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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Iterates of multidimensional approximation operators via Perov theorem

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ABSTRACT. The starting point is an approximation process consisting of linear and positive operators. The purpose of this note is to establish the limit of the iterates of some multidimensional approximation operators. The main tool is a Perov's result which represents a generalization of Banach fixed point theorem. In order to support the theoretical aspects, we present three applications targeting respectively the operators Bernstein, Cheney-Sharma and those of binomial type. The last class involves an incursion into umbral calculus.

1. INTRODUCTION

It is acknowledged that linear positive operators are a useful tool in approximation signals. Referring to classical operators of discrete or continuous type, a constant concern is to highlight their properties such as the rate of convergence for functions belonging to various spaces, preservation of the properties of functions that have been approximated, extensions in q-Calculus, replacement of the classical convergence with statistical convergence.

For a linear positive approximation process $(L_n)_{n\geq 1}$, a distinct direction of study is to investigate the convergence behavior of the sequence $(L_n^j)_{j\geq 1}$, where

$$L_n^1 f = L_n f, \quad L_n^j f = L_n(L_n^{j-1} f), \quad j > 1,$$

assuming that n is fixed and j does not depend on n.

In the following we consider that the space $C([a,b])$ is endowed with the Chebyshev norm $\|\cdot\|$, $\|f\| = \sup\limits_{x\in[a,b]} |f(x)|$.

Best of our knowledge, the first result was obtained in 1967 by Kelisky and Rivlin [11]. It targeted the well-known Bernstein operators

$$B_n : C([0,1]) \to C([0,1]), \quad (B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = \overline{0,n}.$$

The result established [11, *Eq. (2.4)*] is read as follows

$$\lim_{j\to\infty} (B_n^j f)(x) = f(0) + (f(1) - f(0))x, \quad 0 \leq x \leq 1, \tag{1.2}$$

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$f \in C([0,1])$), saying that $((B_n^j f)(x))_{j\ge1}$ converges uniformly to the line segment joining $(0, f(0))$ to $(1, f(1))$.

The above result was reobtained in 2004 by I.A. Rus [17] using another technique based on fixed point theory, more exactly on Banach contraction principle. The idea was taken over and developed in other papers, for example we mention [1], [2], [9].

The purpose of this note is to provide an approach for multidimensional operators. To achieve this, we rely on a specific generalization of the concept of metric space. Perov [13] used the notion of vector-valued metric space and obtained a Banach type fixed theorem on such a complete generalized metric space by using matrices instead of Lipschitz constants. Perov's result have been exploited in various works, see, e.g., [3], [7], [8], [12], [15].

2. Iterates of multidimensional linear positive operators

For a wider information of the reader and the accomplishment of an independent exposition, we will briefly present the method used for the study of the limit of iterations for one-dimensional operators via fixed point theory, as, for example, it appears in [1].

Set

$$X_{\alpha,\beta} := \{f \in C([a,b]) :\ f(a) = \alpha,\ f(b) = \beta\}, \quad (\alpha,\beta) \in \mathbb{R}\times\mathbb{R}. \tag{2.3}$$

Every $X_{\alpha,\beta}$ is a closed subset of $C([a,b])$ and, clearly, the system $X_{\alpha,\beta}$, $(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}$, constitutes a partition of this space. We define the linear positive operators $L_n : C([a,b]) \to C([a,b])$, $n \in \mathbb{N}$, that enjoy the following properties:

(i) they reproduce the Korovkin test function e_0 and e_1, where $e_0(x) = 1$, $e_1(x) = x$, $x \in [a,b]$;

(ii) each $L_n\Big|_{X_{\alpha,\beta}}$, $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$, is a contraction allowing us to say that $\lambda_n \in [0,1)$ exists such that

$$\|L_n f - L_n g\| \le \lambda_n \|f-g\|, \text{ where } f \text{ and } g \text{ belong to } X_{\alpha,\beta}; \tag{2.4}$$

(iii)

$$(L_n f)(a) = f(a) \text{ and } (L_n f)(b) = f(b). \tag{2.5}$$

Since L_n is linear, the first condition implies that it reproduces affine functions. Such operators are also called Markov type operators. Also, the last requirement indicates the interpolation property of the operators at the end of the domain. This condition guarantees that each $X_{\alpha,\beta}$ is an invariant subset of the operators L_n.

Considering the function $p^*_{\alpha,\beta}$ defined on $[a,b]$

$$p^*_{\alpha,\beta} = \alpha + \frac{\beta-\alpha}{b-a}(e_1 - a), \tag{2.6}$$

it is observed that $p^*_{\alpha,\beta} \in X_{\alpha,\beta}$. Since L_n reproduces the affine functions, $p^*_{\alpha,\beta}$ is a fixed point of L_n. For any f belonging to the space $C([a,b])$ one has $f \in X_{f(a),f(b)}$ and, by applying the contraction principle on $X_{f(a),f(b)}$ we deduce

$$\lim_{m\to\infty} L_n^m f = p^*_{f(a),f(b)},$$

uniformly on $[a,b]$.

With these preparations and using Perov's theorem we will extend the above result to the multidimensional case.

In the first step we consider a net on $[a,b]$ named $\Delta_n(a = x_{n,0} < x_{n,1} < \ldots < x_{n,n} = b)$ and a system $\phi_n(\varphi_{n,k})_{k=\overline{0,n}}$ where each function belongs to $C([a,b])$. Defining

$$(L_n f)(x) = \sum_{k=0}^{n} \varphi_{n,k}(x) f(x_{n,k}), \quad x \in [a,b], \tag{2.7}$$

we assume

$$\varphi_{n,k} \geq 0, \quad k = \overline{0,n}, \quad \sum_{k=0}^{n} \varphi_{n,k} = e_0, \quad \sum_{k=0}^{n} x_{n,k}\varphi_{n,k} = e_1,$$

i.e., our system ϕ_n is a blending one related to Δ_n, $n \in \mathbb{N}$. Also we suppose that conditions (2.4) and (2.5) are fulfilled. In order to construct a convex combination of p such operators, let

$$\gamma_{i,j} \in [0,1],\ 1 \leq i,j \leq p,\ \text{such that} \sum_{j=1}^{p} \gamma_{i,j} = 1. \tag{2.8}$$

In the second step, let $p \in \mathbb{N}$ and $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$. Define the operator

$$\mathbf{L} : X^p_{\alpha,\beta} \to X^p_{\alpha,\beta}, \quad \mathbf{L} = (\mathbf{L}_1, \ldots, \mathbf{L}_p), \tag{2.9}$$

$$\mathbf{L}_i(f_1, \ldots, f_p) = \sum_{j=1}^{p} \gamma_{i,j} L_{n_i}(f_j), \quad i = 1, \ldots, p, \tag{2.10}$$

where $X_{\alpha,\beta}$ is given by (2.3). The above combination uses p operators of the form (2.7) having different orders, namely n_i, $i = 1, \ldots, p$. Our main result will be read as follows.

Theorem 2.1. *For any vector-valued function $\mathbf{f} \in X^p_{\alpha,\beta}$, $\mathbf{f} = (f_1, \ldots, f_p)$, one has*

$$\lim_{k\to\infty} \mathbf{L}^k(\mathbf{f}) = \mathbf{f}_0, \tag{2.11}$$

where $\mathbf{L}$ is given by (2.9)-(2.10) and the components of the vector-valued function $\mathbf{f}_0$ are all equal with the affine function $p^*_{\alpha,\beta}$ defined by (2.6).

Proof. We will use the vector version of Banach contraction principle. We only need to show that $\mathbf{L}$ is a Perov contraction on the space $X^p_{\alpha,\beta}$. This involves proving that there is a quadratic matrix, say M, of size p which converges to zero (i.e. M^k tends to the null matrix as k tends to infinity) such that

$$\|\mathbf{L}(\mathbf{f}) - \mathbf{L}(\mathbf{g})\|_C \leq M\|\mathbf{f} - \mathbf{g}\|_C \tag{2.12}$$

for all $\mathbf{f}, \mathbf{g} \in X^p_{\alpha,\beta}$, where by $\|\cdot\|_C$ we mean the vector-valued norm given as follows

$$\|\mathbf{f}\|_C = \begin{bmatrix} \|f_1\| \\ \cdots \\ \|f_p\| \end{bmatrix},$$

$\|\cdot\|$ indicating the Chebyshev norm on $C([a,b])$.

According to our hypothesis, any L_n operator defined by (2.7) is a contraction on $X_{\alpha,\beta}$ with Lipschitz constant λ_n. Taking in view relation (2.10), for any $i \in \{1, \ldots, p\}$ we get

$$\|\mathbf{L}_i(\mathbf{f}) - \mathbf{L}_i(\mathbf{g})\|_C \leq \sum_{j=1}^{p} \gamma_{i,j}\lambda_{n_i}\|f_j - g_j\|. \tag{2.13}$$

Noting $\lambda := \max\{\lambda_{n_i} :\ 1 \leq i \leq p\}$, we have $\lambda \in [0,1)$. We define the matrix M as follows $M = \lambda\Gamma$, where $\Gamma = [\gamma_{i,j}]_{1\leq i,j\leq p}$. Clearly, relation (2.13) yields (2.12).

By induction we prove

$$\Gamma^k \le U, \tag{2.14}$$

where all entries of the matrix U are equal with 1. The key relation used is (2.8). For $k = 2$ we have

$$\sum_{j=1}^{p} \gamma_{i,j}\gamma_{j,k} \le \sum_{j=1}^{p} \gamma_{i,j} = 1 \text{ for all } 1 \le i, k \le p,$$

that is $\Gamma^2 \le U$. Further, assuming that $\Gamma^k \le U$ and observing that $\Gamma U = U$, we obtain that $\Gamma^{k+1} = \Gamma\Gamma^k \le \Gamma U = U$ and the induction is completed.

Based on definition of M and relation (2.14), we can write

$$M^k = \lambda^k \Gamma^k \le \lambda^k U,$$

which implies $\lim\limits_{k\to\infty} M^k = \Theta_p$, null matrix of the order p.

Our statement (2.11) follows from Perov theorem since $\mathbf{f}_0$ is the unique fixed point of the operator $\mathbf{L}$. □

3. Applications

Application 3.1. Let L_n be Bernstein operator B_n, see (1.1). It is known that B_n is a Markov type operator interpolating the functions at the extremities of the domain $[0, 1]$. Moreover, $B_n\Big|_{X_{\alpha,\beta}}$ is a contraction for all $\alpha, \beta \in \mathbb{R}$, where Lipschitz constant is

$$\lambda_n = 1 - 2^{1-n}.$$

Consequently, applying (2.11) we deduce

$$\lim_{k\to\infty} \mathbf{B}^k(\mathbf{f}) = \mathbf{f}_0,$$

where all components of the function $\mathbf{f}_0$ are given by $p^*_{\alpha,\beta} = f(0)e_0 + (f(1) - f(0))e_1$.

For $p = 1$ we reobtain the identity (1.2) which is the classical result of Kelisky and Rivlin [11, *Eq. (2.4)*].

Remark. At first we recall the notion of weakly Picard operator, see e.g., [21]. Let (X, d) be a metric space. An operator $U : X \to X$ is a weakly Picard operator (abbreviated WPO) if the sequence $(U^k(y))_{k\ge1}$ converges for all $y \in X$ and the limit (which may depend on y) is a fixed point of U. Set

$$U^\infty : X \to X, \quad U^\infty(y) = \lim_{k\to\infty} U^k(y), \quad y \in X.$$

By using this concept, Kelisky-Rivlin result can be reformulated as follows: for each $n \in \mathbb{N}$, B_n is WPO and $(B_n^\infty f) = f(0)e_0 + (f(1) - f(0))e_1$.

Recently, following the same route of WPO, in [4], the limit of iterates of modified Bernstein operators in Durrmeyer sense has been approached.

Application 3.2. In our attention is a generalization of Bernstein operators, the basis of its construction being a combinatorial identity of Jensen [10]

$$(x + y + n\gamma)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x(x + \nu\gamma)^{\nu-1}[y + (n - \nu)\gamma]^{n-\nu}. \tag{3.15}$$

The inception of its motivation is Lagrange's formula

$$\frac{u_1(z)}{1 - \dfrac{z u_2'(z)}{u_2(z)}} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{d^\nu}{dz^\nu}((u_2(z))^\nu u_1(z)) \left(\frac{z}{u_2(z)}\right)^\nu$$

and proceeds by setting $u_1(z) = e^{xz}$, $u_2(z) = e^{\gamma z}$. Choosing $y = 1 - x$ in (3.15), Cheney and Sharma [5] have investigated the operators

$$(Q_n f)(x) = \sum_{k=0}^{n} q_{n,k}(x;\gamma) f\left(\frac{k}{n}\right),\ f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}, \tag{3.16}$$

where γ is a non-negative parameter and

$$q_{n,k}(x;\gamma) = (1+n\gamma)^{1-n}\binom{n}{k} x(x+k\gamma)^{k-1}(1-x)(1-x+(n-k)\gamma)^{n-1-k}.$$

Obviously, the Bernstein polynomials represent a particular case of (3.16) obtained by setting $\gamma = 0$.

The authors proved that each Q_n preserves the constant functions. In [19] was shown that Q_n also reproduces the monomial e_1. It is easy to see that for all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$, $X_{\alpha,\beta}$ defined by (2.3) with $a = 0$, $b = 1$ is an invariant set of Q_n. Moreover, $Q_n\Big|_{X_{\alpha,\beta}}$ is a contraction. Indeed, if f_1 and f_2 belong to $X_{\alpha,\beta}$, then we get

$$\begin{aligned}|(Q_n f_1)(x) - (Q_n f_2)(x)| &\le (1 - q_{n,0}(x;\gamma) - q_{n,n}(x;\gamma)) \sup_{x\in[0,1]} |f_1(x) - f_2(x)| \\ &\le (1 - 2^{1-n}(1+n\gamma)^{1-n})\|f_1 - f_2\|,\ x \in [0,1].\end{aligned}$$

Examining (2.4), we can choose $\lambda_n = 1 - 2^{1-n}(1+n\gamma)^{1-n} < 1$ and all requirements for one-dimensional operators $L_n \equiv Q_n$ are met. Thus, we can form the multidimensional operators $\mathbf{L} \equiv \mathbf{Q}$ as in (2.9) and the relation (2.11) takes place ($a = 0$, $b = 1$).

Application 3.3. Here we have in mind operators constructed with the help of binomial polynomials. This involves a foray into umbral calculus, consequently at the beginning we will point out the basics. The first rigorous version of this calculus belongs to Gian-Carlo Rota and his collaborators, see, e.g., [16].

For any $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we denote by Π_n the linear space of polynomials of degree no greater than n and by Π_n^* the set of all polynomials of degree n.

A sequence $p = (p_n)_{n\ge 0}$ such that $p_n \in \Pi_n^*$ for every $n \in \mathbb{N}_0$ is called a polynomial sequence.

A polynomial sequence $b = (b_n)_{n\ge 0}$ is called of binomial type if for any $(x, y) \in \mathbb{R} \times \mathbb{R}$ the following equalities

$$b_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y),\ n \in \mathbb{N}_0, \tag{3.17}$$

hold. We get $b_0(x) = 1$ and by induction we obtain $b_n(0) = 0$ for any $n \in \mathbb{N}$. Set

$$\Pi := \bigcup_{n\ge 0} \Pi_n.$$

The space of all linear operators $T : \Pi \to \Pi$ will be denoted by $\mathcal{L}$. Among these operators an important role will be played by the shift operators, named E^a, defined by

$$(E^a p)(x) = p(x+a),\ p \in \Pi.$$

An operator $T \in \mathcal{L}$ which switches with all shift operators, that is $TE^a = E^a T$ for every $a \in \mathbb{R}$, is called a shift-invariant operator and the set of these operators are denoted by $\mathcal{L}_s$.

An operator Q is called a delta operator if $Q \in \mathcal{L}_s$ and Qe_1 is a nonzero constant. Let $\mathcal{L}_\delta$ denote the set of all delta operators. More generally, according to [16, *Proposition 2*] for every $Q \in \mathcal{L}_\delta$ we have

$$Q(\Pi_n^*) \subset \Pi_{n-1}^*,\ n \in \mathbb{N}.$$

A polynomial sequence $p = (p_n)_{n\geq 0}$ is called the sequence of basic polynomials associated to the delta operator Q if, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we get

(i) $p_0(x) = 1$,

(ii) $p_n(0) = 0$,

(iii) $(Qp_n)(x) = np_{n-1}(x)$.

It was proved [16, *Proposition 3*] that every delta operator has a unique sequence of basic polynomials.

For a good understanding, we collect below some results established in [16] on this topic.

Proposition 3.1. (a) *If $p = (p_n)_{n\geq 0}$ is a basic sequence for some delta operator Q, then it is a sequence of binomial type. Reciprocally, if p is a sequence of binomial type, then it is a basic sequence for some delta operator.*

(b) *Let $T \in \mathcal{L}_s$ and $Q \in \mathcal{L}_\delta$ with the basic sequence $p = (p_n)_{n\geq 0}$. One has*

$$T = \sum_{k\geq 0} \frac{(Tp_k)(0)}{k!} Q^k.$$

(c) *An isomorphism Ψ exists from the ring $(\mathcal{F}, +, \cdot)$ of the formal power series in the variable t over $\mathbb{R}$ field, onto $(\mathcal{L}_s, +, \cdot)$ such that*

$$\Psi(\phi(t)) = T,\ \text{where } \phi(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \text{ and } T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k. \tag{3.18}$$

(d) *An operator $P \in \mathcal{L}_s$ is a delta operator if and only if it corresponds under the isomorphism defined by (3.18), to a formal series $\phi(t)$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$.*

(e) *Let $Q \in \mathcal{L}_\delta$ with $p = (p_n)_{n\geq 0}$ its sequence of basic polynomials. Let $\phi(D) = Q$ and $\tau(t)$ be the inverse formal series of $\phi(t)$, where D indicates the derivative operator. Then one has*

$$\exp(x\tau(t)) = \sum_{n\geq 0} \frac{p_n(x)}{n!} t^n, \tag{3.19}$$

where $\tau(t)$ has the form $c_1 t + c_2 t^2 + \dots$ $(c_1 \neq 0)$.

At this moment we are ready to present a new class of operators. Let Q be a delta operator and $p = (p_n)_{n\geq 0}$ be its sequence of basic polynomials under the additional assumption $p_n(1) \neq 0$ for every $n \in \mathbb{N}$. We define $L_n^Q : C([0,1]) \to C([0,1])$ as follows

$$(L_n^Q f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right),\ n \in \mathbb{N}. \tag{3.20}$$

P. Sablonniere [18] called them Bernstein-Sheffer operators, but as D.D. Stancu and M.R. Occorsio motivated in [20] these operators can be named Popoviciu operators. These operators check all the requirements in order to be able to create the multidimensional operators $\mathbf{L}^Q$, see (2.9).

The operators L_n^Q, $n \in \mathbb{N}$, are linear and reproduce the constants. Indeed, choosing in (3.17) $y := 1 - x$, from (3.20) we obtain $L_n^Q e_0 = e_0$. The positivity of these operators are given by the sign of the coefficients of the series $\tau(t)$ defined at (3.19). Tiberiu Popoviciu [14] and later P. Sablonniere [18, *Theorem 1*] have established the following result.

Lemma 3.1. *L_n^Q defined by (3.20) is a positive operator on $C([0,1])$ for every $n \in \mathbb{N}$ if and only if*

$$c_1 > 0 \text{ and } c_n \geq 0 \text{ for all } n \geq 2. \tag{3.21}$$

In [18, *Theorem 2(i)*] it was shown that if (3.21) takes place, then $L_n^Q e_1 = e_1$.

Further, from the definition of basic polynomials we get $p_k(0) = \delta_{k,0}$, consequently $(L_n^Q f)(x_0) = f(x_0)$ for $x_0 = 0$ and $x_0 = 1$. It remains to check if $L_n^Q\Big|_{X_{\alpha,\beta}}$ is a contraction. Let f_1 and f_2 belong to $X_{\alpha,\beta}$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ are arbitrarily fixed. Using (3.20) we can write

$$\begin{aligned} |(L_n^Q f_1)(x) - (L_n^Q f_2)(x)| &\le \left(1 - \frac{p_n(1-x) + p_n(x)}{p_n(1)}\right) \sup_{x\in[0,1]} |(f_1 - f_2)(x)| \\ &\le \left(1 - \frac{1}{p_n(1)} \min_{x\in[0,1]} (p_n(1-x) + p_n(x))\right) \|f_1 - f_2\| \\ &:= \lambda_n \|f_1 - f_2\|. \end{aligned}$$

Due to positivity of L_n^Q operators, $n \in \mathbb{N}$, see relation (3.21) corroborated with (3.19), we have $0 \le \lambda_n < 1$. Thus, we deduce that $L_n^Q\Big|_{X_{\alpha,\beta}}$ is a contraction and

$$\lim_{m\to\infty} (L_n^Q)^m f = p^*_{f(0),f(1)}.$$

Considering that the additional condition (3.21) occurs, we can apply our result for multidimensional operators. Consequently, we state that (2.11) is valid for $\mathbf{L} \equiv \mathbf{L}^Q$.

Remark. We can show that a particular case of L_n^Q operators defined by (3.20) leads us to Cheney-Sharma operators discussed in Application 3.2. Let γ be a fixed non-negative parameter. We define $\widetilde{\gamma} = (a_n^{\langle\gamma\rangle})_{n\ge 0}$ the following sequence of polynomials

$$a_0^{\langle\gamma\rangle} = 1,\ a_n^{\langle\gamma\rangle}(x) = x(x+n\gamma)^{n-1},\ n \in \mathbb{N}.$$

It represents Abel sequence and verifies relation (3.17). Further, we consider Abel operator $A_\gamma := DE^\gamma$. For every $p \in \Pi$,

$$(A_\gamma p)(x) = \frac{dp}{dx}(x+\gamma).$$

In this case $\widetilde{\gamma}$ forms the sequence of basic polynomials associated to delta operator A_γ. Choosing in (3.20) $Q := A_\gamma$ one obtains Cheney-Sharma operator and $\mathbf{L} \equiv \mathbf{L}^{A_\gamma}$.

We end this application by indicating Crăciun's paper [6]. Here a more complex class of linear operators is built that mixes two sequences $p = (p_n)_{n\ge 0}$, $s = (s_n)_{n\ge 0}$, the first containing basic polynomials associated to a delta operator Q and the second is defined by the identity $s_n = S^{-1}p_n$, where S is an invertible shift invariant operator.

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