RSIP) *and* (RSID), *respectively. If (f, g_T) is generalized convex at y, then*

$$f(x) \geq f(y) + \sum_{t\in T} \lambda_t g_t(y, v_t) - \varepsilon\|y - x\|.$$

Proof. Let x and (y, v, λ) be feasible solution of (RSIP) and (RSID), respectively. Since (y, v, λ) is feasible solution of (RSID), there exist $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, $v_t \in \mathcal{V}_t$, $t \in T$, $\xi \in \partial_c f(y)$, $\gamma_t \in \partial_c g_t(y, v_t)$ and $b \in \mathbb{B}$ such that

$$0 = \xi + \sum_{t\in T} \lambda_t \gamma_t + \varepsilon b. \tag{4.9}$$

Since (f, g_T) is generalized convex at y, there exists $d \in \mathbb{R}^n$ such that

$$f(x) - f(y) \geq \langle \xi, d\rangle, \tag{4.10}$$

$$g_t(x, v_t) - g_t(y, v_t) \geq \langle \gamma_t, d\rangle, \ \ \forall t \in T(\bar{x}), \tag{4.11}$$

$$\|d\| \leq \|x - y\|. \tag{4.12}$$

Then, by multiplying both sides of (4.11) by λ_t, $t \in T$, and summing up the obtained inequalities, we obtain that $\sum_{t\in T} \lambda_t g_t(x, v_t) - \sum_{t\in T} \lambda_t g_t(y, v_t) \geq \sum_{t\in T} \lambda_t \langle \gamma_t, d\rangle$. By virtue of (4.12), it holds that $\langle b, d\rangle \leq \|d\| \leq \|x - y\|$. This together with (4.9) and (4.10), it entails especially that

$$f(x) - \left(f(y) + \sum_{t\in T} \lambda_t g_t(y, v_t) \right) \geq \langle \xi, d\rangle - \sum_{t\in T} \lambda_t g_t(y, v_t)$$

$$= \left\langle - \sum_{t\in T} \lambda_t \gamma_t - \varepsilon b, d \right\rangle - \sum_{t\in T} \lambda_t g_t(y, v_t) = - \sum_{t\in T} \lambda_t \langle \gamma_t, d\rangle - \varepsilon\langle b, d\rangle - \sum_{t\in T} \lambda_t g_t(y, v_t)$$

$$\geq - \sum_{t\in T} \lambda_t \langle \gamma_t, d\rangle - \varepsilon\|x - y\| - \sum_{t\in T} \lambda_t g_t(y, v_t) \geq - \sum_{t\in T} \lambda_t \left(g_t(x, v_t) - g_t(y, v_t)\right) - \varepsilon\|y - x\|$$

$$-\sum_{t\in T}\lambda_t g_t(y,v_t) = -\sum_{t\in T}\lambda_t g_t(x,v_t) - \varepsilon\|y-x\| \geq -\varepsilon\|y-x\|,$$

and consequently, $f(x) \geq f(y) + \sum_{t\in T}\lambda_t g_t(y,v_t) - \varepsilon\|y-x\|$. □

Now, under the condition RKTCQ, we give the following approximate strong duality theorem, which holds between (RSIP) and (RSID).

Theorem 4.6. (Approximate strong duality theorem) *Assume that the RKTCQ holds and cone*$(Z(\bar{x}))$ *is closed. Let* $\bar{x}$ *be a quasi* ε*-solution of* (RSIP)*. If* (f, g_T) *is generalized convex at* $\bar{x}$*, then there exists* $(\bar{v},\bar{\lambda}) \in \mathcal{V}\times\mathbb{R}_+^{(T)}$ *such that* $(\bar{x},\bar{v},\bar{\lambda})$ *is a quasi* ε*-solution of* (RSID).

Proof. Let $\bar{x}$ be a quasi ε-solution of (RSIP). Then, by Theorem 3.2 (ii), for any $t \in T(\bar{x})$ there exists $\bar{v}_t \in \mathcal{V}_t(\bar{x})$ and $\bar{\lambda}_t \geq 0$, which not all zero such that

$$0 \in \partial_c f(\bar{x}) + \sum_{t\in T(\bar{x})}\bar{\lambda}_t\partial_c g_t(\bar{x},\bar{v}_t) + \varepsilon\mathbb{B}.$$

So $(\bar{x},\bar{v},\bar{\lambda})$ is feasible solution of (RSID). By approximate weak duality theorem, for any feasible (y,v,λ) of (RSID),

$$f(\bar{x}) \geq f(y) + \sum_{t\in T(\bar{x})}\lambda_t g_t(y,v_t) - \varepsilon\|y-\bar{x}\|.$$

For any $t\in T(\bar{x}),\ \bar{v}_t \in \mathcal{V}_t(\bar{x})$, we obtain that $g_t(\bar{x},\bar{v}_t) = 0$, so

$$f(\bar{x}) + \sum_{t\in T(\bar{x})}\bar{\lambda}_t g_t(\bar{x},\bar{v}_t) \geq f(y) + \sum_{t\in T(\bar{x})}\lambda_t g_t(y,v_t) - \varepsilon\|y-\bar{x}\|.$$

Therefore $(\bar{x},\bar{v},\bar{\lambda})$ is a quasi ε-solution of (RSID). □

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