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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Eigenvalues of the (p, q, r)-Laplacian with a parametric boundary condition

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ABSTRACT. Consider in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$ the following nonlinear eigenvalue problem

$$\begin{cases} -\sum_{\alpha\in\{p,q,r\}} \rho_\alpha \Delta_\alpha u = \lambda a(x) \mid u \mid^{r-2} u \;\; \text{in } \Omega, \\ \left(\sum_{\alpha\in\{p,q,r\}} \rho_\alpha \mid \nabla u \mid^{\alpha-2}\right) \frac{\partial u}{\partial\nu} = \lambda b(x) \mid u \mid^{r-2} u \;\text{ on } \partial\Omega, \end{cases}$$

where $p, q, r \in (1, +\infty)$, $q < p$, $r \notin \{p, q\}$; ρ_p, ρ_q, ρ_r are positive constants; Δ_α is the usual α-Laplacian, i.e., $\Delta_\alpha u = \text{ div } (|\nabla u|^{\alpha-2}\nabla u)$; ν is the unit outward normal to $\partial\Omega$; $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying $\int_\Omega a\, dx + \int_{\partial\Omega} b\, d\sigma > 0$. Such a triple-phase problem is motivated by some models arising in mathematical physics.

If $r \notin (q, p)$, we determine a positive number λ_r such that the set of eigenvalues of the above problem is precisely $\{0\} \cup (\lambda_r, +\infty)$. On the other hand, in the complementary case $r \in (q, p)$ with $r < q(N-1)/(N-q)$ if $q < N$, we prove that there exist two positive constants $\lambda_* < \lambda^*$ such that any $\lambda \in \{0\} \cup [\lambda^*, \infty)$ is an eigenvalue of the above problem, while the set $(-\infty, 0) \cup (0, \lambda_*)$ contains no eigenvalue λ of the problem.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} \mathcal{A}u := -\Delta_p u - \Delta_q u - \Delta_r u = \lambda a(x) \mid u \mid^{r-2} u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} = \lambda b(x) \mid u \mid^{r-2} u \;\text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

under the following hypotheses

(h_{pqr}) $\;p, q, r \in (1, +\infty),\; q < p,\; r \notin \{p, q\}$;

(h_{ab}) $\;a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega a(x)\, dx + \int_{\partial\Omega} b(\sigma)\, d\sigma > 0. \tag{1.2}$$

In the boundary condition $(1.1)_2$, $\frac{\partial u}{\partial \nu_{\mathcal{A}}}$ denotes the conormal derivative corresponding to the differential operator $\mathcal{A}$, i.e.,

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} := \Big(\sum_{\alpha\in\{p,q,r\}} \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu},$$

where ν is the unit outward normal to $\partial\Omega$. As usual, for every $\alpha \in (1, \infty)$, we denote by Δ_α the α-Laplacian, i.e., $\Delta_\alpha u = \text{ div } (|\nabla u|^{\alpha-2}\nabla u)$.

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In fact, one can consider a more general eigenvalue problem, with

$$\mathcal{B}u := \rho_p \Delta_p u + \rho_q \Delta_q u + \rho_r \Delta_r u, \;\; \rho_p,\, \rho_q,\, \rho_r > 0,$$

instead of $\mathcal{A}$, and with

$$\frac{\partial u}{\partial \nu_{\mathcal{B}}} := \Big(\sum_{\alpha \in \{p,q,r\}} \rho_\alpha \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu}$$

instead of $\frac{\partial u}{\partial \nu_{\mathcal{A}}}$. However, for the sake of simplicity, we restrict our analysis to the case $\rho_p = \rho_q = \rho_r = 1$. For the general case we have similar results, as shown in Section 4 below.

Such a triple-phase eigenvalue problem is motivated by some models arising in mathematical physics. More exactly, let us consider the operator

$$Qu := -\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1- \mid \nabla u \mid^2}} \Big).$$

This operator appears in the electrostatic Born–Infeld equation (see [5]), in string theory, in particular in the study of D-branes (see, e.g., [11]), and in classical relativity, where Q represents the mean curvature operator in Lorentz–Minkowski space (see, e.g., [4] and [9]). A second order approximation of Q is $\mathcal{B} := -\triangle u - \triangle_4 u - \frac{3}{2}\triangle_6 u$ (see [12]), which is a negative $(2,4,6)$-Laplacian.

Under assumption (h_{pqr}), the appropriate Sobolev space for problem (1.1) is $W := W^{1,\max\{p,r\}}(\Omega)$. We seek the solutions u of problem (1.1) in the space W, so that the conormal derivative $\frac{\partial u}{\partial \nu_{\mathcal{A}}}$ exists in a trace sense. Using a Green type formula (see Casas-Fernández [8, p. 71]) one can define the eigenvalues of problem (1.1) as follows.

Definition 1.1. $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u_\lambda \in W \setminus \{0\}$ such that

$$\begin{aligned} &\int_\Omega \Big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2} + \mid \nabla u_\lambda \mid^{r-2} \Big) \nabla u_\lambda \cdot \nabla w \, dx \\ &\quad = \lambda \Big(\int_\Omega a \mid u_\lambda \mid^{r-2} u_\lambda w \, dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{r-2} u_\lambda w \, d\sigma \Big) \; \forall\, w \in W. \end{aligned} \tag{1.3}$$

Conversely, if λ is an eigenvalue then any eigenfunction $u \in W \setminus \{0\}$ corresponding to it satisfies problem (1.1) in the distribution sense.

We first note that no number $\lambda < 0$ can be an eigenvalue of problem(1.1). Indeed, choosing $w = u_\lambda$ in (1.3) we can see that the eigenvalues of problem (1.1) cannot be negative. It is also obvious that $\lambda_0 = 0$ is an eigenvalue of this problem and the corresponding eigenfunctions are the nonzero constant functions. So any other eigenvalue belongs to $(0, \infty)$.

If u is an eigenfunction corresponding to a positive eigenvalue λ then, by choosing $w \equiv 1$ in (1.3), we find that

$$\int_\Omega a \mid u_\lambda \mid^{r-2} u_\lambda \, dx + \int_{\partial\Omega} b \mid u_\lambda \mid^{r-2} u_\lambda \, d\sigma = 0. \tag{1.4}$$

This shows that all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

$$\mathcal{C} := \Big\{ u \in W; \; \int_\Omega a \mid u \mid^{r-2} u \, dx + \int_{\partial\Omega} b \mid u \mid^{r-2} u \, d\sigma = 0 \Big\}. \tag{1.5}$$

This set is a symmetric cone and using the Lebesgue Dominated Convergence Theorem (see also [7, Theorem 4.9]) we can see that $\mathcal{C}$ is also a weakly closed subset of W which contains non-zero elements (see [3, Section 2]).

Now, let us introduce the notations

$$K_\alpha(u) := \int_\Omega \mid \nabla u \mid^\alpha dx, \ \alpha \in \{p, q, r\},$$
$$k_r(u) := \int_\Omega a \mid u \mid^r dx + \int_{\partial\Omega} b \mid u \mid^r d\sigma \ \forall \, u \in W. \tag{1.6}$$

Remark 1.1. Obviously, any eigenfunction u_λ corresponding to an eigenvalue $\lambda > 0$ cannot be a constant function (see (1.3) with $v = u_\lambda$ and (1.2)). In addition, as $k_r(u_\lambda) > 0$, all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

$$\mathcal{C} \setminus \mathcal{Z}, \ \mathcal{Z} := \{v \in W; \ k_r(v) = 0\}.$$

In order to state our main results, let us define

$$\lambda_r := \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \frac{K_r(v)}{k_r(v)}, \tag{1.7}$$

$$\lambda_* := \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \left(\Gamma \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right),$$
$$\lambda^* := \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \left(\Gamma \frac{r}{p^{1-\gamma} q^\gamma} \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \frac{K_r(v)}{k_r(v)} \right), \tag{1.8}$$
$$\gamma := \frac{p-r}{p-q}, \ \Gamma := \frac{p-q}{(r-q)^{1-\gamma}(p-r)^\gamma}.$$

We can now state our main results

Theorem 1.1. *Assume that* (h_{pqr}) *and* (h_{ab}) *above are fulfilled. If* $r \notin (q, p)$, *then* $\lambda_r > 0$ *and the set of eigenvalues of problem* (1.1) *is precisely* $\{0\} \cup (\lambda_r, \infty)$, *where* λ_r *is the constant defined by* (1.7).

Theorem 1.2. *Assume that* (h_{pqr}) *and* (h_{ab}) *above are fulfilled,* $r \in (q, p)$, *and in addition* $r < q(N-1)/(N-q)$ *if* $q < N$. *Then* $0 < \lambda_* < \lambda^*$, *every* $\lambda \in \{0\} \cup [\lambda^*, \infty)$ *is an eigenvalue of problem* (1.1), *and for any* $\lambda \in (-\infty, \lambda_*) \setminus \{0\}$ *problem* (1.1) *has only the trivial solution.*

Moreover, the constants λ_*, λ^* *can be expressed as follows*

$$\lambda_* = \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \frac{K_p(v) + K_q(v) + K_r(v)}{k_r(v)}, \ \lambda^* = \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \frac{\frac{1}{p}K_p(v) + \frac{1}{q}K_q(v) + \frac{1}{r}K_r(v)}{\frac{1}{r}k_r(v)}. \tag{1.9}$$

Remark 1.2. Regarding the restriction $r < q(N-1)/(N-q)$ if $q < N$ in Theorem 1.2, we point out that this is directly related to the well-known compact embedding $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ which holds when $1 < r < q^*$, where

$$q^* = \begin{cases} \frac{qN}{N-q} & \text{if } 1 < q < N, \\ \infty & \text{if } q \geq N, \end{cases}$$

and the trace compact embedding $W^{1,q}(\Omega) \hookrightarrow L^r(\partial\Omega)$ if $1 < r < \widetilde{q}$ (see [1], [7, Section 9.3]), where

$$\widetilde{q} = \begin{cases} \frac{q(N-1)}{N-q} & \text{if } q < N, \\ \infty & \text{if } q \geq N. \end{cases}$$

We also note that if $1 < q < p < N$, then $q^* < p^*$ and $\widetilde{q} < \widetilde{p}$, $\widetilde{q} < q^*$.

If $b \equiv 0$ (i.e., the boundary condition is of Neumann type), Theorem 1.2 still holds if in the case $q < N$ the condition $r < q(N-1)/(N-q)$ is replaced by the weaker condition $r < qN/(N-q)$ since in this case we need only the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$.

2. Preliminary results

For $\alpha, \beta \in (1, \infty)$ and $\beta < \widetilde{\alpha}$ if $\alpha < N$, let $\mathcal{C}_{\alpha,\beta}$ be the following symmetric cone

$$\mathcal{C}_{\alpha,\beta} := \{u \in W^{1,\alpha}(\Omega);\ \int_\Omega a \mid u \mid^{\beta-2} u\, dx + \int_{\partial\Omega} b \mid u \mid^{\beta-2} u\, d\sigma = 0\}, \tag{2.10}$$

which is weakly closed in $W^{1,\alpha}(\Omega)$.

The following lemmas will be useful in the proof of our main results.

Lemma 2.1. *Assume that* (h_{ab}) *is fulfilled,*

$$\alpha, \beta \in (1, \infty) \text{ and } \beta < \widetilde{\alpha} \text{ if } \alpha < N. \tag{2.11}$$

Then the norm $\| u \|_{\alpha,\beta} := \| \nabla u \|_{L^\alpha(\Omega)} + \big(k_\beta(u)\big)^{\frac{1}{\beta}}$ $\forall\, u \in W^{1,\alpha}(\Omega)$ *is equivalent with the usual norm of the Sobolev space* $W^{1,\alpha}(\Omega)$.

Proof. This fact follows from [10, Proposition 3.9.55]. Indeed, $\big(k_\beta(u)\big)^{\frac{1}{\beta}}$ is a seminorm which satisfies the two requirements of that proposition, namely
(j) $\exists d_{\alpha,\beta} > 0$ such that $k_\beta(u)^{\frac{1}{\beta}} \le d_{\alpha,\beta} \| u \|_{W^{1,\alpha}(\Omega)}$ $\quad\forall u \in W^{1,\alpha}(\Omega)$ and
(jj) if $u =$ constant, then $k_\beta(u) = 0$ implies $u \equiv 0$. □

Lemma 2.2. *Assume that* (h_{ab}) *and* (2.11) *are fulfilled. Then, there exist a positive constant* L, *which depends on* α, β, N *and* Ω, *such that*

$$k_\beta(u)^{\frac{1}{\beta}} \le L K_\alpha(u)^{\frac{1}{\alpha}}\ \forall\, u \in \mathcal{C}_{\alpha,\beta}. \tag{2.12}$$

Proof. Assume the contrary: $\forall\, n \ge 1$ there exists $u_n \in \mathcal{C}_{\alpha,\beta}$ such that $k_\beta(u_n) = 1$ and

$$K_\alpha(u_n)^{\frac{1}{\alpha}} \le \frac{1}{n}. \tag{2.13}$$

By Lemma 2.1 and (2.13), $\big(u_n\big)_n$ is a bounded sequence in $W^{1,\alpha}(\Omega)$. Hence, after passing to a subsequence if necessary, we can assume that there exists $u_0 \in W^{1,\alpha}(\Omega)$ such that $u_n \rightharpoonup u_0$ in $W^{1,\alpha}(\Omega)$. Since $W^{1,\alpha}(\Omega)$ is compactly embedded in both $L^\beta(\Omega)$ and $L^\beta(\partial\Omega)$, we have $u_n \to u_0$ in $L^\beta(\Omega)$, $u_n \to u_0$ in $L^\beta(\partial\Omega)$. As $k_\beta(u_n) = 1\ \forall\, n \ge 1$ and $\big(u_n\big)_n \subset \mathcal{C}_{\alpha,\beta}$, we have $k_\beta(u_0) = 1$ and $u_0 \in \mathcal{C}_{\alpha,\beta}$. Now, it follows from (2.13) that $K_\alpha(u_n) \to 0$ as $n \to \infty$ in $L^\beta(\Omega)$, hence $\nabla u_0 \equiv 0$, so u_0 is a constant function. Since $u_0 \in \mathcal{C}_{\alpha,\beta}$ we have $u_0 \equiv 0$ which implies $k_\beta(u_0) = 0$, which contradicts $k_\beta(u_0) = 1$. Therefore, (2.12) holds true. □

For $\alpha \in (1, \infty)$ consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_\alpha u = \lambda a(x) \mid u \mid^{\alpha-2} u \text{ in } \Omega, \\ \mid \nabla u \mid^{\alpha-2} \frac{\partial u}{\partial \nu} = \lambda b(x) \mid u \mid^{\alpha-2} u \text{ on } \partial\Omega. \end{cases} \tag{2.14}$$

We recall the following result (see [2, Section 2]).

Lemma 2.3. *Assume that* (h_{ab}) *is fulfilled. Then,* $\widehat{\lambda}_\alpha := \inf_{v \in \mathcal{C}_{\alpha,\alpha} \setminus \mathcal{Z}} \frac{K_r(v)}{k_r(v)}$ *is the lowest positive eigenvalue of problem* (2.14).

3. Proof of the main results

Let us first state the following simple result.

Lemma 3.4. *Assume that* (h_{ab}), (h_{pqr}) *are fulfilled. Then* $\lambda = 0$ *is an eigenvalue of problem* (1.1), *and there is no negative eigenvalue of problem* (1.1).

Proof. We have seen in Section 1 that $\lambda = 0$ is an eigenvalue of problem (1.1) with the nonzero constant functions as the corresponding eigenfunctions. As has already been pointed out, any other possible eigenvalue of this problem necessarily belong to $(0, \infty)$. Indeed, let $\lambda \in \mathbb{R} \setminus \{0\}$ be an eigenvalue of problem (1.1) and let $u_\lambda \in W \setminus \{0\}$ be a corresponding eigenfunction. Choosing $w = u_\lambda$ in (1.3), we obtain

$$K_p(u_\lambda) + K_q(u_\lambda) + K_r(u_\lambda) = \lambda k_r(u_\lambda).$$

Since $k_r(u_\lambda) > 0$ (by Remark 1.1), it is obvious from the above equality that there is no negative eigenvalue of problem (1.1). $\square$

Now, for $\lambda > 0$ define the energy functional for problem (1.1), $\mathcal{J}_\lambda : W \to \mathbb{R}$,

$$\mathcal{J}_\lambda(u) = \frac{1}{p}K_p(u) + \frac{1}{q}K_q(u) + \frac{1}{r}K_r(u) - \frac{\lambda}{r}k_r(u)\ \forall\, u \in W. \tag{3.15}$$

This is a C^1 functional whose derivative is given by

$$\langle \mathcal{J}'_\lambda(u), w\rangle = \sum_{\alpha\in\{p,q,r\}} \frac{1}{\alpha}\langle K'_\alpha(u), w\rangle - \frac{\lambda}{r}\langle k'_r(u), w\rangle,$$

where

$$\langle K'_\alpha(u), w\rangle = \alpha \int_\Omega \mid \nabla u \mid^{\alpha-2} \nabla u \cdot \nabla w\, dx,\ \alpha \in \{p, q, r\},$$
$$\langle k'_r(u), w\rangle = r \int_\Omega a \mid u \mid^{r-2} uw\, dx + r \int_{\partial\Omega} b \mid u \mid^{r-2} uw\, d\sigma\ \forall u, w \in W.$$

Obviously, λ is an eigenvalue of problem (1.1) if and only if there exists a critical point $u_\lambda \in W \setminus \{0\}$ of $\mathcal{J}_\lambda$, i. e. $\mathcal{J}'_\lambda(u_\lambda) = 0$.

The following lemma shows that if $r < p$ the functional defined in (3.15), restricted to the subset $\mathcal{C} \subset W$, is coercive for every $\lambda > 0$.

Lemma 3.5. *Assume that (h_{ab}), (h_{pqr}) are fulfilled, and $r < p$. For every $\lambda > 0$, we have* $\lim_{\|u\|_W\to\infty, u\in\mathcal{C}} \mathcal{J}_\lambda(u) = \infty$.

Proof. By Lemma 2.2 there exists a positive constant L such that (2.12) holds for $\alpha = p, \beta = r$. Therefore,

$$k_r(u) \le L^r K_p(u)^{\frac{r}{p}}\ \forall\, u \in \mathcal{C}_{p,r} = \mathcal{C}. \tag{3.16}$$

On the other hand, from (3.15) and (3.16) we easily deduce that

$$\mathcal{J}_\lambda(u) \ge \frac{1}{p}K_p(u) - \frac{\lambda}{r}L^r K_p(u)^{\frac{r}{p}}\ \forall\, u \in \mathcal{C}. \tag{3.17}$$

Taking into account Lemma 2.1 (for $\alpha = p > \beta = r$) and (3.16), we can see that $\|u\|_W \to \infty,\ u \in \mathcal{C}$ if and only if $K_p(u) \to \infty$. As $r < p$, we derive from (3.17) that $\mathcal{J}_\lambda(u) \to \infty$ if $\|u\|_W \to \infty,\ u \in \mathcal{C}$, as claimed. $\square$

3.1. **Proof of Theorem 1.1.** The conclusion of Theorem 1.1 will follow from several lemmas which are all based on the assumptions specified in the statement of Theorem 1.1. These assumptions will no longer be explicitly mentioned in the statements of the next lemmas.

Lemma 3.6. *The constant λ_r defined by* (1.7) *is positive and is equal with the Rayleigh-type quotient associated to the eigenvalue problem* (1.1)

$$\lambda_r := \inf_{w\in\mathcal{C}\setminus\mathcal{Z}} \frac{\frac{1}{p}K_p(u_\lambda) + \frac{1}{q}K_q(u_\lambda) + \frac{1}{r}K_r(u_\lambda)}{\frac{1}{r}k_r(u_\lambda))}. \tag{3.18}$$

Moreover, for any $\lambda \in (0, \lambda_r]$, problem (1.1) *has only the trivial solution.*

Proof. First we check that $\lambda_r > 0$. Indeed, we have $\mathcal{C} = \mathcal{C}_{r,r}$ if $p < r$, and $\mathcal{C} = \mathcal{C}_{p,r}$ if $p > r$, which implies that $\mathcal{C} \setminus \mathcal{Z} \subseteq \mathcal{C}_{r,r} \setminus \{0\}$. Taking into account definition (1.7) and Lemma 2.3 for $\alpha = r$ we find that $\lambda_r \geq \widehat{\lambda}_r > 0$.

Next, let us prove (3.18). From (1.7), the inequality $\lambda_r \leq \widetilde{\lambda}_r$ is obvious. On the other hand, for each $v \in \mathcal{C} \setminus \mathcal{Z}$, $t > 0$, we have $tv \in \mathcal{C} \setminus \mathcal{Z}$ and

$$\widetilde{\lambda}_1 = \inf_{w \in \mathcal{C} \setminus \mathcal{Z}} \frac{\frac{1}{p} K_p(w) + \frac{1}{q} K_q(w) + \frac{1}{r} K_r(w)}{\frac{1}{r} k_r(w)} \leq \frac{\frac{rt^{p-r}}{p} K_p(v) + \frac{rt^{q-r}}{q} K_q(v) + K_r(v)}{k_r(v)}.$$

Now letting $t \to \infty$ if $r > p$, and $t \to 0_+$ if $r < q$, then passing to infimum for $v \in \mathcal{C} \setminus \mathcal{Z}$ we get the desired inequality. Hence λ_r can be expressed in two different ways (see (1.7) and (3.18)).

To complete the proof we need to show that no eigenvalue belongs to $(0, \lambda_r]$. Indeed, if there were an eigenvalue $\lambda \in (0, \lambda_r]$ with a corresponding eigenfunction $u_\lambda \in W \setminus \{0\}$, then from (1.3) we would have $K_p(u_\lambda) + K_q(u_\lambda) + K_r(u_\lambda) = \lambda k_r(u_\lambda)$. On the other hand, as $u_\lambda \in \mathcal{C} \setminus \mathcal{Z}$, we derive from (1.7) that

$$\lambda \leq \lambda_r \leq \frac{K_r(u_\lambda)}{k_r(u_\lambda)} = \frac{\lambda k_r(u_\lambda) - K_p(u_\lambda) - K_q(u_\lambda)}{k_r(u_\lambda)} = \lambda - \frac{K_p(u_\lambda) + K_q(u_\lambda)}{k_r(u_\lambda)} < \lambda.$$

which is obviously a contradiction. □

In what follows we shall prove that every $\lambda > \lambda_r$ is an eigenvalue of problem (1.1). We distinguish two cases which are complementary to each other.

Case 1: $r \in (1, q)$.

In this case $W = W^{1,p}(\Omega)$ and $\mathcal{C} = \mathcal{C}_{p,r}$ in the next lemma.

Lemma 3.7. *Every number $\lambda \in (\lambda_r, \infty)$ is an eigenvalue of problem* (1.1).

Proof. Let $\lambda > \lambda_r$ be an arbitrary but fixed number. We know from Lemma 3.5 that $\mathcal{J}_\lambda$ is coercive on $\mathcal{C}$. Moreover, $\mathcal{C}$ is a weakly closed subset of the reflexive Banach space W, and functional $\mathcal{J}_\lambda$ is weakly lower semicontinuous on $\mathcal{C}$ with respect to the norm of W. Thus, we can apply a standard result in the calculus of variations (see, e.g., Struwe [13, Theorem 1.2]) in order to obtain the existence of a global minimum point of $\mathcal{J}_\lambda$ over $\mathcal{C}$, say $u_* \in \mathcal{C}$ (which depends of λ), i.e., $\mathcal{J}_\lambda(u_*) = \min_{\mathcal{C}} \mathcal{J}_\lambda$.

Now, from (3.18), as $\lambda > \lambda_r$, we obtain that there exists $u_{0\lambda} \in \mathcal{C} \setminus \mathcal{Z}$ such that $\mathcal{J}_\lambda(u_{0\lambda}) < 0$. We have $\mathcal{J}_\lambda(u_*) \leq \mathcal{J}_\lambda(u_{0\lambda}) < 0$, which implies that $u_* \not\equiv 0$.

Next, we are going to show that the minimizer u_* for $\mathcal{J}_\lambda$ restricted to $\mathcal{C} \setminus \mathcal{Z}$ is a critical point of $\mathcal{J}_\lambda$ considered on the whole space W, i.e., $\mathcal{J}_\lambda'(u_*) = 0$, thus, u_* is an eigenfunction of problem (1.1) corresponding to λ.

In order to show this, we make use of an argument similar to that used in [2, Lemma 3]. Let $v \in \mathrm{Lip}(\Omega)$ be arbitrary but fixed. For all $n \in \mathbb{N}$ we define the C^1 convex function $\varphi_n : \mathbb{R} \to \mathbb{R}$,

$$\varphi_n(s) := k_r\Big(u_* + \frac{1}{n} v + s\Big) \; \forall\, s \in \mathbb{R}.$$

Note that φ_n is coercive, since we have

$$\varphi_n(s) \geq \frac{|\, s\,|^r}{2^{r-1}} \Big(\int_\Omega a \, dx + \int_{\partial\Omega} b \, d\sigma \Big) - k_r\Big(u_* + \frac{1}{n} v\Big).$$

Consequently, for all $n \in \mathbb{N}$, φ_n admits a minimizer s_n, such that $\varphi_n'(s_n) = 0$, i.e. $u_n \in \mathcal{C}$, where

$$u_n := u_* + \frac{1}{n}v + s_n \, \forall \, n \in \mathbb{N}. \tag{3.19}$$

In addition, the sequence $\big(ns_n\big)_n$ is bounded. Otherwise, up to a subsequence, $ns_n \to \infty$ or $ns_n \to -\infty$ as $n \to \infty$. Since $v \in \mathrm{Lip}(\Omega)$ there exists N_1 large enough such that we have either

$$v(\cdot) + ns_n > 0 \text{ in } \Omega, \text{ or } v(\cdot) + ns_n < 0 \text{ in } \Omega \, \forall \, n \geq N_1.$$

Since the function $\tau \mapsto | \, u_* + \tau \, |^{r-2} \, (u_* + \tau)$ is strictly increasing on $\mathbb{R}$, we get

$$\begin{aligned} 0 &= \int_\Omega a \mid u_n \mid^{r-2} u_n \, dx + \int_{\partial\Omega} b \mid u_n \mid^{r-2} u_n \, d\sigma \\ &> \int_\Omega a \mid u_* \mid^{r-2} u_* \, dx + \int_{\partial\Omega} b \mid u_* \mid^{r-2} u_* \, d\sigma = 0 \, \forall n \geq N_1, \end{aligned} \tag{3.20}$$

if $v(\cdot) + ns_n > 0$ in Ω, or the reverse inequality in the later case, when $v(\cdot) + ns_n < 0$ in Ω. In both cases we get a contradiction.

We point out that the inequality in (3.20) is strict. Indeed, (1.2) implies that either $|\{x \in \Omega; \; a(x) > 0\}|_N > 0$ or $a = 0$ a.e. in Ω and $|\{x \in \partial\Omega; \; b(x) > 0\}|_{N-1} > 0$, hence we cannot have equality above. Here $| \cdot |_N, \; | \cdot |_{N-1}$ denote the Lebesgue measures of the corresponding sets.

Therefore, $\big(ns_n\big)_n$ must be bounded. This implies that there exists $S \in \mathbb{R}$ such that, on a subsequence, $ns_n \to S$ as $n \to \infty$. So we have

$$n\big(u_n - u_*\big) \to v + S \text{ and } u_n \to u_* \text{ in } W \text{ as } n \to \infty. \tag{3.21}$$

Since $u_* \in \mathcal{C} \setminus \mathcal{Z}$, there exists $N_2 \in \mathbb{N}$ such that $\big(u_n\big)_n \subset \mathcal{C} \setminus \mathcal{Z} \; \forall \; n \geq N_2$. By using the minimality of u_* we obtain that

$$0 \leq \lim_{n\to\infty} n\big(\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)\big) \; \forall \; n \geq N_2. \tag{3.22}$$

On the other hand,

$$n\big(\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)\big) = \langle \mathcal{J}_\lambda'(u_*), n(u_n - u_*)\rangle + o(n; u_*, v), \tag{3.23}$$

where $o(n; u_*, v)$ is a notation for the term which tends to zero in the definition of the Fréchet deriative of $\mathcal{J}_\lambda$ at u_*, that is $o(n; u_*, v) \to 0$ as $n \to \infty$. It follows from (3.21)-(3.23) in combination with $u_* \in \mathcal{C}$ that

$$\begin{aligned} 0 \leq \lim_{n\to\infty} n\big(\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda(u_*)\big) &= \lim_{n\to\infty} \langle \mathcal{J}_\lambda'(u_*), n(u_n - u_*)\rangle + o(n; u_*, v) \\ = \langle \mathcal{J}_\lambda'(u_*), v + S\rangle &= \langle \mathcal{J}_\lambda'(u_*), v\rangle. \end{aligned}$$

A similar reasoning with $-v$ instead of v and the density of Lipschitz functions in W yield $\mathcal{J}_\lambda'(u_*) = 0$, which concludes the proof. □

Case 2: $r \in (p, \infty)$.

In this case $W = W^{1,r}(\Omega)$ and $\mathcal{C} = \mathcal{C}_{r,r}$. Let $\lambda > \lambda_r$ be an argitrary but fixed number. Under the present assumption, $r \in (p, \infty)$, the functional $\mathcal{J}_\lambda$ is not necessarily coercive on $\mathcal{C}$.

Choosing $w \equiv u_\lambda$ in (1.3), we see that any eigenfunction u_λ corresponding to λ satisfies $\langle J_\lambda'(u_\lambda), u_\lambda\rangle = 0$. We are going to show that J_λ has a nonzero critical point (which will be an eigenfunction corresponding to λ), therefore it is natural to investigate the restriction of J_λ to the Nehari type manifold (see [14]) defined by

$$\begin{aligned}\mathcal{N}_\lambda &= \{v \in \mathcal{C} \setminus \{0\}; \langle \mathcal{J}_\lambda'(v), v\rangle = 0\} \\ &= \{v \in \mathcal{C} \setminus \{0\}; K_p(v) + K_q(v) + K_r(v) = \lambda k_r(v)\}.\end{aligned}$$

Note that, by virtue of Remark 1.1, $\mathcal{N}_\lambda \subset \mathcal{C} \setminus \mathcal{Z}$. We shall prove that $\mathcal{J}_\lambda$ attains its minimum $m_\lambda := \inf_{w\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$ at some point $u_* \in \mathcal{N}_\lambda$ (which depends on λ) and $\mathcal{J}_\lambda'(u_*) = 0$.

First, we show that $\mathcal{N}_\lambda$ is non empty. Indeed, since $\lambda > \lambda_r$, we deduce from (1.7) that there exists $u_0 \in \mathcal{C} \setminus \mathcal{Z}$ such that $K_r(u_0) < \lambda k_r(u_0)$. The condition $tu_0 \in \mathcal{N}_\lambda$, $t > 0$, reads

$$h(t; u_0) := t^{p-r} K_r(u_0) + t^{q-r} K_q(u_0) + K_r(u_0) - \lambda k_r(u_0) = 0.$$

Obviously, the map defined by $t \mapsto h_\lambda(t; \cdot)$ is continuous on $(0, \infty)$, satisfies

$$h_\lambda(t; u_0) \to K_r(u_0) - \lambda k_r(u_0) < 0 \text{ as } t \to \infty, \; h_\lambda(t; u_0) \to \infty \text{ as } t \to 0_+,$$

so there exists $t_0 \in (0, \infty)$ such that $h_\lambda(t_0; u_0) = 0$. Hence, for this t_0 we have $t_0 u_0 \in \mathcal{N}_\lambda$.

Lemma 3.8. *There exists a point $u_* \in \mathcal{N}_\lambda$ where $\mathcal{J}_\lambda$ attains its minimal value, $m_\lambda := \inf_{w\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(w) >$ 0.*

Proof. Note that on $\mathcal{N}_\lambda$ functional $\mathcal{J}_\lambda$ has the form

$$\mathcal{J}_\lambda(u) = \frac{r-p}{pr} K_p(u) + \frac{r-q}{qr} K_q(u) > 0, \tag{3.24}$$

therefore $m_\lambda \geq 0$.

First, let us check that every minimizing sequence $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ for $\mathcal{J}_\lambda$ is bounded in W. Since $u_n \in \mathcal{N}_\lambda$ for all n, taking into account (3.24), we obtain

$$\mathcal{J}_\lambda(u_n) = \frac{r-p}{pr} K_p(u_n) + \frac{r-q}{qr} K_q(u_n) \to m_\lambda, \text{ as } n \to \infty. \tag{3.25}$$

Also, we have

$$0 < \lambda k_r(u_n) - K_r(u_n) = K_p(u_n) + K_q(u_n). \tag{3.26}$$

Assume by contradiction that $\big(u_n\big)_n$ is unbounded in W. Then, after passing to a subsequence if necessary, we have $\|u_n\|_{r,p} \to \infty$ (see Lemma 2.1). Now, from (3.25) we obtain that the sequences $\big(K_p(u_n)\big)_n$ and $\big(K_q(u_n)\big)_n$ are bounded and, taking into account (3.26), $\delta_n := k_r(u_n)^{\frac{1}{r}} \to \infty$ as $n \to \infty$.

Set $v_n = u_n/\delta_n$, $n \in \mathbb{N}$. From (3.26) we have $K_r(v_n) \leq \lambda$ for all n, therefore $\big(v_n\big)_n$ is bounded in W. Thus there exists a $v_0 \in W$ such that $v_n \rightharpoonup v_0$ in W (also in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, by continuous inclusions) and $v_n \to v_0$ in boyh $L^r(\Omega)$ and $L^r(\partial\Omega)$. As $\mathcal{C}$ is weakly closed in W and $\big(v_n\big)_n \subset \mathcal{C}$ we obtain that $v_0 \in \mathcal{C}$. Now, dividing (3.26) by $k_r(u_n)$, we get

$$\lambda - K_r(v_n) = k_r(u_n)^{\frac{r}{p}} K_p(v_n) + k_r(u_n)^{\frac{r}{p}} K_p(v_n) \; \forall\, n.$$

As $k_r(u_n) \to \infty$, $r > p$, and the left hand side term is bounded, we obtain that $K_p(v_n) \to 0$, and so

$$\int_\Omega |\nabla v_0|^p \; dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \; dx = 0.$$

Therefore v_0 is a constant function. In fact, $v_0 \equiv 0$ since $v_0 \in \mathcal{C}$. On the other hand, $1 = k_r(v_n) \to k_r(v_0) = 0$ as $n \to \infty$, which is a contradiction

Next, we show that $m_\lambda > 0$. Suppose the contrary, that $m_\lambda = 0$ and let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $\mathcal{J}_\lambda$, i.e. $\mathcal{J}_\lambda(u_n) \to 0$ as $n \to \infty$. We have already proved that

$\big(u_n\big)_n$ is bounded in W, thus $u_n \rightharpoonup u_0$ (on a subsequence if necessary) for some $u_0 \in W$ (also in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$), and $u_n \to u_0$ in $L^r(\Omega)$ and also in $L^r(\partial\Omega)$.

From (3.25) we obtain that $K_p(u_n) \to 0$ as $n \to \infty$, thus u_0 is a constant function. Since $u_0 \in \mathcal{C}$, we get $u_0 \equiv 0$, hence $k_r(u_n) \to 0$. As $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ we have $\delta_n := k_r(u_n)^{\frac{1}{r}} \neq 0$ so we can define as before, $v_n = u_n/\delta_n$ for all n. We can conclude that $\big(v_n\big)_n$ is bounded in W and there exists $v_0 \in \mathcal{C}$ such that, on a subsequence, $v_n \rightharpoonup v_0$ in W and $v_n \to v_0$ in $L^r(\Omega)$ as well as in $L^r(\partial\Omega)$, hence $k_r(v_0) = 1$. Dividing (3.26) by δ_n^q and taking into account that $q < p < r,\ \delta_r(u_n) \to 0$, we find

$$\delta_n^{r-q}\big(K_r(v_n) - \lambda\big) - \delta_n^{p-q} K_p(v_n) = K_q(v_n) \to 0\,.$$

Next, since $v_n \rightharpoonup v_0$ in W (also in $W^{1,q}(\Omega)$ and $W^{1,p}(\Omega)$), we infer that

$$\int_\Omega |\nabla v_0|^q dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^q \ dx = 0.$$

Therefore v_0 is a constant function and in fact $v_0 \equiv 0$ since $v_0 \in \mathcal{C}$. This contradicts $k_r(v_0) = 1$.

To complete the proof, we need to show that $\mathcal{J}_\lambda(u_*) = m_\lambda$ for some $u_* \in \mathcal{N}_\lambda$. Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence, i.e. $\mathcal{J}_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. In particular, as we have proved that the sequence $\big(u_n\big)_n$ is bounded in W, $\big(u_n\big)_n$ converges weakly in W to some $u_* \in W$ and strongly in both $L^r(\Omega)$ and $L^r(\partial\Omega)$. We have

$$\mathcal{J}_\lambda(u_*) \leq \liminf_{n\to\infty} \mathcal{J}_\lambda(u_n) = m_\lambda. \tag{3.27}$$

Since $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ we have

$$K_p(u_n) + K_q(u_n) + K_r(u_n) = \lambda k_r(u_n)\ \forall\, n \in \mathbb{N}. \tag{3.28}$$

It is easily seen that u_* is not the null function. Indeed, assuming that $u_* \equiv 0$, we infer by (3.28) that $\big(u_n\big)_n$ converges strongly to 0 in W, hence also in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$. Then (3.25) will give $m_\lambda = 0$ which is a contradiction. Thus, $u_* \in \mathcal{C} \setminus \{0\}$.

Letting $n \to \infty$ in (3.28) yields

$$K_p(u_*) + K_q(u_*) + K_r(u_*) \leq \lambda k_r(u_*). \tag{3.29}$$

If (3.29) holds with equality then $u_* \in \mathcal{N}_\lambda$ and we are done (cf. (3.27)). If we assume that strict inequality holds in (3.29), we have $t_0 u_* \in \mathcal{N}_\lambda$ for some $t_0 \in (0,1)$. Indeed, if we define $j : (0,\infty) \to \mathbb{R}$,

$$j(t) := t^r\Big(t^{p-r}K_p(u_*) + t^{q-r}K_q(u_*) + K_r(u_*) - \lambda k_r(u_*)\Big)$$

we have $j(1) < 0$ (from the strict inequality (3.29)) and $t^{-r}j(t) \to \infty$ as $t \to 0_+$. Therefore, there exists $t_0 \in (0,1)$ such that $j(t_0) = 0$, which implies $t_0 u_* \in \mathcal{N}_\lambda$.

Next, using (3.24), we get

$$\mathcal{J}_\lambda(t_0 u_*) = \frac{t_0^p(r-p)}{pr} K_p(u_*) + \frac{t_0^q(r-q)}{qr} K_q(u_*). \tag{3.30}$$

Therefore

$$\begin{aligned} 0 < m_\lambda \leq \mathcal{J}_\lambda(t_0 u_*) &= \frac{t_0^p(r-p)}{pr} K_p(u_*) + \frac{t_0^q(r-q)}{qr} K_q(u_*) \\ &< \frac{r-p}{pr} K_p(u_*) + \frac{r-q}{qr} K_q(u_*) \\ &\leq \frac{r-p}{pr} \liminf_{n\to\infty} K_p(u_n) + \frac{r-q}{qr} \liminf_{n\to\infty} K_q(u_n) \leq \liminf_{n\to\infty} \mathcal{J}_\lambda(u_n) = m_\lambda, \end{aligned}$$

which is impossible. □

The next result states that the minimizer u_*, given by Lemma 3.8, is a critical point of $\mathcal{J}_\lambda$ considered on the whole space W.

Lemma 3.9. *The minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 3.8 is an eigenfunction of problem* (1.1) *with corresponding eigenvalue λ.*

Proof. It suffices to prove that $\mathcal{J}_\lambda'(u_*) = 0$. Let $v \in \mathrm{Lip}(\Omega)$ be an arbitrary but fixed function and let $u_* \in \mathcal{N}_\lambda$ be the minimizer of $\mathcal{J}_\lambda$ over $\mathcal{N}_\lambda$. Now, using arguments similar to thouse in Lemma 3.7, we are able to obtain a sequence $\big(u_n\big)_n \subset \mathcal{C} \setminus \mathcal{Z}$,

$$u_n := u_* + \frac{1}{n}v + s_n \; \forall\, n \geq 1. \tag{3.31}$$

The sequence $\big(ns_n\big)_n$ is also bounded, so it converges, on a subsequence, to some $S \in \mathbb{R}$. Therefore, we have

$$n\big(u_n - u_*\big) \to v + S,\; u_n \to u_* \text{ in } W \text{ as } n \to \infty. \tag{3.32}$$

Since $u_* \in \mathcal{N}_\lambda$, we have $K_p(u_*) + K_q(u_*) + K_r(u_*) = \lambda k_r(u_*)$, thus $K_r(u_*) - \lambda k_r(u_*) < 0$. Also, $k_r(u_n) \to k_r(u_*) > 0$, thus (on a subsequence if necessary) we can assume that

$$K_r(u_n) - \lambda k_r(u_n) < 0,\; k_r(u_n) > 0 \; \forall\, n \geq 1. \tag{3.33}$$

Using the sequence $\big(u_n\big)_n$, we shall construct a sequence $\big(t_n\big)_n \subset \mathbb{R} \setminus \{0\}$ such that $\big(t_n u_n\big)_n \subset \mathcal{N}_\lambda$, i.e.,

$$t_n^{p-r} K_p(u_n) + t_n^{q-r} K_q(u_n) + K_r(u_n) = \lambda k_r(u_n). \tag{3.34}$$

Define

$$h_n : (0,\infty) \to \mathbb{R},\; h_n(t) := t^{p-r} K_p(u_n) + t^{q-r} K_q(u_n) + K_r(u_n) - \lambda k_r(u_n).$$

Obviously, $h_n(t) \to \infty$ as $t \to 0+$ and $h_n(t) \to K_r(u_n) - \lambda k_r(u_n) < 0$ (see (3.33)). So there exists $t_n > 0$ such that $h_n(t_n) = 0 \; \forall\, n \geq 1$, hence (3.34) holds, as claimed.

In what follows we shall prove that the sequence $\big(n(t_n - 1)\big)_n$ is bounded. To this purpose, we rewrite (3.34) in the equivalent form

$$\begin{aligned} &n\big(t_n^{p-r} - 1\big)K_p(u_n) + n\big(t_n^{q-r} - 1\big)K_q(u_n) \\ &\quad = n\big(\lambda k_r(u_n) - K_p(u_n) - K_q(u_n) - K_r(u_n)\big). \end{aligned} \tag{3.35}$$

We shall prove first that the sequence $\big(n(\lambda k_r(u_n) - K_p(u_n) - K_q(u_n) - K_r(u_n))\big)_n$ is convergent. To this purpose, let us define the C^1 functional

$$\mathcal{K}_\lambda : W \to \mathbb{R},\; \mathcal{K}_\lambda(u) = \lambda k_r(u) - K_p(u) - K_q(u) - K_r(u) \; \forall\, u \in W. \tag{3.36}$$

For all $u, w \in W$

$$\langle \mathcal{K}_\lambda'(u), w\rangle = -\langle K_p'(u), w\rangle - \langle K_q'(u), w\rangle - \langle K_r'(u), w\rangle + \lambda\langle k_r'(u), w\rangle. \tag{3.37}$$

Since $u_* \in \mathcal{N}_\lambda$, we infer that $\mathcal{K}_\lambda(u_*) = 0$ and taking into account (3.36) we get

$$n\big(\lambda k_r(u_n) - K_p(u_n) - K_q(u_n) - K_r(u_n)\big) = n\big(\mathcal{K}_\lambda(u_n) - \mathcal{K}_\lambda(u_*)\big). \tag{3.38}$$

We have

$$n\big(\mathcal{K}_\lambda(u_n) - \mathcal{K}_\lambda(u_*)\big) \to \langle \mathcal{K}_\lambda'(u_*), v + S\rangle \text{ as } n \to \infty. \tag{3.39}$$

From (3.38) and (3.39) we deduce that the sequence $\big(n(\lambda k_r(u_n) - K_p(u_n) - K_q(u_n) - K_r(u_n)\big)_n$ has a finite limit.

Returning to (3.35), we observe that $\big(K_p(u_n)\big)_n$, $\big(K_q(u_n)\big)_n$, $\big(K_r(u_n)\big)_n$ are bounded sequences with positive limits. If we assume the contrary, that the sequence $\big(n(t_n^{p-r} - 1)\big)_n$

has an unbounded subsequence converging, e.g., to $+\infty$, then the corresponding subsequence of $\big(n(t_n^{q-r}-1)\big)_n$ will have positive terms (since $q-r<0$ and $p-r<0$), so the sequence defined by the left hand side of (3.35) will be unbounded, thus contradicting the fact that the right hand side defines a convergent sequence. A similar argument works in the case of a subsequence converging to $-\infty$. Therefore, $\big(n(t_n^{p-r}-1)\big)_n$ is a bounded sequence. Hence, there is $M>0$ such that for all $n\ge 1$, $n\mid t_n^{p-r}-1\mid\le M$, which implies $1-M/n\le t_n^{p-r}\le 1+M/n\ \forall\, n\ge 1$. Since, there exists $N_1\in\mathbb{N}$ such that $1-M/n>0\ \forall\, n\ge N_1$, we have

$$n\Big(\big(1+M/n\big)^{\frac{1}{p-r}}-1\Big)\le n(t_n-1)\le n\Big(\big(1-M/n\big)^{\frac{1}{p-r}}-1\Big)\ \forall\, n\ge N_1. \tag{3.40}$$

Taking into account the relations

$$\lim_{x\to 0}\frac{(1+Mx)^{1/(p-r)}-1}{x}=M/(p-r),\ \lim_{x\to 0}\frac{(1-Mx)^{1/(p-r)}-1}{x}=-M/(p-r),$$

we infer from (3.40) that the sequence $\big(n(t_n-1)\big)_n$ is bounded, thus, by possibly passing to a subsequence, there exists $T\in\mathbb{R}$, such that $n(t_n-1)\to T$ as $n\to\infty$. We define

$$z_n:=t_n\Big(u_*+\frac{1}{n}v+s_n\Big)=t_nu_n\ \forall\, n\ge N_1, \tag{3.41}$$

with $\big(z_n\big)_n\subset\mathcal{N}_\lambda$. In addition, as $\big(n(t_n-1)\big)_n$ is a bounded sequence, we can see that

$$t_n\to 1 \text{ in } \mathbb{R},\ z_n\to u_* \text{ in } W \text{ as } n\to\infty. \tag{3.42}$$

By using the minimality of u_* and the fact that $\big(z_n\big)_n\subset\mathcal{N}_\lambda$ we obtain that

$$0\le\lim_{n\to\infty}n\big(\mathcal{J}_\lambda(z_n)-\mathcal{J}_\lambda(u_*)\big). \tag{3.43}$$

Since functional $\mathcal{J}_\lambda\in C^1(W;\mathbb{R})$, we can write

$$n\big(\mathcal{J}_\lambda(z_n)-\mathcal{J}_\lambda(u_*)\big)=\big(\langle\mathcal{J}_\lambda'(u_*),n(z_n-u_*)\rangle+o(n;u_*,v), \tag{3.44}$$

with $o(n;u_*,v)\to 0$ as $n\to\infty$. Taking into account (3.41) and (3.42), we can see that

$$n(z_n-u_*)=n\big(t_n-1\big)u_*+v+ns_n\to Tu_*+v+S \text{ as } n\to\infty \text{ in } W. \tag{3.45}$$

It follows from (3.43) and (3.45) that

$$0\le\langle\mathcal{J}_\lambda'(u_*),v+S+Tu_*\rangle. \tag{3.46}$$

Since $u_*\in\mathcal{N}_\lambda$, we obtain that $\langle\mathcal{J}_\lambda'(u_*),u_*\rangle=0,\ \langle\mathcal{J}_\lambda'(u_*),S\rangle=0$, hence (3.46) implies $0\le\langle\mathcal{J}_\lambda'(u_*),v\rangle$. A similar reasoning with $-v$ instead of v shows that the converse inequality holds, hence $0=\langle\mathcal{J}_\lambda'(u_*),v\rangle$. Finally, using the density of Lipschitz functions in W we obtain that $\mathcal{J}_\lambda'(u_*)=0$, which concludes the proof. □

Finally, the conclusion of Theorem 1.1 follows from Lemmas 3.4, 3.6, 3.7 and 3.9.

3.2. **Proof of Theorem 2.** The conclusions of Theorem 1.2 will follow from Lemma 3.4 and the next several lemmas which rely on the assumptions specified in the statement of Theorem 1.2. These assumptions will not be explicitly mentioned again in the statements of the next lemmas.

In the pesent case $W=W^{1,p}(\Omega)$ and $\mathcal{C}=\mathcal{C}_{p,r}$. Note that, under the present assumptions, including $r\in(q,p)$, functional $\mathcal{J}_\lambda$, is coercive on $\mathcal{C}$ (see Lemma 3.5). The main difficulty is to show that the global minimizer of $\mathcal{J}_\lambda$ over $\mathcal{C}$ is not zero.

First, let us check that $0<\lambda_*<\lambda^*$.

Lemma 3.10. *The constants $\lambda_*,\ \lambda^*$ defined by $(1.8)_{1,2}$ are positive and $\lambda_*<\lambda^*$.*

Proof. From $(1.8)_1$, taking into account that $r < p$ and Lemma 2.3, we infer that

$$\lambda_* \geq \inf_{v\in\mathcal{C}\setminus\mathcal{Z}} \frac{K_r(v)}{k_r(v)} \geq \inf_{v\in\mathcal{C}_{r,r}\setminus\mathcal{Z}} \frac{K_r(v)}{k_r(v)} = \widehat{\lambda}_r > 0. \tag{3.47}$$

It remains to prove that $\lambda_* < \lambda^*$. In order to show this inequality, taking into account $(1.8)_{1,2}$, it is enough to check that $r/\big(p^{1-\gamma}q^{\gamma}\big) > 1 \ \Leftrightarrow\ r^{p-q} > p^{r-q}q^{p-r}$, which can be rewritten as

$$\Big(1+\frac{p-q}{q}\Big)^{\frac{q}{p-q}} < \Big(1+\frac{r-q}{q}\Big)^{\frac{q}{r-q}}.$$

So the desired inequality follows, since the function $x \to (1+x)^{\frac{1}{x}}$ is decreasing on $(0,\infty)$ and $p-q > r-q$. $\square$

The next result shows that the equalities (1.9) from Theorem 1.2 hold true.

Lemma 3.11. *The constants λ_* and λ^* defined in* $(1.8)_{1,2}$ *can be equivalently expressed by* (1.9).

Proof. First we show the equalities

$$\begin{aligned} \lambda_* &= \inf_{v\in\mathcal{C}\setminus\mathcal{Z}} \inf_{t>0} \left(\frac{K_p(tv)+K_q(tv)+K_r(tv)}{k_r(tv)}\right),\\ \lambda^* &= \inf_{v\in\mathcal{C}\setminus\mathcal{Z}} \inf_{t>0} \left(\frac{\frac{r}{p}K_p(tv)+\frac{r}{q}K_q(tv)+K_r(tv)}{k_r(tv)}\right). \end{aligned} \tag{3.48}$$

Define for all $v\in\mathcal{C}\setminus\mathcal{Z}$,

$$R_*(v) := \frac{K_p(v)+K_q(v)+K_r(v)}{k_r(v)},\ R^*(v) := \frac{\frac{r}{p}K_p(v)+\frac{r}{q}K_q(v)+K_r(v)}{k_r(v)}, \tag{3.49}$$

$$\begin{aligned} g_v(t) &:= R_*(tv) = \frac{t^{p-r}K_p(v)+t^{q-r}K_q(v)+K_r(v)}{k_r(v)},\\ h_v(t) &:= R^*(tv) = \frac{\frac{r}{p}t^{p-r}K_p(v)+\frac{r}{q}t^{q-r}K_q(v)+K_r(v)}{k_r(v)}\ \forall\, t>0. \end{aligned} \tag{3.50}$$

It is easy to check that the function g_v achieves its minimal value $\lambda(v) > 0$ on $(0,\infty)$ for $t = t(v) > 0$, where

$$\lambda(v) = \Gamma\frac{K_p(v)^{1-\gamma}K_q(v)^{\gamma}}{k_r(v)} + \frac{K_r(v)}{k_r(v)},\ t(v) = \Big(\frac{(r-q)K_q(v)}{(p-r)K_p(v)}\Big)^{\frac{1}{p-q}}. \tag{3.51}$$

But then,

$$\lambda(v) = R_*\big(t(v)v\big) = \inf_{t>0}\frac{K_p(tv)+K_q(tv)+K_r(tv)}{k_r(tv)}$$

and, taking the infimum over all $v\in\mathcal{C}\setminus\mathcal{Z}$, we obtain the first equality in (3.48).

Define

$$\mu_* := \inf_{v\in\mathcal{C}\setminus\mathcal{Z}} R_*(v),\ \mu^* := \inf_{v\in\mathcal{C}\setminus\mathcal{Z}} R^*(v).$$

According to the definition of μ_* and $(3.48)_1$, we have $\mu_* \geq \lambda_*$. Let us prove the converse inequality. Pick an arbitrary $v\in\mathcal{C}\setminus\mathcal{Z}$. As $t(v)v\in\mathcal{C}\setminus\mathcal{Z}$, we derive that $\lambda(v)\geq\mu_*$, and taking the infimum over all $v\in\mathcal{C}\setminus\mathcal{Z}$, we infer that $\mu_*\leq\lambda_*$, therefore the desired equality holds true. By a similar reasoning, as the h_v achieves its minimal value $\widehat{\lambda}(v) = r/\big(p^{1-\gamma)q^{\gamma}}\big)\lambda(v) > 0$ on $(0,\infty)$ for $t=\widehat{t}(v) = (p/q)^{1/(p-q)}t(v) > 0$, we first get the second equality in (3.48) and then $\lambda^* = \mu^*$. $\square$

Lemma 3.12. *There exists $u^*\in\mathcal{C}\setminus\mathcal{Z}$ such that $\lambda^* = R^*(u^*)$. In addition, u^* is an eigenfunction of problem* (1.1) *corresponding to the eigenvalue λ^* and $J_{\lambda^*}(u^*) = 0$.*

Proof. First, we check that $\lambda^* = R^*(u^*)$ for some $u^* \in \mathcal{C} \setminus \mathcal{Z}$. Let $\big(u_n\big)_n \subset \mathcal{C} \setminus \mathcal{Z}$ be a minimizing sequence for λ^*, that is

$$\frac{\frac{r}{p}K_p(u_n) + \frac{r}{q}K_q(u_n) + K_r(u_n)}{k_r(u_n)} \to \lambda^* \text{ as } n \to \infty. \tag{3.52}$$

In particular, from (3.52) we obtain that the sequence

$$\Big(\frac{K_p(u_n)}{k_r(u_n)}\Big)_n \text{ is bounded.} \tag{3.53}$$

Since $r < p < \widetilde{p}$, we get from Lemma 3.10, that there exists a positive constant L such that $k_r(u_n) \le L^r K_p(u_n)^{r/p}$ for all $n \ge 1$, which implies

$$\frac{K_p(u_n)}{L^r K_p(u_n)^{r/p}} \le \frac{K_p(u_n)}{k_r(u_n)} \;\; \forall\, n \ge 1.$$

This inequality combined with (3.53) and the assumption $r < p$ shows that the sequence $\big(K_p(u_n)\big)_n$ is bounded. So, taking into account Lemma 3.10 with $\alpha = \beta = p$, we infer that the sequence $\big(u_n\big)_n$ is bounded in W. Hence, there exists $u_* \in W$ such that, on a subsequence, $u_n \rightharpoonup u_*$ in W, $k_r(u_n) \to k_r(u_*)$ as $n \to \infty$. Obviously, $u_* \in \mathcal{C}$.
Next, we check that $k_r(u_*) \neq 0$ which will imply that $u_* \neq 0$. Let us assume, by way of contradiction, that $k_r(u_n) \to k_r(u_*) = 0$. Define the sequence $v_n := u_n/k_r(u_n)^{1/r}$ $\forall\, n \ge 1$. From (3.52) we obtain that

$$\Big(\frac{K_q(u_n)}{k_r(u_n)}\Big)_n \text{ is bounded.} \tag{3.54}$$

Taking into account the relation $K_q(u_n)/k_r(u_n) = k_r^{\frac{q-r}{r}}(u_n)K_q(v_n)$ and the assumption $k_r(u_n) \to 0$, we deduce from (3.54) that $K_q(v_n) \to 0$ as $n \to \infty$.

Since $r < \tilde{q}$ and $\big(v_n\big)_n \subset \mathcal{C}$, according to Lemma 2.2, there exists a positive constant L_1 such that

$$\big(k_r(v_n)\big)^{\frac{1}{r}} \le L_1\big(K_q(v_n)\big)^{\frac{1}{q}} \;\forall\, n.$$

By passing to the limit as $n \to \infty$ and taking into account that $k_r(v_n) = 1$ we obtain a contradiction, therefore we conclude that $u^* \neq 0$.

Now, let us show that $\lambda^* = R^*(u^*)$. Indeed, since the numerator of the fraction from the definition of R^* is a weakly lower semicontinuous function, we have

$$R^*(u^*) \le \frac{\liminf\limits_{n\to\infty}\Big(\frac{r}{p}K_p(u_n) + \frac{r}{q}K_q(u_n) + K_r(u_n)\Big)}{\lim\limits_{n\to\infty} k_r(u_n)} \le \liminf_{n\to\infty} R^*(u_n) = \lambda^*,$$

which implies that $\lambda^* = R^*(u^*)$ and also $J_{\lambda^*}(u^*) = 0$. Hence, $\mathcal{J}_{\lambda^*}(u^*) = \min_{\mathcal{C}\setminus\mathcal{Z}} \mathcal{J}_{\lambda^*} = 0$.

Finally, by an argument similar to that used in the proof of Lemma 3.7, one can show that $\mathcal{J}'_{\lambda^*}(u_*) = 0$ so in particular λ^* is an eigenvalue of problem (1.1). $\square$

Lemma 3.13. *Every number $\lambda \in (\lambda^*, \infty)$ is an eigenvalue of problem* (1.1). *In addition, for every $\lambda > \lambda^*$ there exists an eigenfunction u_λ corresponding to the eigenvalue λ such that $J_\lambda(u_\lambda) < 0$.*

Proof. Choose an arbitrary $\lambda > \lambda^*$. Taking into account Lemma 3.5 and using arguments similar to those from the proof of Lemma 3.7, we obtain the existence of a global minimizer $u_\lambda \in \mathcal{C}$ for $\mathcal{J}_\lambda$ restricted to $\mathcal{C}$, i.e., $\mathcal{J}_\lambda(u_\lambda) = \min_{\mathcal{C}} \mathcal{J}_\lambda$.

Now, let us prove that $u_\lambda \not\equiv 0$.

Since $\lambda > \lambda^*$ and $k_r(u^*) \neq 0$ (see Remark 1.1), we infer, based on Lemma 3.12, that

$$\begin{aligned} 0 = &\mathcal{J}_{\lambda^*}(u^*) = \frac{1}{p}K_p(u^*) + \frac{1}{q}K_q(u) + \frac{1}{q}K_r(u) - \frac{\lambda^*}{r}k_r(u^*) \\ &> \frac{1}{p}K_p(u^*) + \frac{1}{q}K_q(u^*) + \frac{1}{q}K_r(u) - \frac{\lambda}{r}k_r(u^*) = \mathcal{J}_\lambda(u^*). \end{aligned}$$

As $u^* \in \mathcal{C}$, using the fact that $\mathcal{J}_\lambda(u_\lambda) = \min_{\mathcal{C}} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(u^*) < 0$, we deduce that $u_\lambda \not\equiv 0$.

Then, as in the proof of Lemma 3.7, we deduce that the global minimizer u_λ for $\mathcal{J}_\lambda$ restricted to $\mathcal{C}$ is a critical point of $\mathcal{J}_\lambda$ considered on the whole space W, i.e. $\mathcal{J}'_\lambda(u_\lambda) = 0$. In other words, u_λ is an eigenfunction of problem (1.1) corresponding to λ. □

Next, let us prove that there is no eigenvalue of problem (1.1) in the interval $(0, \lambda_*)$.

Lemma 3.14. *Problem* (1.1) *has no nontrivial solution for* $\lambda \in (0, \lambda_*)$.

Proof. The result is a simple consequence of Lemma 3.11. Assume, by way of contradiction, that there exists a $\lambda \in (0, \lambda_*)$ and $u_\lambda \in \mathcal{C} \setminus \mathcal{Z}$ which satisfy the relation (1.3) from Definition 1.1. Choosing here $w = u_\lambda$ yields

$$\lambda = \frac{K_p(u_\lambda) + K_q(u_\lambda) + K_r(u_\lambda)}{k_r(u_\lambda)}, \tag{3.55}$$

which, by virtue of the equivalent definition of λ_* in $(1.8)_1$, implies that $\lambda \geq \lambda_*$. This contradicts the choice of λ. □

Summarizing, the conclusions of Theorem 1.2 follow from Lemmas 3.4 and 3.10-3.14.

4. The general case

Both Theorem 1.1 and Theorem 1.2 above remain valid for the general problem

$$\begin{cases} \mathcal{B}u := -\rho_p\Delta_p u - \rho_q\Delta_q u - \rho_r\Delta_r u = \lambda a(x) \mid u \mid^{r-2} u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{B}}} = \lambda b(x) \mid u \mid^{r-2} u \ \text{ on } \partial\Omega, \end{cases} \tag{4.56}$$

where ρ_p, ρ_q, ρ_r are given positive constants, and

$$\frac{\partial u}{\partial \nu_{\mathcal{B}}} := \Big(\sum_{\alpha \in \{p,q,r\}} \rho_\alpha \mid \nabla u \mid^{\alpha-2} \Big) \frac{\partial u}{\partial \nu}.$$

Definition 1.1 is modified by replacing the left hand side of (1.3) with

$$\sum_{\alpha \in \{p,q,r\}} \rho_\alpha \int_\Omega \mid \nabla u_\lambda \mid^{\alpha-2} \nabla u_\lambda \cdot \nabla w \, dx.$$

Define

$$\overline{\lambda}_r := \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \rho_r \frac{K_r(v)}{k_r(v)}, \tag{4.57}$$

$$\begin{aligned} \overline{\lambda}_* &:= \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \left(\overline{\Gamma} \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \rho_r \frac{K_r(v)}{k_r(v)} \right), \\ \overline{\lambda}^* &:= \inf_{v \in \mathcal{C}\setminus\mathcal{Z}} \left(\overline{\Gamma} \frac{r}{p^{1-\gamma}q^\gamma} \frac{K_p(v)^{1-\gamma} K_q(v)^\gamma}{k_r(v)} + \rho_r \frac{K_r(v)}{k_r(v)} \right), \\ \gamma &:= \frac{p-r}{p-q}, \ \overline{\Gamma} := \frac{\rho_p^{1-\gamma}\rho_q^\gamma (p-q)}{(r-q)^{1-\gamma}(p-r)^\gamma}. \end{aligned} \tag{4.58}$$

Theorems 1.1 and 1.2 can be reformulated as follows.

Theorem 4.3. *Assume that ρ_p, ρ_q, ρ_r are positive constants, and (h_{pqr}), (h_{ab}) are fulfilled. If $r \notin (q, p)$, then $\overline{\lambda}_r > 0$ and the set of eigenvalues of problem* (4.56) *is precisely $\{0\} \cup (\overline{\lambda}_r, \infty)$, where $\overline{\lambda}_r$ is the constant defined by* (4.57).

Theorem 4.4. *Assume that ρ_p, ρ_q, ρ_r are positive constants, (h_{pqr}) and (h_{ab}) are fulfilled, $r \in (p, q)$, and in addition $r < q(N-1)/(N-q)$ if $q < N$. Then $0 < \overline{\lambda}_* < \overline{\lambda}^*$, every $\lambda \in \{0\} \cup [\overline{\lambda}^*, \infty)$ is an eigenvalue of problem* (4.56) *and for any $\lambda \in (-\infty, \overline{\lambda}_*) \setminus \{0\}$, problem* (4.56) *has only the trivial solution.*

Moreover, the constants $\overline{\lambda}_*$, $\overline{\lambda}^*$ can be expressed as follows

$$\overline{\lambda}_* = \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \frac{\sum\limits_{\alpha \in \{p,q,r\}} \rho_\alpha K_\alpha(v)}{k_r(v)}, \quad \overline{\lambda}^* = \inf_{v \in \mathcal{C} \setminus \mathcal{Z}} \frac{\sum\limits_{\alpha \in \{p,q,r\}} \frac{\rho_\alpha}{\alpha} K_\alpha(v)}{\frac{1}{r} k_r(v)}. \tag{4.59}$$

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