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New asymptotic results for half-linear differential equations with deviating argument

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ABSTRACT. In the paper, we study the oscillation of the half-linear second-order differential equations with deviating argument of the form

$$(E) \qquad \big(r(t)(y'(t))^{\alpha}\big)' = p(t)y^{\alpha}(\tau(t)).$$

We introduce new monotonic properties of nonoscillatory solutions and use them to offer new criteria for elimination of certain types of solutions. The presented results essentially improve existing ones even for linear differential equations.

1. INTRODUCTION

In this paper, we shall study the asymptotic and oscillation behavior of the solutions for half-linear second order delay differential equations

$$(E) \qquad (r(t)(y'(t))^{\alpha})' = p(t)y^{\alpha}(\tau(t)).$$

We shall assume that

(H_1) $p, r \in C([t_0,\infty))$, $p(t) > 0$, $r(t) > 0$, α is the ratio of two positive odd integers,
(H_2) $\tau \in C^1([t_0,\infty))$, $\tau'(t) > 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $y(t) \in C^1([T_y,\infty))$, $T_y \ge t_0$, such that $r(t)(y'(t))^{\alpha} \in C^1([T_y,\infty))$ and $y(t)$ satisfies Eq. (E) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

Throughout the paper we consider (E) in canonical form, that is,

$$R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s \to \infty \text{ as } t \to \infty.$$

The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades (see [1]–[15]).

It is known that the equation

$$(1.1) \qquad y''(t) = p(t)y(t)$$

always possesses both positive decreasing and positive increasing solution. The situation for equation with deviating argument

$$(1.2) \qquad y''(t) = p(t)y(\tau(t))$$

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is different. Koplatadze and Chanturia [12] have shown that for $\tau(t) \le t$ the condition

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s-\tau(t))p(s)\mathrm{d}s > 1 \tag{1.3}$$

does not allow the presence of positive decreasing solutions.

On the other hand, (1.2) does not possess positive increasing solutions if $\tau(t) \ge t$ and

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} (\tau(s)-\tau(t))p(s)\mathrm{d}s > 1. \tag{1.4}$$

The aim of this paper is to establish corresponding results for second-order half-linear functional differential equation (E), which not only generalize, but also improve the existing ones given in the linear case (1.2).

We have been motivated by the observation that there are very few effective criteria for (1.2) and its half-linear analogue (E), although both have been the object of intensive investigations in case of $p(t) < 0$ (see e.g. [1]-[13]).

The structure of this paper is following. In the first part, we investigate the phenomena of the nonexistence of positive decreasing solutions for (E) with delay argument. Next, we propose the analogous results for the nonexistence of positive increasing solutions for (E) with advanced argument. Finally, we combine the results to obtain oscillation of the equation involving both delayed and advanced arguments. Our approach is based on establishing new monotonic properties of nonoscillatory solutions and their derivatives.

2. MAIN RESULTS

It follows from a generalization of lemma of Kiguradze [9] that the set of positive solutions of (E) has the following structure.

Lemma 2.1. *Assume that $y(t)$ is an eventually positive solution of (E). Then $y(t)$ satisfies one of the following conditions*

$$(N_0): \; r(t)(y'(t))^\alpha < 0, \quad (r(t)(y'(t))^\alpha)' > 0,$$
$$(N_2): \; r(t)(y'(t))^\alpha > 0, \quad (r(t)(y'(t))^\alpha)' > 0 \; \textit{for} \; t \ge t_1 \ge t_0.$$

If we denote by N the set of all positive solutions of (E), then it has the following decomposition

$$N = N_0 \cup N_2,$$

where the class N_i involves solutions satisfying conditions (N_i), $i = 0, 2$.

We start with some useful lemma concerning monotonic properties of nonoscillatory solutions for studied equations.

Lemma 2.2. *Let $\tau(t) \le t$. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_0). If there exists some positive constant β such that*

$$R(t) \left[\int_{t}^{\tau^{-1}(t)} p(s)\mathrm{d}s \right]^{1/\alpha} \ge \beta, \; \textit{for} \; t \ge t_0, \tag{2.5}$$

then $R^\beta(t)y(t)$ is decreasing.

Proof. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_0). Since $r(t)(y'(t))^\alpha < 0$, then $y(t)$ is decreasing. Consequently, an integration of (E) from $\tau(t)$ to t yields

$$-r^{1/\alpha}(\tau(t))y'(\tau(t)) \ge y(\tau(t)) \left[\int_{\tau(t)}^{t} p(s)\mathrm{d}s \right]^{1/\alpha},$$

that is

$$-r^{1/\alpha}(t)y'(t)R(t) \geq y(t)R(t)\left[\int_t^{\tau^{-1}(t)} p(s)\mathrm{d}s\right]^{1/\alpha}.$$

Applying (2.5), we are led to

$$-r^{1/\alpha}(t)y'(t)R(t) \geq y(t)\beta,$$

which implies that

$$\left[R^{\beta}(t)y(t)\right]' = \frac{R^{\beta-1}}{r^{1/\alpha}(t)}\left[\beta y(t) + R(t)r^{1/\alpha}(t)y'(t)\right] \leq 0$$

and we conclude that function $R^{\beta}(t)y(t)$ is decreasing. The proof is complete. □

Theorem 2.1. *Let $\tau(t) \leq t$ and (2.5) hold. If*

$$\limsup_{t\to\infty} R^{\beta}(\tau(t))\int_{\tau(t)}^{t} \frac{1}{r^{1/\alpha}(v)}\left[\int_v^t \frac{p(s)}{R^{\beta\alpha}(\tau(s))}\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v > 1, \tag{2.6}$$

then $N_0 = \emptyset$.

Proof. Assume on the contrary, that (E) possesses an eventually positive solution $y(t)$ satisfying condition (N_0). Integrating (E) from u to t and using monotonic property of $R^{\beta}(t)y(t)$, we obtain

$$-r(u)(y'(u))^{\alpha} \geq R^{\beta\alpha}(\tau(t))y^{\alpha}(\tau(t))\int_u^t \frac{p(s)}{R^{\beta\alpha}(\tau(s))}\mathrm{d}s.$$

Integrating once more from u to t, we get

$$y(u) \geq R^{\beta}(\tau(t))y(\tau(t))\int_u^t \frac{1}{r^{1/\alpha}(v)}\left[\int_v^t \frac{p(s)}{R^{\beta\alpha}(\tau(s))}\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq R^{\beta}(\tau(t))y(\tau(t))\int_{\tau(t)}^t \frac{1}{r^{1/\alpha}(v)}\left[\int_v^t \frac{p(s)}{R^{\beta\alpha}(\tau(s))}\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v,$$

which contradicts to condition (2.6) and we conclude, that class N_0 is empty. □

Now we turn our attention to monotonic properties for possible positive increasing solutions of (E).

Lemma 2.3. *Let $\tau(t) \geq t$. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_2). If there exists a positive constant ω such that*

$$R(t)\left[\int_{\tau^{-1}(t)}^{t} p(s)\mathrm{d}s\right]^{1/\alpha} \geq \omega, \text{ for } t \geq t_0, \tag{2.7}$$

then $\frac{y(t)}{R^{\omega}(t)}$ is increasing.

Proof. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_2). Then $r(t)(y'(t))^{\alpha} > 0$ and $y(t)$ is increasing. Using this property and integrating (E) from t to $\tau(t)$, we have

$$r^{1/\alpha}(\tau(t))y'(\tau(t)) \geq y(\tau(t))\left[\int_t^{\tau(t)} p(s)\mathrm{d}s\right]^{1/\alpha}$$

and so in view of (2.7)

$$r^{1/\alpha}(t)y'(t)R(t) \geq y(t)R(t)\left[\int_{\tau^{-1}(t)}^{t} p(s)\mathrm{d}s\right]^{1/\alpha} \geq y(t)\omega.$$

Therefore,

$$\left[\frac{y(t)}{R^{\omega}(t)}\right]' = \frac{R^{-\omega-1}}{r^{1/\alpha}(t)}\left[-\omega y(t) + R(t)r^{1/\alpha}(t)y'(t)\right] \geq 0$$

and we see that function $\frac{y(t)}{R^{\omega}(t)}$ is increasing. The proof is complete. $\square$

Theorem 2.2. *Let $\tau(t) \geq t$ and (2.7) hold. If*

$$\limsup_{t\to\infty} \frac{1}{R^{\omega}(\tau(t))}\int_{t}^{\tau(t)} \frac{1}{r^{1/\alpha}(v)}\left[\int_{t}^{v} p(s)R^{\omega\alpha}(\tau(s))\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v > 1, \tag{2.8}$$

then $N_2 = \emptyset$.

Proof. Assume on the contrary that (E) has some positive solution $y(t)$ satisfying (N_2). An integration of (E) from t to u and using monotonic property of $\frac{y(t)}{R^{\omega}(t)}$ yield

$$r(u)(y'(u))^{\alpha} \geq \frac{y^{\alpha}(\tau(t))}{R^{\omega\alpha}(\tau(t))}\int_{t}^{u} p(s)R^{\omega\alpha}(\tau(s))\mathrm{d}s.$$

Integrating once more from t to u, we obtain

$$y(u) \geq \frac{y(\tau(t))}{R^{\omega}(\tau(t))}\int_{t}^{u}\frac{1}{r^{1/\alpha}(v)}\left[\int_{t}^{v} p(s)R^{\omega\alpha}(\tau(s))\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v$$

and putting $u = \tau(t)$, we have

$$y(\tau(t)) \geq \frac{y(\tau(t))}{R^{\omega}(\tau(t))}\int_{t}^{\tau(t)}\frac{1}{r^{1/\alpha}(v)}\left[\int_{t}^{v} p(s)R^{\omega\alpha}(\tau(s))\mathrm{d}s\right]^{1/\alpha}\mathrm{d}v$$

which contradicts to condition (2.8) and we conclude, that class N_2 is empty. $\square$

In Theorems 2.1 and 2.2 we have introduced new criteria for emptying classes N_0 and N_2. Those criteria are based on monotonic properties of nonoscillatory solutions. Our next considerations are intended to present new monotonic properties for the first derivatives of nonoscillatory solutions that will be applied for obtaining alternative criteria to those presented Theorems 2.1 and 2.2.

Lemma 2.4. *Let $\tau(t) \leq t$. Assume that $y(t)$ is a positive solution of (E) satisfying (N_0). If there exists positive constant γ such that*

$$p(t)\left[R(t) - R(\tau(t))\right]^{\alpha} R(t)r^{1/\alpha}(t) \geq \gamma, \text{ for } t \geq t_0 \tag{2.9}$$

then $-R^{\gamma}(t)r(t)(y'(t))^{\alpha}$ is decreasing.

Proof. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_0). Since $-y'(t)r^{1/\alpha}(t)$ is positive and decreasing function, one can see that

$$\begin{aligned} y(\tau(t)) &\geq \int_{\tau(t)}^{t} -y'(u)\mathrm{d}u \geq -y'(t)r^{1/\alpha}(t)\int_{\tau(t)}^{t}\frac{1}{r^{1/\alpha}}(u)\mathrm{d}u = \\ &-y'(t)r^{1/\alpha}(t)\left[R(t) - R(\tau(t))\right]. \end{aligned} \tag{2.10}$$

Substituting (2.10) into (E), we obtain

$$\left(r(t)(y'(t))^{\alpha}\right)' \geq p(t)(-y'(t))^{\alpha}r(t)\left[R(t) - R(\tau(t))\right]^{\alpha},$$

which in view of (2.9), yields

$$(r(t)(y'(t))^\alpha)' \, r^{1/\alpha}(t)R(t) \geq (-y'(t))^\alpha r(t)\gamma. \tag{2.11}$$

Consequently,

$$[-R^\gamma(t)r(t)(y'(t))^\alpha]' = \frac{R^{\gamma-1}(t)}{r^{1/\alpha}(t)} \left[(-y'(t))^\alpha r(t)\gamma - (r(t)(y'(t))^\alpha)' \, r^{1/\alpha}(t)R(t)\right] \leq 0,$$

which implies that $-R^\gamma(t)r(t)(y'(t))^\alpha$ is decreasing and the proof is complete. □

Theorem 2.3. *Let $\tau(t) \leq t$ and (2.9) hold. If*

$$\limsup_{t\to\infty} R^\gamma(\tau(t)) \int_{\tau(t)}^t p(s) \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]^\alpha \mathrm{d}s > \frac{(\alpha-\gamma)^\alpha}{\alpha^\alpha}, \tag{2.12}$$

then $N_0 = \emptyset$.

Proof. Assume on the contrary, that (E) possesses an eventually positive solution $y(t)$ satisfying (N_0). It follows from the fact that $-R^\gamma(t)r(t)(y'(t))^\alpha$ is positive and decreasing function:

$$\begin{aligned} y(\tau(s)) &\geq \int_{\tau(s)}^{\tau(t)} -y'(u)\frac{R^{\gamma/\alpha}(u)r^{1/\alpha}(u)}{R^{\gamma/\alpha}(u)r^{1/\alpha}(u)}\mathrm{d}u \\ &\geq -y'(\tau(t))R^{\gamma/\alpha}(\tau(t))r^{1/\alpha}(\tau(t)) \int_{\tau(s)}^{\tau(t)} \frac{1}{R^{\gamma/\alpha}(u)r^{1/\alpha}(u)}\mathrm{d}u \\ &= -y'(\tau(t))R^{\gamma/\alpha}(\tau(t))r^{1/\alpha}(\tau(t))\frac{\alpha}{\alpha-\gamma} \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]. \end{aligned}$$

Therefore,

$$y^\alpha(\tau(s)) \geq (-y'(\tau(t)))^\alpha R^\gamma(\tau(t))r(\tau(t)) \left(\frac{\alpha}{\alpha-\gamma}\right)^\alpha \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]^\alpha.$$

Integrating equation (E) from $\tau(t)$ to t and applying the last estimate, we obtain

$$\begin{aligned} &- r(\tau(t))(y'(\tau(t))^\alpha \geq \int_{\tau(t)}^t p(s)y^\alpha(\tau(s))\mathrm{d}s \\ &\geq \int_{\tau(t)}^t p(s)(-y'(\tau(t)))^\alpha R^\gamma(\tau(t))r(\tau(t)) \left(\frac{\alpha}{\alpha-\gamma}\right)^\alpha \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]^\alpha \mathrm{d}s \\ &= (-y'(\tau(t)))^\alpha R^\gamma(\tau(t))r(\tau(t)) \left(\frac{\alpha}{\alpha-\gamma}\right)^\alpha \int_{\tau(t)}^t p(s) \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]^\alpha \mathrm{d}s, \end{aligned}$$

which contradicts to condition (2.12) and we conclude, that class N_0 is empty. □

Lemma 2.5. *Let $\tau(t) \geq t$. Assume that $y(t)$ is a positive solution of (E) satisfying (N_2). If there exists positive constant δ such that*

$$p(t)\left[R(\tau(t)) - R(t)\right]^\alpha R(t)r^{1/\alpha}(t) \geq \delta, \; \textit{for } t \geq t_0 \tag{2.13}$$

then $\frac{r(t)(y'(t))^\alpha}{R^\delta(t)}$ is increasing.

Proof. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_2) of Lemma 1. Since $y'(t)r^{1/\alpha}(t)$ is positive and increasing function, it is easy to verify that

$$\begin{aligned} y(\tau(t)) &\geq \int_t^{\tau(t)} y'(u)\mathrm{d}u \geq y'(t)r^{1/\alpha}(t) \int_t^{\tau(t)} \frac{1}{r^{1/\alpha}}(u)\mathrm{d}u \\ &= y'(t)r^{1/\alpha}(t)\left[R(\tau(t)) - R(t)\right]. \end{aligned} \tag{2.14}$$

Setting (2.14) into (E), we are led to

$$\left(r(t)(y'(t))^\alpha\right)' \geq p(t)(y'(t))^\alpha r(t)\left[R(\tau(t)) - R(t)\right]^\alpha.$$

Taking (2.13) into account, we have

$$\left(r(t)(y'(t))^\alpha\right)' r^{1/\alpha}(t)R(t) \geq (y'(t))^\alpha r(t)\delta. \tag{2.15}$$

Thus,

$$\left[\frac{r(t)(y'(t))^\alpha}{R^\delta(t)}\right]' = \frac{R^{-\delta-1}(t)}{r^{1/\alpha}(t)}\left[\left(r(t)(y'(t))^\alpha\right)' r^{1/\alpha}(t)R(t) - r(t)(y'(t))^\alpha\delta\right] \geq 0,$$

which implies that $\frac{r(t)(y'(t))^\alpha}{R^\delta(t)}$ is increasing and the proof is completed. □

Theorem 2.4. *Let $\tau(t) \geq t$ and (2.13) hold. If*

$$\limsup_{t\to\infty} \frac{1}{R^\delta(\tau(t))}\int_t^{\tau(t)} p(s)\left[R^{\delta/\alpha+1}(\tau(s)) - R^{\delta/\alpha+1}(\tau(t))\right]^\alpha \mathrm{d}s > \frac{(\alpha+\delta)^\alpha}{\alpha^\alpha},$$

then $N_2 = \emptyset$.

Proof. Assume that $y(t)$ is a positive solution of (E) satisfying condition (N_2). Using that $\frac{r(t)(y'(t))^\alpha}{R^\delta(t)}$ is positive and increasing function, we have

$$\begin{aligned} y(\tau(s)) &\geq \int_{\tau(t)}^{\tau(s)} y'(u)\frac{R^{\delta/\alpha}(u)r^{1/\alpha}(u)}{R^{\delta/\alpha}(u)r^{1/\alpha}(u)}\mathrm{d}u \\ &\geq \frac{y'(\tau(t))r^{1/\alpha}(\tau(t))}{R^{\delta/\alpha}(\tau(t))}\int_{\tau(t)}^{\tau(s)} \frac{R^{\delta/\alpha}(u)}{r^{1/\alpha}(u)}\mathrm{d}u \\ &= \frac{y'(\tau(t))r^{1/\alpha}(\tau(t))\alpha}{R^{\delta/\alpha}(\tau(t))(\delta+\alpha)}\left[R^{1+\delta/\alpha}(\tau(s)) - R^{1+\delta/\alpha}(\tau(t))\right]. \end{aligned}$$

Integrating equation (E) from t to $\tau(t)$ and substituting the above inequality, we get

$$\begin{aligned} & r(\tau(t))(y'(\tau(t))^\alpha \geq \int_t^{\tau(t)} p(s)y^\alpha(\tau(s))\mathrm{d}s \\ &\geq \int_t^{\tau(t)} p(s)\frac{(y'(\tau(t)))^\alpha r(\tau(t))\alpha^\alpha}{R^\delta(\tau(t))(\delta+\alpha)^\alpha}\left[R^{1+\delta/\alpha}(\tau(s)) - R^{1+\delta/\alpha}(\tau(t))\right]^\alpha \mathrm{d}s \\ &= \frac{(y'(\tau(t)))^\alpha r(\tau(t))\alpha^\alpha}{R^\delta(\tau(t))(\delta+\alpha)^\alpha}\int_t^{\tau(t)} p(s)\left[R^{1+\delta/\alpha}(\tau(s)) - R^{1+\delta/\alpha}(\tau(t))\right]^\alpha \mathrm{d}s, \end{aligned}$$

which contradicts to assumption of the theorem and we conclude, that class N_2 is empty. □

Remark 2.1. The technique used in the proofs of Theorems 1,2 is based on properties of possible nonoscillatory solutions, while the technique applied for Theorems 3,4 employs properties of derivative of solutions. So the results presented in theorems are independent.

In view of our above results, it is natural to expect that there will be no nonoscillatory solutions, or equivalently all solutions will be oscillatory for certain functional differential equation involving both advanced and delayed arguments. The purpose of the following theorem is to show that this is indeed the case for

$$(r(t)(y'(t))^{\alpha})' = p(t)y^{\alpha}(\tau(t)) + q(t)y^{\alpha}(\sigma(t)). \tag{2.16}$$

where $p(t)$, $\tau(t)$, $r(t)$ and α are subjects of the conditions (H_1) and (H_2) and moreover,

(H_3) $q \in C([t_0, \infty))$, $q(t) > 0$,
(H_4) $\sigma \in C([t_0, \infty))$, $\sigma'(t) > 0$ $\lim_{t\to\infty} \sigma(t) = \infty$.

Theorem 2.5. *Let $\tau(t) \le t$, $\sigma(t) \ge t$ and (2.5), (2.6) hold. Assume that there exists positive constant ω_0 such that*

$$R(t)\left[\int_{\sigma^{-1}(t)}^{t} q(s)\mathrm{d}s\right]^{1/\alpha} \ge \omega_0, \text{ for } t \ge t_0. \tag{2.17}$$

If

$$\limsup_{t\to\infty} \frac{1}{R^{\omega_0}(\sigma(t))}\int_t^{\sigma(t)} \frac{1}{r^{1/\alpha}(v)}\left[\int_t^v q(s)R^{\omega_0\alpha}(\sigma(s))\mathrm{d}s\right]^{1/\alpha} \mathrm{d}v > 1, \tag{2.18}$$

then (2.16) *is oscillatory.*

Proof. Assume that $y(t)$ is an eventually positive solution of (2.16). Then $y(t)$ satisfies either (N_0) or (N_2). Assume at first, that $y(t)$ is from the class N_0. It is easy to see that (2.16) implies

$$(r(t)(y'(t))^{\alpha})' \ge p(t)y^{\alpha}(\tau(t))$$

and proceeding exactly as in the proof of Theorem 2.1, we can verify that (2.6), guarantees that $N_0 = \emptyset$.

On the other hand, if $y(t)$ is from the class N_2, then it follows from (2.16) that

$$(r(t)(y'(t))^{\alpha})' \ge q(t)y^{\alpha}(\sigma(t)).$$

Proceeding as in proof of Theorem 2.2, we can see that $N_2 = \emptyset$. The proof is complete. □

Theorem 2.6. *Let $\tau(t) \le t$, $\sigma(t) \ge t$ and (2.9), (2.12) hold. Assume that there exists positive constant δ_0 such that*

$$q(t)\left[R(\sigma(t)) - R(t)\right]^{\alpha} R(t)r^{1/\alpha}(t) \ge \delta_0, \text{ for } t \ge t_0. \tag{2.19}$$

If

$$\frac{1}{R^{\delta_0}(\sigma(t))}\int_t^{\sigma(t)} q(s)\left[R^{\delta_0/\alpha+1}(\sigma(s)) - R^{\delta_0/\alpha+1}(\sigma(t))\right]^{\alpha} \mathrm{d}s > \frac{(\alpha+\delta_0)^{\alpha}}{\alpha^{\alpha}}, \tag{2.20}$$

then (2.16) *is oscillatory.*

The proof is similar to that of Theorem 2.5 and so, we it can be omitted.

3. Examples

We support our results with the set of illustrative examples. In the first example we compare our criteria with those of Koplatadze and Chanturia [12].

Example 3.1. We consider the Euler differential equation

$$(E_{x1}) \qquad y''(t) = \frac{a}{t^2} y(\lambda t),$$

with $a > 0$ and $\lambda > 0$. Assume that $\lambda \in (0,1)$. According to Koplatadze and Chanturia's criterion (see (1.3)) the class $N_0 = \emptyset$ for Eq.(E_{x1}) provided that

$$a > 17.6445.$$

On the other hand, Theorem 2.1 implies that this situation occurs if

$$a > 11.21,$$

which is remarkable better result.

Now assume that $\lambda > 1$. Condition (1.4) of Koplatadze and Chanturia guarantees that the class $N_2 = \emptyset$ provided that

$$a > 9.2423,$$

while by Theorem 2.2, it is sufficient to request

$$a > 6.557.$$

The following example is intended to show how our criteria work for general half-linear equations.

Example 3.2. We consider the differential equation

$$(E_{x2}) \qquad (t^{-1/3}(y'(t))^{1/3})' = \frac{a}{t^{5/3}} y^{1/3}(\lambda t), \qquad a > 0, \quad \lambda > 0.$$

We set $\lambda = 0.7$ and $a = 3.952$. Then $\beta_0 = 0.9872$. Condition (2.6) reduces to

$$\limsup_{t\to\infty} R^{\beta_0}(\tau(t)) \int_{\tau(t)}^{t} \frac{1}{r^{1/\alpha}(v)} \left[\int_v^t \frac{p(s)}{R^{\beta_0\alpha}(\tau(s))} \mathrm{d}s\right]^{1/\alpha} \mathrm{d}v = 1.0011 > 1$$

and Theorem 2.1 ensures that the class $N_0 = \emptyset$.
Now we set $\lambda = 1.465$, and $a = 3.952$, which implies $\gamma_0 = 2.5379$. Condition (2.8) works down to

$$\limsup_{t\to\infty} \frac{1}{R^{\gamma_0}(\tau(t))} \int_t^{\tau(t)} \frac{1}{r^{1/\alpha}(v)} \left[\int_t^v p(s) R^{\gamma_0\alpha}(\tau(s)) \mathrm{d}s\right]^{1/\alpha} \mathrm{d}v = 1.0025 > 1$$

and by Theorem 2.2 class $N_2 = \emptyset$.

For (E_{x2}) it is more convenient to use Theorem 2.1 and 2.2 instead of Theorem 2.3 and 2.4, because evaluation of corresponding integrals is much simpler.

Example 3.3. 'We consider the differential equation

$$(E_{x3}) \qquad (t(y'(t))^3)' = \frac{a}{t^3} y^3(\lambda t), \qquad a > 0, \quad \lambda > 0.$$

We set $\lambda = 0.4$ and $a = 5.2$. Then $\gamma = 2.5145$ and $\frac{(\alpha-\gamma)^\alpha}{\alpha^\alpha} = 0,0042$ Condition (2.12) reduces to

$$\lim_{t\to\infty} R^{\gamma}(\tau(t)) \int_{\tau(t)}^{t} p(s) \left[R^{1-\gamma/\alpha}(\tau(t)) - R^{1-\gamma/\alpha}(\tau(s))\right]^{\alpha} \mathrm{d}s = 0.0044 > 0.0042$$

and Theorem 2.3 ensures that the class $N_0 = \emptyset$.
Now we set $\lambda = 1.7$ and $a = 12.52$, which implies that $\delta = 4.8451$ and $\frac{(\alpha+\delta)^\alpha}{\alpha^\alpha} = 17.8826$. Since

$$\lim_{t\to\infty} \frac{1}{R^{\delta}(\tau(t))} \int_t^{\tau(t)} p(s) \left[R^{\delta/\alpha+1}(\tau(s)) - R^{\delta/\alpha+1}(\tau(t))\right]^{\alpha} \mathrm{d}s = 17.927 > 17.882,$$

then by Theorem 2.4 class $N_2 = \emptyset$.

For (E_{x3}) it is better to apply Theorem 2.3 and 2.4 instead of Theorem 2.1 and 2.2, due to evaluation of corresponding integrals.

The following example is intended to illustrate Theorems 5 applied to equations involving both delayed and advanced arguments.

Example 3.4. We consider the differential equation

$$(E_{x4}) \qquad (t^{-1/3}(y'(t))^{1/3})' = \frac{3.952}{t^{5/3}}\left(y^{1/3}(0.7\,t) + y^{1/3}(1.465\,t)\right)$$

In view of Example 2 conditions (2.5), (2.6) and (2.17), (2.18) are satisfied, so by Theorem 2.4 equation (E_{x4}) is oscillatory.

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