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On the real projections of zeros of analytic almost periodic functions

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ABSTRACT. This paper deals with the sets of real projections of zeros of analytic almost periodic functions defined in a vertical strip. By using our equivalence relation introduced in the context of the complex functions which can be represented by a Dirichlet-like series, this work provides practical results in order to determine whether a real number belongs to the closure of such a set. Its main result shows that, in the case that the Fourier exponents of an analytic almost periodic function are linearly independent over the rational numbers, such a set has no isolated points.

1. INTRODUCTION

The theory of almost periodic functions, which was created and developed in its main features by H. Bohr during the 1920's, opened a way to study a wide class of trigonometric series of the general type and even exponential series. This theory shortly acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations. In the case of functions that are defined on the real numbers, the notion of almost periodicity is a generalization of purely periodic functions and, in fact, as in classical Fourier analysis, every almost periodic function is associated with a Fourier series with real frequencies.

Let us briefly recall some notions concerning the theory of the almost periodic functions of a complex variable, which was theorized in [4] (see also [3, 5, 8, 12]). A function $f(s)$, $s=\sigma+it$, analytic in a vertical strip $U=\{s=\sigma+it\in\mathbb{C}:\alpha<\sigma<\beta\}$ $(-\infty\leq\alpha<\beta\leq\infty)$, is called almost periodic in U if to any $\varepsilon>0$ there exists a number $l=l(\varepsilon)$ such that each interval $t_0<t<t_0+l$ of length l contains a number τ satisfying

$$|f(s+i\tau)-f(s)|\leq\varepsilon\ \forall s\in U.$$

We will denote as $AP(U,\mathbb{C})$ the space of analytic almost periodic functions in a vertical strip U. It is known that every almost periodic function in $AP(U,\mathbb{C})$ is determined by an exponential series of the form $\sum_{n\geq 1}a_ne^{\lambda_n s}$ with complex coefficients a_n and real exponents λ_n, called Fourier exponents of f. This associated series is called the Dirichlet series of the given analytic almost periodic function (see [3, p.147], [8, p.77] or [12, p.312]).

Moreover, the set of analytic almost periodic functions in a vertical strip U coincides with the set of the functions which can be approximated uniformly in every reduced strip by exponential polynomials of the form

$$a_1e^{\lambda_1 s}+\ldots+a_ne^{\lambda_n s},\ a_j\in\mathbb{C},\ n\geq 2 \tag{1.1}$$

where $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is an ordered set of real numbers (see for example [8, Theorem 3.18]). In fact, it is convenient to recall that, even in the case that the sequence of the partial

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sums of its Dirichlet series does not converge uniformly, there exists a sequence of exponential polynomials, called Bochner-Fejér polynomials, of the type $P_k(s) = \sum_{j\geq 1} p_{j,k}a_je^{\lambda_j s}$ where for each k only a finite number of the factors $p_{j,k}$ differ from zero, which converges uniformly to f in every reduced strip in U and converges formally to the Dirichlet series on U [3, Polynomial approximation theorem, pp. 50,148].

In the context of the complex functions which can be represented by a Dirichlet-like series (in particular those almost periodic functions in $AP(U, \mathbb{C})$), we established in 2018 a new equivalence relation among them, say the *-equivalence, which led to refining Bochner's result that characterizes the almost periodicity (see [19, Theorem 5]) and a thorough extension of Bohr's equivalence theorem (see [22, Theorem 1]). This new equivalence relation, which is widely used in this paper, coincides with that of Bohr for the particular case of general Dirichlet series whose sets of exponents have an integral basis (see [22, Proposition 1]). Other important results derived from this equivalence relation can be seen in [19, 20, 21, 22, 23].

On the other hand, the study of the zeros of the class of exponential polynomials of type (1.1) has become a topic of increasing interest, see for example [2, 6, 10, 11, 13, 14, 15, 17, 18, 19, 21]. In this paper, we will study certain properties on the zeros of an analytic almost periodic function $f(s)$ in its vertical strip of almost periodicity $U = \{s = \sigma + it : \alpha < \sigma < \beta\}$. Specifically, consider the values a_f and b_f defined as

$$a_f := \inf \{\operatorname{Re} s : f(s) = 0,\ s \in U\} \tag{1.2}$$

and

$$b_f := \sup \{\operatorname{Re} s : f(s) = 0,\ s \in U\}\,. \tag{1.3}$$

In general, if f has at least one zero in U, it is satisfied $-\infty \leq a_f \leq b_f \leq \infty$ (it also depends on U). Given such a function $f(s)$ with $a_f, b_f \in \mathbb{R}$, the bounds a_f and b_f allow us to define an interval $I_f := [a_f, b_f]$ which contains the closure of the set of the real parts of the zeros of $f(s)$ in U. If either $a_f = \infty$ or $b_f = \infty$, the interval I_f is of the form $(-\infty, b_f]$, $[a_f, \infty)$ or $(-\infty, \infty)$. In this paper, we will focus our attention on the set

$$R_f := \overline{\{\operatorname{Re} s : f(s) = 0,\ s \in U\}} \cap (\alpha, \beta). \tag{1.4}$$

In this respect, the density properties of the zeros of several groups of exponential polynomials have also become a topic of increasing interest. In particular, the topological properties of the set $R_{\zeta_n} = \overline{\{\operatorname{Re} s : \zeta_n(s) = 0\}}$ associated with the partial sums $\zeta_n(s) = 1 + 2^{-s} + \ldots + n^{-s}$, $n \geq 2$, of the Riemann zeta function has been studied from different approaches. For example, an auxiliary function associated with $\zeta_n(s)$ was used in [17, Theorem 9] in order to establish conditions to decide whether a real number is in the set R_{ζ_n}. This auxiliary function, which is called the "companion function" of ζ_n in [24, p. 163], can also be adapted from a known result of C.E. Avellar and J.K. Hale [2, Theorem 3.1] in order to obtain analytical criterions about R_P in the more general case of exponential polynomials $P(s)$ of type (1.1).

In this paper, by analogy to the case of exponential polynomials, we first introduce an auxiliary function, of countably many real variables, which is associated with a prefixed almost periodic function $f(s)$ in a vertical strip U (see section 2). Secondly, we will see that this auxiliary function leads us to a practical characterization of the points of the set R_f associated with $f(s)$ (see Theorem 3.2). See also Theorem 3.1 which provides another characterization of the points in R_f and extends other results such as [7, Lemma 3] or [16, Lemma 4]. Thirdly, we study the closure set of the real parts of the zeros of almost periodic functions $f(s)$ whose Fourier exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$ are linearly independent over the rational numbers (see section 4). Under these hypothesis, this study provides a new pointwise characterization of the set R_f in terms of the inequalities (4.8) (see Theorem 4.3), which facilitates the obtaining of Proposition 4.1, about the boundary points of R_f, and corollaries 4.1 and 4.2 about some extra conditions under which we can state that $R_f \neq \emptyset$ (concerning this topic, see also Example 4.1). Finally, also under $\mathbb{Q}$-linear independence of $\{\lambda_1, \lambda_2, \ldots \lambda_k, \ldots\}$, with $k > 2$, we prove that the set of the real projections of the zeros of $f(s)$ has no isolated point in (α, β) (see Theorem 4.4), which generalizes [13, Theorem 7].

2. Auxiliary functions associated with almost periodic functions

Let $\mathcal{S}_\Lambda$ denote the class consisting of exponential sums of the form

$$\sum_{j\geq 1} a_j e^{\lambda_j p},\ a_j \in \mathbb{C},\ \lambda_j \in \Lambda,$$

where $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ is an arbitrary countable set of distinct real numbers (not necessarily unbounded), which are called a set of exponents or frequencies.

Also, let $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ be a basis of the vector space over the rationals generated by a set Λ of exponents, which implies that G_Λ is linearly independent over the rational numbers and each λ_j is expressible as a finite linear combination of terms of G_Λ, say

$$\lambda_j = \sum_{k=1}^{q_j} r_{j,k} g_k,\ \text{for some } r_{j,k} \in \mathbb{Q}.$$

By abuse of notation, we will say that G_Λ is a basis for Λ. Moreover, we will say that G_Λ is an integral basis for Λ when $r_{j,k} \in \mathbb{Z}$ for any j, k. Finally, we will say that G_Λ is the *natural basis* for Λ, and we will denote it as G_Λ^*, when it is constituted by elements in Λ as follows. Firstly if $\lambda_1 \neq 0$ then $g_1 := \lambda_1 \in G_\Lambda^*$. Secondly, if $\{\lambda_1, \lambda_2\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_2 \in G_\Lambda^*$. Otherwise, if $\{\lambda_1, \lambda_3\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_3 \in G_\Lambda^*$, and so on.

Let $A_1(p)$ and $A_2(p)$ be two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. It is said that A_1 is *-equivalent to A_2, and it is denoted as $A_1 \overset{*}{\sim} A_2$, if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$ (cardinal of Λ), there exists a $\mathbb{Q}$-linear map $\psi_n : \operatorname{span}_{\mathbb{Q}}(\{\lambda_1, \ldots, \lambda_n\}) \to \mathbb{R}$ such that

$$b_j = a_j e^{i\psi_n(\lambda_j)},\ j = 1, \ldots, n.$$

The important aspect in the *-equivalence, which is inspired by that of Bohr given for the case of general Dirichlet series, is the condition of existence of the $\mathbb{Q}$-linear maps ψ_n for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$ (see [19]).

If the formal series in $\mathcal{S}_\Lambda$ are handled as exponential sums of a complex variable on which we fix a summation procedure, we can introduce an auxiliary function which will be an important tool in this paper. To do this, we first consider the definition of the classes $\mathcal{D}_\Lambda$ of almost periodic functions in the following terms.

Definition 2.1. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ be an arbitrary countable set of distinct real numbers. We will say that a function $f : U \subset \mathbb{C} \to \mathbb{C}$ is in the class $\mathcal{D}_\Lambda$ if it is an almost periodic function in $AP(U, \mathbb{C})$ whose associated Dirichlet series is of the form

$$\sum_{j\geq 1} a_j e^{\lambda_j s},\ a_j \in \mathbb{C},\ \lambda_j \in \Lambda, \tag{2.5}$$

where U is a strip of the type $\{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$.

Now, with respect to our particular case of almost periodic functions with the Bochner-Fejér summation method (see, in this regard, [3, Chapter 1, Section 9]), to every almost periodic function $f \in \mathcal{D}_\Lambda$ we can associate an auxiliary function F_f of countably many real variables as follows (see [22, Definition 5]). For this, let $2\pi\mathbb{Z}^m = \{(c_1, c_2, \ldots, c_m) \in \mathbb{R}^m : c_k = 2\pi n_k, \text{with } n_k \in \mathbb{Z},\ k = 1, 2, \ldots, m\}$.

Definition 2.2. Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in $\{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, $-\infty \leq \alpha < \beta \leq \infty$, whose Dirichlet series is given by $\sum_{j\geq 1} a_j e^{\lambda_j s}$. For each $j \geq 1$ let $\mathbf{r}_j$ be the vector of rational components satisfying the equality $\lambda_j =< \mathbf{r}_j, \mathbf{g} >= \sum_{k=1}^{q_j} r_{j,k} g_k$, where $\mathbf{g} := (g_1, \ldots, g_k, \ldots)$ is the vector of the elements of the natural basis G_Λ^* for Λ. Then we define the auxiliary function $F_f : (\alpha, \beta) \times [0, 2\pi)^{\sharp G_\Lambda^*} \times \prod_{j\geq 1} 2\pi\mathbb{Z}^{\sharp G_\Lambda^*} \to \mathbb{C}$ associated with f, relative to the basis G_Λ^*, as

$$F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) := \sum_{j\geq 1} a_j e^{\lambda_j \sigma} e^{<\mathbf{r}_j, \mathbf{x}+\mathbf{p}_j>i}, \tag{2.6}$$

where $\sigma \in (\alpha, \beta)$, $\mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*}$, $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$ and series (2.6) is summed by Bochner-Fejér procedure, applied at $t = 0$ to the sum $\sum_{j\geq 1} a_j e^{<\mathbf{r}_j, \mathbf{x}+\mathbf{p}_j>i} e^{\lambda_j s}$.

We first note that, if $\sum_{j\geq 1} a_j e^{\lambda_j s}$ is the Dirichlet series of $f \in AP(U, \mathbb{C})$, for every choice of $\mathbf{x} \in \mathbb{R}^{\sharp G_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda}$, $j = 1, 2, \ldots$, the sum $\sum_{j\geq 1} a_j e^{<\mathbf{r}_j, \mathbf{x}+\mathbf{p}_j>i} e^{\lambda_j s}$ represents the Dirichlet series of an almost periodic function which is connected with f through the *-equivalence (see [19, Lemma 3] and [22, Proposition 2]).

We next introduce the following notation which will be used to show the direct relation between an almost periodic function and the auxiliary function associated with it.

Definition 2.3. Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, let $f(s) \in \mathcal{D}_\Lambda$ be an almost periodic function in an open vertical strip U, and $\sigma_0 = \operatorname{Re} s_0$ with $s_0 \in U$. We define $\operatorname{Img}(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots))$ to be the set of values in the complex plane taken on by the auxiliary function $F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)$ when $\sigma = \sigma_0$; that is $\operatorname{Img}(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)) = \{s \in \mathbb{C} : \exists \mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*} \text{ and } \mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*} \text{ such that } s = F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)\}$.

The notation $\operatorname{Img}(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots))$ is well-posed because this set is independent of the basis G_Λ (see [22, Lemma 1]). Also, given a function $f(s)$, take the notation

$$\operatorname{Img}(f(\sigma_0 + it)) = \{s \in \mathbb{C} : \exists t \in \mathbb{R} \text{ such that } s = f(\sigma_0 + it)\}.$$

It is convenient to remark that, concerning the classes $\mathcal{D}_\Lambda$ and the *-equivalence, it was proved in [22, Proposition 4, ii)] that $\operatorname{Img}(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)) = \bigcup_{f_k \overset{*}{\sim} f} \operatorname{Img}(f_k(\sigma_0 + it))$. Moreover, it was proved in [22, Theorem 1] that if E is an open set of real numbers included in (α, β), then

$$\bigcup_{\sigma\in E} \operatorname{Img}(f_1(\sigma + it)) = \bigcup_{\sigma\in E} \operatorname{Img}(f_2(\sigma + it)),$$

where $f_1, f_2 \in \mathcal{D}_\Lambda$ are two equivalent almost periodic functions in U. That is, the functions f_1 and f_2 take the same set of values on the region $\{s = \sigma + it \in \mathbb{C} : \sigma \in E\}$.

3. The closure set of the real projections of the zeros in the general case

This section is mainly devoted to show two point-wise characterizations of the sets R_f defined in (1.4) associated with an almost periodic function $f(s)$.

We first extend and improve some results in the style of [7, Lemma 3] or [16, Lemma 4].

Theorem 3.1. *Let $f(s)$ be an almost periodic function in an open vertical strip $U = \{\sigma + it \in \mathbb{C} : \alpha < \sigma < \beta\}$, and $\sigma_0 \in (\alpha, \beta)$. Then $\sigma_0 \in R_f$ if and only if there exists a sequence $\{t_j\}_{j=1,2,\ldots}$ of real numbers such that*

$$\lim_{j\to\infty} f(\sigma_0 + it_j) = 0. \tag{3.7}$$

Proof. Note that $f(s)$, and its derivatives, are bounded on every reduced strip of U [3, pp. 142-144]. Suppose first the existence of $\{t_j\}_{j=1,2,\ldots} \subset \mathbb{R}$ satisfying (3.7). In order to apply [14, Lemma, p.73], we next prove that there exist positive numbers δ and l such that on any segment of length l of the line $x = \sigma_0$ there is a point $\sigma_0 + iM$ such that $|f(\sigma_0 + iM)| \geq \delta$. Indeed, let t_0 be a real number such that $|f(\sigma_0 + it_0)| > 0$ and take $\delta = \dfrac{|f(\sigma_0 + it_0)|}{2}$. Then, since $f(s)$ is an almost periodic function in U, there exists a positive real number $l = l(\delta)$ such that every interval of length l on the imaginary axis contains at least one translation number iT, associated with δ, satisfying $|f(s + iT) - f(s)| \leq \delta$ for all $s \in U$. Thus, by taking $s = \sigma_0 + it_0$, we have $|f(\sigma_0 + i(t_0 + T)) - f(\sigma_0 + it_0)| \leq \delta$ and, according to the choice of δ, it follows that $|f(\sigma_0 + i(t_0 + T))| \geq \delta$. Therefore, $|f(\sigma_0 + iM)| \geq \delta$, with $M := t_0 + T$. Consequently, the function $f(s)$ has the properties needed to apply [14, Lemma, p.73] and thus $f(s)$ has zeros in every strip

$$S_\epsilon := \{s \in \mathbb{C} : \sigma_0 - \epsilon < \operatorname{Re} s < \sigma_0 + \epsilon\},$$

for any arbitrary $\epsilon > 0$, which proves that $\sigma_0 \in R_f$.

Conversely suppose that $\sigma_0 \in R_f$. This means that there exists a sequence $\{s_j\}_{j\geq 1} \subset U$, with $s_j = \sigma_j + it_j$, such that $f(s_j) = 0$ and $\lim_{j\to\infty} \sigma_j = \sigma_0$. Consider the sequence of functions given by $f_j(s) := f(s + t_j)$, $s \in U$, $j = 1, 2, \ldots$ which is uniformly bounded on every strip $S_\epsilon \subset U$. By Montel's theorem [1, Section 5.1.10], $\{f_j(s)\}_{j\geq 1}$ has a subsequence, which we denote again by $\{f_j(s)\}$, converging uniformly on compact subsets of S_ϵ to a function $h(s)$ analytic in S_ϵ. In this way, it is clear that

$$h(\sigma_0) = \lim_{j\to\infty} f_j(\sigma_0) = \lim_{j\to\infty} f(\sigma_0 + it_j) = 0.$$

Indeed, note that $h(\sigma_0) = 0$ in virtue from $f_j(\sigma_j) = 0$, $j = 1, 2, \ldots$, and $\lim_{j\to\infty} \sigma_j = \sigma_0$. $\square$

We next give a characterization of the sets R_f by means of an *ad hoc* version of [2, Theorem 3.1], which is obtained through the auxiliary function F_f analysed in the previous section (see (2.6)).

Theorem 3.2. *Let $f(s)$ be an almost periodic function in a vertical strip $U = \{s = \sigma + it : \alpha < \sigma < \beta\}$. Consider $\sigma \in (\alpha, \beta)$. Then $\sigma \in R_f$ if and only if there exist some vectors $\mathbf{x} \in [0, 2\pi)^{\sharp G_\Lambda^*}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G_\Lambda^*}$ such that $F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) = 0$, where Λ is the set of Fourier exponents of $f(s)$.*

Proof. Let $\sigma_0 \in R_f$. Then there exists a sequence $\{s_j\}_{j=1,2,\ldots} \subset U$, with $s_j = \sigma_j + it_j$, of zeros of $f(s)$ such that $\sigma_0 = \lim_{j\to\infty} \sigma_j$. Consider the sequence of functions $\{f_j(s)\}_{j\geq 1}$ defined as $f_j(s) := f(s + it_j)$, which are analytic in U. Then it is clear that each $f_j(s)$ is equivalent to $f(s)$ (see [19, Lemma 1]), that is $f_j \overset{*}{\sim} f$ where $\overset{*}{\sim}$ is the equivalence relation considered in [19, 22]. Now, by taking into account [19, Propositions 3 and 4], we can extract a subsequence of $\{f_j(s)\}_{j\geq 1}$ which converges uniformly on every reduced strip of U to a function $h(s)$ in the same equivalence class as f. Moreover, we have $h(\sigma_0) = 0$. Indeed $h(\sigma_0) = \lim_{j\to\infty} f_j(\sigma_0) = 0$. Otherwise, there would exist $D(\sigma_0, \varepsilon)$ such that $h(s) \neq 0\ \forall s \in \overline{D}(\sigma_0, \varepsilon)$ and, by Hurwitz's theorem [1, Section 5.1.3], there would exist $j_0 \in \mathbb{N}$ such that $f_j(s) \neq 0\ \forall s \in \overline{D}(\sigma_0, \varepsilon)$ and each $j \geq j_0$, which is a contradiction because $f_j(\sigma_j) = 0$.

Consequently, as $h(\sigma_0) = 0$ and $\mathrm{Img}\,(F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)) = \bigcup_{f_k \overset{*}{\sim} f} \mathrm{Img}\,(f_k(\sigma_0 + it))$ [22, Proposition 4, ii)], we have that there exist $\mathbf{x} \in [0, 2\pi)^{\sharp G^*_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$ such that $0 = F_f(\sigma_0, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots)$.

Conversely, suppose that $F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) = 0$ for some real number $\sigma \in (\alpha, \beta)$ and some vectors $\mathbf{x} = (x_1, x_2, \ldots, x_k, \ldots) \in [0, 2\pi)^{\sharp G^*_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$. Again by [22, Proposition 4, ii)], we have that $\mathrm{Img}\,(F_f(\sigma, \mathbf{x})) = \bigcup_{f_k \overset{*}{\sim} f} \mathrm{Img}\,(f_k(\sigma + it))$. Hence there exists $f_k \overset{*}{\sim} f$ such that $f_k(\sigma + it) = 0$ for some real number t. Furthermore, by [22, Theorem 1], for any $\varepsilon > 0$ sufficiently small it is accomplished that the functions f_k and f take the same set of values on the region $\{s \in \mathbb{C} : \mathrm{Re}\, s \in (\sigma - \varepsilon, \sigma + \varepsilon)\}$. This means that $\sigma \in R_f$. □

4. The closure set of the real projections of the zeros under $\mathbb{Q}$-linear independence of the Fourier exponents

We first recall that if the Fourier exponents of an almost periodic function f in a vertical strip $U = \{s = \sigma + it : \alpha < \sigma < \beta\}$ are linearly independent over the rational numbers, then the Dirichlet series expansion of f converges to f itself and, in fact, it converges absolutely in the open strip U ([9, Theorem 3.6] and [3, p. 154]). Moreover, in this case it is obvious that the set of Fourier exponents has an integral basis.

We next prove the following characterization of the points in the set R_f associated with an almost periodic function f whose Fourier exponents are $\mathbb{Q}$-linearly independent, that is, when its exponents are linearly independent over the rational numbers.

Theorem 4.3. *Let $f(s)$ be an almost periodic function in a vertical strip $U = \{s = \sigma + it : \alpha < \sigma < \beta\}$ whose Dirichlet series is given by $\sum_{n\geq 1} a_n e^{\lambda_n s}$ with $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$ $\mathbb{Q}$-linearly independent and $k > 2$. Let $\sigma_0 \in (\alpha, \beta)$. Then $\sigma_0 \in R_f$ if and only if*

$$|a_j|\, e^{\sigma_0 \lambda_j} \leq \sum_{i\geq 1,\, i\neq j} |a_i|\, e^{\sigma_0 \lambda_i} \ \ (j = 1, 2, \ldots, k, \ldots)\,. \tag{4.8}$$

Proof. Without loss of generality, take $G^*_\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ as the basis of the vector space over the rationals generated by the set of Fourier exponents of f. Suppose that $\sigma_0 \in R_f$, then by Theorem 3.2 there exist some vectors $\mathbf{x} \in [0, 2\pi)^{\sharp G^*_\Lambda}$ and $\mathbf{p}_j \in 2\pi\mathbb{Z}^{\sharp G^*_\Lambda}$ such that $F_f(\sigma, \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \ldots) = 0$ or, equivalently, $\sum_{n\geq 1} a_n e^{\lambda_n \sigma_0} e^{x_n i} = 0$ (by taking $\mathbf{g} = (\lambda_1, \lambda_2, \ldots)$ and hence $r_{n,k} = 0$ if $k \neq n$ and $r_{n,n} = 1$). Therefore,

$$a_j e^{\lambda_j \sigma_0} e^{x_j i} = -\sum_{k\geq 1, k\neq j} a_k e^{\sigma_0 \lambda_k} e^{x_k i},\ j = 1, 2, \ldots$$

and, by taking the modulus, we get

$$|a_j| e^{\lambda_j \sigma_0} \leq \sum_{k\geq 1, k\neq j} |a_k| e^{\sigma_0 \lambda_k},\ j = 1, 2, \ldots.$$

Conversely, suppose that the positive real numbers $|a_j| e^{\sigma_0 \lambda_j}$, $j = 1, 2, \ldots$, satisfy inequalities (4.8). We recall that, by [9, Theorem 3.6] or [3, p.154], it is accomplished that $\sum_{j\geq 1} |a_j| e^{\sigma_0 \lambda_j} < \infty$. Thus, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{j\geq n_0} |a_j| e^{\sigma_0 \lambda_j} < \varepsilon$. Hence, for $\varepsilon > 0$ sufficiently small we can index the terms in decreasing order so that m_1 is such that $|a_{m_1}| e^{\sigma_0 \lambda_{m_1}} := \max\{|a_k| e^{\sigma_0 \lambda_k} : k = 1, 2, \ldots, n_0 - 1\}$, m_2 such that $|a_{m_2}| e^{\sigma_0 \lambda_{m_2}} := \max\{|a_k| e^{\sigma_0 \lambda_k} : k = 1, 2, \ldots, n_0 - 1, k \neq m_1\}$, etc. Therefore, by taking $r := \sum_{j\geq n_0} |a_j| e^{\sigma_0 \lambda_j}$, there is at least one n_0-sided polygon whose sides have the lengths

$|a_{m_j}|e^{\sigma_0\lambda_{m_j}}$, $j=1,2,\ldots,n_0-1$ and r [14, p.71]. That means that there exist real numbers $\theta_1,\theta_2,\ldots,\theta_{n_0}$ satisfying

$$\sum_{k=1}^{n_0-1}|a_k|e^{\sigma_0\lambda_k}e^{i\theta_k}+re^{i\theta_{n_0}}=0.$$

Consequently, by taking $\mathbf{x}\in\mathbb{R}^{\sharp G_\Lambda}$ the vector with components $x_k=\theta_k-\operatorname{Arg}(a_k)$ for $k=1,\ldots,n_0-1$, and $x_k=\theta_{n_0}-\operatorname{Arg}(a_k)$ for each $k\geq n_0$, where $\operatorname{Arg}(a_k)$ denotes the principal argument of a_k, we have

$$F_f(\sigma_0,\mathbf{x},\mathbf{0},\mathbf{0},\ldots)=\sum_{k\geq 1}a_ke^{\lambda_k\sigma_0}e^{x_ki}=0.$$

Hence, from Theorem 3.2, $\sigma_0\in R_f$. □

From now on we will analyse some properties of the set R_f defined by

$$\overline{\{\operatorname{Re}s:f(s)=0,\ s\in U\}}\cap(\alpha,\beta),$$

associated with an almost periodic function $f(s)$ in a vertical strip $U=\{s=\sigma+it:\alpha<\sigma<\beta\}$ with rationally independent Fourier exponents.

Concerning inequalities (4.8), given such a function $f(s)$ and a boundary point σ_0 of the set R_f, we next prove that the equality is attained at σ_0 in only one of these inequalities.

Proposition 4.1. *Let $f(s)$ be an almost periodic function in a vertical strip $U=\{s=\sigma+it:\alpha<\sigma<\beta\}$ whose Fourier exponents $\{\lambda_1,\lambda_2,\ldots,\lambda_k,\ldots\}$, with $k>2$, are $\mathbb{Q}$-linearly independent. Let $\sigma_0\in(\alpha,\beta)$. If σ_0 is a boundary point of R_f, then it satisfies all the inequalities (4.8) and only one of them is an equality.*

Proof. Let $f(s)$ be an almost periodic function in a vertical strip $U=\{s=\sigma+it:\alpha<\sigma<\beta\}$ whose Dirichlet series is given by $\sum_{n\geq1}a_ne^{\lambda_ns}$, with $\{\lambda_1,\lambda_2,\ldots,\lambda_k,\ldots\}$ $\mathbb{Q}$-linearly independent and $k>2$. As R_f is closed in (α,β), the boundary of R_f is a subset of R_f itself. Then $\sigma_0\in R_f$ and, by Theorem 4.3, inequalities (4.8) are obviously satisfied for σ_0. Moreover, if some of the inequalities (4.8) is an equality, as any couple of equalities are incompatible, the lemma follows. Otherwise we have the following strict inequalities

$$|a_j|\,e^{\sigma_0\lambda_j}<\sum_{i\geq1,\,i\neq j}|a_i|\,e^{\sigma_0\lambda_i},\ (j=1,2,\ldots,k,\ldots)\,. \tag{4.9}$$

Now, as σ_0 is a boundary point of $R_f=\overline{\{\operatorname{Re}s:f(s)=0,\ s\in U\}}\cap(\alpha,\beta)$, there exists $\varepsilon>0$ such that either $R_f^c\supset(\sigma_0-\varepsilon,\sigma_0)$ or $R_f^c\supset(\sigma_0,\sigma_0+\varepsilon)$, where R_f^c denotes $(\alpha,\beta)\setminus R_f$. Let σ_1 be a point in $(\sigma_0-\varepsilon,\sigma_0+\varepsilon)\cap R_f^c$. In virtue of Theorem 4.3, it is plain that there exists a single $j_0\geq1$ so that

$$|a_{j_0}|\,e^{\sigma_1\lambda_{j_0}}>\sum_{i\geq1,\,i\neq j_0}|a_i|\,e^{\sigma_1\lambda_i}. \tag{4.10}$$

Thus, by continuity, we deduce from (4.9) and (4.10), for this j_0, that there exists σ_2 between σ_0 and σ_1 such that

$$|a_{j_0}|\,e^{\sigma_2\lambda_{j_0}}=\sum_{i\geq1,\,i\neq j_0}|a_i|\,e^{\sigma_2\lambda_i}. \tag{4.11}$$

Moreover, by (4.11) it is clear that

$$|a_j|\,e^{\sigma_2\lambda_j}\leq\sum_{i\geq1,\,i\neq j}|a_i|\,e^{\sigma_2\lambda_i},\ (j=1,2,\ldots,k,\ldots)\,.$$

which yields, again by Theorem 4.3, that $\sigma_2\in R_f$. This is a contradiction and hence the result holds. □

We first prove that the set R_f associated with an almost periodic function in $U = \mathbb{C}$, whose Fourier exponents are linearly independent over the rational numbers, is not the empty set.

Corollary 4.1. *Let $f(s)$ be an almost periodic function in $\mathbb{C}$ whose Fourier exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent. Then $R_f \neq \emptyset$.*

Proof. Let $f(s)$ be an almost periodic function in $\mathbb{C}$ whose Dirichlet series is given by $\sum_{n\geq 1} a_n e^{\lambda_n s}$. By reductio ad absurdum, suppose $R_f = \emptyset$. By Theorem 4.3, there exists $j_0 \geq 1$ so that

$$|a_{j_0}|\, e^{\sigma\lambda_{j_0}} > \sum_{i\geq 1,\, i\neq j_0} |a_i|\, e^{\sigma\lambda_i}, \text{ for all } \sigma \in \mathbb{R}. \tag{4.12}$$

Otherwise (if the inequality is not true for all $\sigma \in \mathbb{R}$), by continuity, there would exist $\sigma_0 \in \mathbb{R}$ such that $|a_{j_0}|\, e^{\sigma_0\lambda_{j_0}} = \sum_{i\geq 1,\, i\neq j_0} |a_i|\, e^{\sigma_0\lambda_i}$ and hence, by Theorem 4.3, it is clear that σ_0 would be in R_f.

However, we next show that (4.12) is a contradiction. Indeed, take $k \neq j_0$. Since $\lambda_k \neq \lambda_{j_0}$, it is plain that there exists $\sigma_0 \in \mathbb{R}$ such that $|a_k| e^{\sigma_0\lambda_k} = |a_{j_0}| e^{\sigma_0\lambda_{j_0}}$, which yields that

$$\sum_{i\geq 1,\, i\neq j_0} |a_i|\, e^{\sigma_0\lambda_i} > |a_k| e^{\sigma_0\lambda_k} = |a_{j_0}| e^{\sigma_0\lambda_{j_0}},$$

which contradicts (4.12). Now the result holds. □

It is worth noting that the condition $U = \mathbb{C}$ in Corollary 4.1 is necessary. That is, an almost periodic function $f(s)$ in a vertical strip U, with $U \neq \mathbb{C}$, such that its Fourier exponents are $\mathbb{Q}$-linearly independent could satisfy $R_f = \emptyset$.

Example 4.1. Let $\{1, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots, \rho_1, \rho_2, \ldots, \rho_n, \ldots\}$ be an ordered set of positive real numbers which are linearly independent over the rational numbers and satisfy $\lambda_n > n^2$ (hence $\rho_n > n^2$) for each $n = 1, 2, \ldots$. It is clear that such a set can be constructed from an arbitrary Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ in virtue of the density of rational numbers in $\mathbb{R}$.

On the one hand, consider the exponential sum

$$S_1(s) = \sum_{n\geq 1} \frac{1}{\lambda_n} e^{\lambda_n s}.$$

Note that the exponents of S_1 are linearly independent over the rational numbers. Moreover, $S_1(s)$ converges absolutely on $U_1 = \{s = \sigma + it \in \mathbb{C} : \sigma < 0\}$. Indeed, for any $\sigma < 0$, it is satisfied that

$$\sum_{n\geq 1} \frac{1}{\lambda_n} e^{\lambda_n \sigma} \leq \sum_{n\geq 1} \frac{1}{\lambda_n} \leq \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and the derivative of $S_1(s)$ accomplishes

$$S_1'(\sigma) = \sum_{n\geq 1} e^{\lambda_n \sigma},$$

which implies that $S_1(s)$ diverges when $\sigma > 0$.

On the other hand, consider the exponential sum

$$S_2(s) = \sum_{n\geq 1} \frac{1}{\rho_n} e^{-\rho_n(s+1)} = \sum_{n\geq 1} \frac{1}{\rho_n} e^{-\rho_n} e^{-\rho_n s}.$$

Note that the exponents of S_2 are linearly independent over the rational numbers. Moreover, $S_2(s)$ converges absolutely on $U_2 = \{s = \sigma + it \in \mathbb{C} : \sigma > -1\}$. Indeed, for any $\sigma > -1$, it is satisfied that

$$\sum_{n\geq 1} \frac{1}{\rho_n} e^{-\rho_n} e^{-\rho_n \sigma} \leq \sum_{n\geq 1} \frac{1}{\rho_n} \leq \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and the derivative accomplishes

$$S_2'(\sigma) = \sum_{n\geq 1} -e^{-\rho_n(\sigma+1)},$$

which implies that $S_2(s)$ diverges when $\sigma < -1$.

In this way, let $S(s) := S_1(s) + S_2(s) + \frac{e\pi^2}{3} e^s$, $s \in \mathbb{C}$. By above, it is accomplished that $S(s)$ is an exponential sum, whose exponents are linearly independent over the rationals, which converges uniformly in $U = U_1 \cap U_2 = \{s = \sigma + it \in \mathbb{C} : -1 < \sigma < 0\}$. In fact, U is the largest open vertical strip of almost periodicity of $S(s)$. Moreover,

$$|S_1(s) + S_2(s)| \leq 2\frac{\pi^2}{6} = \frac{\pi^2}{3} < \left|\frac{e\pi^2}{3} e^s\right| \ \forall s \in U.$$

Consequently, $S(s)$ has no zeros in U and hence $R_S = \emptyset$.

In this respect, we next study the following more general result.

Corollary 4.2. *Let $f(s)$ be an almost periodic function in a vertical strip $U = \{s = \sigma + it : \alpha < \sigma < \beta\}$ whose Dirichlet series is given by $\sum_{n\geq 1} a_n e^{\lambda_n s}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent. Suppose that U is the largest open vertical strip of almost periodicity of f and some of the following conditions is satisfied:*

a) $\lim_{\sigma\to\alpha^+} \sum_{i\geq 1} |a_i| e^{\sigma\lambda_i} > 2\sup\{|a_i| e^{\alpha\lambda_i} : i \geq 1\}$;

b) $\lim_{\sigma\to\beta^-} \sum_{i\geq 1} |a_i| e^{\sigma\lambda_i} > 2\sup\{|a_i| e^{\beta\lambda_i} : i \geq 1\}$.

Then $R_f \neq \emptyset$.

Proof. Let $f(s)$ be an almost periodic function in U whose Dirichlet series is given by $\sum_{n\geq 1} a_n e^{\lambda_n s}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent. By [3, p. 154, first theorem], $\sum_{n\geq 1} a_n e^{\lambda_n s}$ is absolutely convergent in U and hence $f(s) = \sum_{n\geq 1} a_n e^{\lambda_n s}$ for any $s \in U$. Let $f_1(s) := \sum_{n\geq 1} |a_n| e^{\lambda_n s}$, which is clearly anaytic on U, then U is the largest open vertical strip of almost periodicity of f_1. Indeed, it is plain that f_1 is almost periodic on U (it converges absolutely on U). Moreover, if there was an open vertical strip $V \supset U$, with $U \neq V$ and f_1 almost periodic on V, then

$$\left|\sum_{n\geq 1} a_n e^{\lambda_n s}\right| \leq \sum_{n\geq 1} |a_n| e^{\lambda_n \sigma} = f_1(\sigma) < \infty \ \forall s = \sigma + it \in V$$

and hence V would be an open vertical strip where f converges absolutely, which is a contradiction.

Now, by reductio ad absurdum, suppose $R_f = \emptyset$. By Theorem 4.3, there exists $j_0 \geq 1$ so that

$$|a_{j_0}| e^{\sigma\lambda_{j_0}} > \sum_{i\geq 1,\, i\neq j_0} |a_i| e^{\sigma\lambda_i}, \text{ for all } \sigma \in (\alpha, \beta). \tag{4.13}$$

Otherwise (if the inequality is not true for all $\sigma \in (\alpha, \beta)$), by continuity, there would exist $\sigma_0 \in (\alpha, \beta)$ such that $|a_{j_0}|e^{\sigma_0 \lambda_{j_0}} = \sum_{i \geq 1,\, i \neq j_0} |a_i| e^{\sigma_0 \lambda_i}$ and hence, by Theorem 4.3, it is clear that σ_0 would be in R_f. Let $g(\sigma) := \sum_{i \geq 1,\, i \neq j_0} |a_i| e^{\sigma \lambda_i} - |a_{j_0}| e^{\sigma \lambda_{j_0}},\ \sigma \in (\alpha, \beta)$. We deduce from (4.13) that

$$g(\sigma) < 0 \text{ for all } \sigma \in (\alpha, \beta). \tag{4.14}$$

However, we next show that this is a contradiction. Note first that, by taking into account [3, p. 154, second theorem], α and β are singular points of $f_1(s)$. By hypothesis (condition a)), given $\varepsilon > 0$, there exists $\sigma_1 \in (\alpha, \alpha + \varepsilon)$ such that

$$\sum_{i \geq 1} |a_i| e^{\sigma \lambda_i} > 2 \sup\{|a_i| e^{\alpha \lambda_i} : i \geq 1\} \geq 2 |a_{j_0}| e^{\alpha \lambda_{j_0}}\ \forall \sigma \in (\alpha, \sigma_1)$$

or (condition b)) there exists $\sigma_2 \in (\beta - \varepsilon, \beta)$ such that

$$\sum_{i \geq 1} |a_i| e^{\sigma \lambda_i} > 2 \sup\{|a_i| e^{\beta \lambda_i} : i \geq 1\} \geq 2 |a_{j_0}| e^{\beta \lambda_{j_0}}\ \forall \sigma \in (\sigma_2, \beta).$$

Therefore, by continuity, there exists $0 < \tau < \min\{\sigma_1 - \alpha, \beta - \sigma_2\}$, sufficiently small, such that

$$\sum_{i \geq 1} |a_i| e^{\sigma \lambda_i} > 2 |a_{j_0}| e^{(\alpha + \tau) \lambda_{j_0}}\ \forall \sigma \in (\alpha, \sigma_1)$$

or

$$\sum_{i \geq 1} |a_i| e^{\sigma \lambda_i} > 2 |a_{j_0}| e^{(\beta - \tau) \lambda_{j_0}}\ \forall \sigma \in (\sigma_2, \beta).$$

In particular, we get

$$\sum_{i \geq 1} |a_i| e^{(\alpha + \tau) \lambda_i} > 2 |a_{j_0}| e^{(\alpha + \tau) \lambda_{j_0}} \tag{4.15}$$

or

$$\sum_{i \geq 1} |a_i| e^{(\beta - \tau) \lambda_i} > 2 |a_{j_0}| e^{(\beta - \tau) \lambda_{j_0}}. \tag{4.16}$$

Since (4.15) and (4.16) imply $g(\alpha + \tau) > 0$ and $g(\beta - \tau) > 0$, respectively, we get a contradiction with (4.14). Now the result follows. $\square$

We next focus our attention on the real solutions of the equations

$$|a_j|\, e^{\lambda_j \sigma} = \sum_{i \geq 1,\, i \neq j} |a_i|\, e^{\lambda_i \sigma},\ j = 1, 2, \ldots, k, \ldots. \tag{4.17}$$

Lemma 4.1. *Let $f(s)$ be an almost periodic function in a vertical strip $U = \{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$ whose Fourier exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent. Then each equation (4.17) has at most 2 real solutions in (α, β).*

Proof. Let $f(s)$ be an almost periodic function in $U = \{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$ whose Dirichlet series is given by $\sum_{n \geq 1} a_n e^{\lambda_n s}$ with $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$ $\mathbb{Q}$-linearly independent and $k > 2$. Fixed $j = 1, 2, ..., k, \ldots$, we define the real function

$$f_j(\sigma) := \sum_{i \geq 1, i \neq j} |a_i|\, e^{\lambda_i \sigma} - |a_j|\, e^{\lambda_j \sigma},\ \sigma \in (\alpha, \beta).$$

Also, by dividing by $|a_j| e^{\lambda_j \sigma}$, consider

$$B_j(\sigma) := \frac{f_j(\sigma)}{|a_j| e^{\lambda_j \sigma}} = \sum_{i \geq 1, i \neq j} \frac{|a_i|}{|a_j|} e^{(\lambda_i - \lambda_j)\sigma} - 1,\ \sigma \in (\alpha, \beta).$$

It is worth noting that, since $f(s)$ is analytic in U, then it is uniformly continuous in every open interval interior to U together with all its derivatives [3, p. 142]. Thus it is easy to check that $B_j''(\sigma) \geq 0$ for all $\sigma \in (\alpha, \beta)$, i.e. $B_j(\sigma)$ is convex in (α, β). Consequently, equation $B(\sigma) = 0$, $\sigma \in (\alpha, \beta)$, has at most two solutions. Thus the result holds. □

At this point, we prove the following important result which generalizes [13, Theorem 7].

Theorem 4.4. *The set of the real projections of the zeros of an almost periodic function in an open vertical strip $U = \{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$, whose Fourier exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent, has no isolated point in (α, β).*

Proof. If the real projection of a zero $s_0 \in U$ of $f(s)$, say σ_0, were an isolated point of the set $\{\operatorname{Re} s : f(s) = 0,\ s \in U\}$, necessarily σ_0 would be a boundary point of the set $R_f = \overline{\{\operatorname{Re} s : f(s) = 0,\ s \in U\}} \cap (\alpha, \beta)$, with $\sigma_0 \in (\alpha, \beta)$. By Proposition 4.1, it satisfies all the inequalities (4.8), that is,

$$|a_j|\, e^{\sigma_0 \lambda_j} \leq \sum_{i \geq 1,\, i \neq j} |a_i|\, e^{\sigma_0 \lambda_i},\ (j = 1, 2, \ldots, k, \ldots),$$

and only one of them is an equality, say

$$|a_k|\, e^{\lambda_k \sigma} = \sum_{i \geq 1,\, i \neq k} |a_i|\, e^{\lambda_i \sigma}. \tag{4.18}$$

Now, from Lemma 4.1, we have that equation (4.18), which is satisfied by $\sigma_0 \in (\alpha, \beta)$, has 1 or 2 solutions in (α, β). This means that the equation $B_k(\sigma) = 0$ has 1 or 2 solutions in (α, β), where

$$B_k(\sigma) := \sum_{i \geq 1, i \neq k} \frac{|a_i|}{|a_k|} e^{(\lambda_i - \lambda_k)\sigma} - 1,\ \sigma \in (\alpha, \beta).$$

Thus, since $B_k(\sigma)$ is continuous and convex in (α, β) (see also the proof of Lemma 4.1), we can assure the existence of some $\varepsilon > 0$ such that any σ in the interval $(\sigma_0 - \varepsilon, \sigma_0) \subset (\alpha, \beta)$ or $(\sigma_0, \sigma_0 + \varepsilon) \subset (\alpha, \beta)$ satisfies $B_k(\sigma) \geq 0$, which implies that σ satisfies inequalities (4.8). Then, by Theorem 4.3, the interval $(\sigma_0 - \varepsilon, \sigma_0)$ or $(\sigma_0, \sigma_0 + \varepsilon)$ is in R_f, which is a contradiction because σ_0 is an isolated point in R_f. □

Under the conditions of the previous result, it is now clear that the set R_f is the union of a denumerable amount of disjoint nondegenerate intervals. In this respect, the following result concernes the gaps of the set R_f.

Corollary 4.3. *Let $f(s)$ be an almost periodic function in a vertical strip $U = \{s \in \mathbb{C} : \alpha < \operatorname{Re} s < \beta\}$ whose Fourier exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots\}$, with $k > 2$, are $\mathbb{Q}$-linearly independent. Then the gaps of R_f are produced by those equations (4.17) having two real solutions in (α, β).*

Proof. Let $\sigma_0 \in (\alpha, \beta)$ be a boundary point of R_f. Then, by Proposition 4.1, σ_0 satisfies only one of equalities (4.17), say

$$|a_k|\, e^{\lambda_k \sigma} = \sum_{i \geq 1,\, i \neq k} |a_i|\, e^{\lambda_i \sigma}. \tag{4.19}$$

If we suppose that equation (4.19) has only the solution σ_0 in (α, β), it follows from theorems 4.3 and 4.4 that

$$|a_k|\, e^{\lambda_k \sigma} < \sum_{i \geq 1,\, i \neq k} |a_i|\, e^{\lambda_i \sigma}\ \forall \sigma \in (\alpha, \beta) \setminus \{\sigma_0\}.$$

Therefore, in virtue from continuity and Theorem 4.3, we can assure the existence of some $\varepsilon > 0$ such that the interval $(\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$ is in R_f, which is a contradiction because σ_0 is a boundary point in R_f. Consequently, by Lemma 4.1, equation (4.19) has two solutions in (α, β), which means that the gaps of R_f are only produced by those equations (4.17) having two real solutions in (α, β). □

5. Conclusions and further developments

Our goal was to verify to what extent several known results on the sets of real projections of zeros of certain classes of exponential polynomials can be extended to the case of analytic almost periodic functions defined in vertical strips of the form $U = \{s = \sigma + it \in \mathbb{C} : \alpha < \sigma < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. This approach is possible in the general case because there exists a way of summation, called Bochner-Fejér procedure, which gives rise to a sequence of finite exponential sums that converges uniformly to a prefixed almost periodic function $f(s)$ in every reduced strip in U. In fact, given such a function $f(s)$, this procedure allows us to introduce an auxiliar function associated with it, of countably many real variables, which is a key tool in order to characterize the sets $R_f := \overline{\{\operatorname{Re} s : f(s) = 0,\ s \in U\}} \cap (\alpha, \beta)$. The proof of this first characterization is based on an equivalence relation which we previously introduced in the context of the complex functions which can be represented by a Fourier-like or Dirichlet-like series (as in the case of the spaces of Bohr's almost periodic functions).

For the case that the Fourier exponents of such an almost periodic function $f(s)$ are linearly independent over the rational numbers, we know that the Dirichlet series expansion of $f(s)$ is absolutely convergent in U (in fact, it is also uniformly convergent to $f(s)$). Due to the circumstances surrounding this particular case, we have provided a new pointwise characterization of the set R_f in terms of the inequalities (4.8), which leads to determining the possible boundary points of R_f. Under this condition and with at least three Fourier exponents, our main result in this paper shows that the set of the real projections of the zeros of $f(s)$ has no isolated point in (α, β), which extends a known result which was previously demonstrated for the case of exponential polynomials.

Remark 5.1. Every result in Section 4 has been formulated for the case where the set of Fourier exponents has at least three elements and all of them are $\mathbb{Q}$-linearly independent. However, these results are also certain for other cases such as that where one of the Fourier exponents is $\lambda_1 = 0$ and the set of the remaining Fourier exponents has at least two elements and all of them are $\mathbb{Q}$-linearly independent. Indeed, if $f(s)$ is an almost periodic function whose Fourier exponents $\{0, \lambda_2, \lambda_3, \ldots\}$ satisfy these new conditions, then the function $g(s) = f(s)e^{\mu s}$, where μ is chosen so that it is not in the $\mathbb{Q}$-vector space generated by $\{\lambda_2, \lambda_3, \ldots\}$, has the same set of zeros as that of $f(s)$ and its Fourier exponents satisfy the conditions of the results of this section.

Finally, note that for the case where the set of Fourier exponents of an almost periodic function f has only two elements (and they are $\mathbb{Q}$-linearly independent) it is clear that the zeros of $f(s)$ are located on a vertical line, and consequently Theorem 4.4 is not satisfied in this case.

It would be interesting to study in more detail the more general case where the sets of Fourier exponents associated with an almost periodic function $f(s)$ are not necessarily linearly independent over the rational numbers. In particular, the consideration of other specific conditions on the Fourier exponents or coefficients could also lead to sets R_f without isolated points. In general, it is proposed the search of specific results on the possible gaps of the sets R_f and the reasons why they are produced.

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