In this case one has the operator for an operator of \( H \) acting between two complex Hilbert spaces. Similarly, the operator \( J \) is similar to the \( T \)-isometry \( P \) for its square root. According to the terminology of [19] we say that an operator \( T \) is a contraction if \( T^* T \leq I \) and it is an isometry if \( T^* T = I \). Also, \( T \) is unitary if \( T \) and \( T^* \) are isometries, and \( T \) is expansive if \( T^* T \geq I \), or in other words \( \Delta_T := T^* T - I \geq 0 \).

In this paper we denote by \( B(\mathcal{H}, \mathcal{H}') \) the Banach space of all bounded linear operators acting between two complex Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \) and \( B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H}) \) is considered a \( C^* \)-algebra with the identity operator \( I = I_\mathcal{H} \). For \( T \in B(\mathcal{H}, \mathcal{H}', \mathcal{R}(T) \subset \mathcal{H}' \) and \( \mathcal{N}(T) \subset \mathcal{H} \) stand for the range and the kernel of \( T \), while \( T^* \in B(\mathcal{H}', \mathcal{H}) \) means the adjoint operator of \( T \). If \( M \) is a subspace of \( \mathcal{H} \) we write \( \overline{M} \) for the closure of \( M \) in \( \mathcal{H} \). When \( M \) is closed we denote by \( P_M \in B(\mathcal{H}) \) the orthogonal projection with \( \mathcal{R}(P_M) = M \), and by \( P_{H,M} \in B(\mathcal{H}, M) \) the projection of \( \mathcal{H} \) onto \( M \). The (closed) subspace \( M \) is invariant (resp. reducing) for \( T \in B(\mathcal{H}) \) if \( TP_M = P_M TP_M \) (resp. \( T P_M = P_M T \)). When \( M \) is invariant for \( T \), the operator \( T|_M = T|_M \in B(M) \) is the restriction of \( T \) to \( M \), while \( T \) is an extension for \( T_\mathcal{M} \). In this case \( \mathcal{K} \ominus M \) is an invariant subspace for \( T^* \).

Let \( \mathcal{K}, \mathcal{K}' \) be Hilbert spaces which contain \( \mathcal{H} \) respectively \( \mathcal{H}' \) as closed subspaces. An operator \( S \in B(\mathcal{K}, \mathcal{K}') \) is a lifting for \( T \in B(\mathcal{H}, \mathcal{H}') \) if \( P_{\mathcal{K}, \mathcal{H}} S = TP_{\mathcal{K}, \mathcal{H}} \). When this occurs one has \( S(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K}' \ominus \mathcal{H}' \). Equivalently, \( S \) is a lifting for \( T \) if and only if \( S^* J_{\mathcal{K}', \mathcal{K}} = J_{\mathcal{K}, \mathcal{H}} T^* \) where \( J_{\mathcal{K}, \mathcal{H}} = P_{\mathcal{K}, \mathcal{H}}^* \) is the embedding mapping of \( \mathcal{H} \) into \( \mathcal{K} \), and similarly \( J_{\mathcal{K}', \mathcal{H}} = P_{\mathcal{K}', \mathcal{H}}^* \). It is obvious that if we take \( \mathcal{K}' = \mathcal{K} \) and \( \mathcal{H}' = \mathcal{H} \), the relation \( S^* J_{\mathcal{K}, \mathcal{H}} = J_{\mathcal{H}, \mathcal{K}} T^* \) exactly means that \( S^* \) is an extension for \( T^* \), and in this case \( S(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K}' \ominus \mathcal{H}' \). More generally, we can say that \( S \in B(\mathcal{K}) \) is a dilation of \( T \in B(\mathcal{H}) \) if \( T^n = P_{\mathcal{K}, \mathcal{H}} S^n J_{\mathcal{K}, \mathcal{H}} \) for every integer \( n \geq 0 \). When this happens we also say that \( T \) is a compression of \( S \).

An operator \( A \in B(\mathcal{H}) \) is said to be positive (in notation \( A \geq 0 \)) if \( \langle Ah, h \rangle \geq 0 \) for any \( h \in \mathcal{H} \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in any Hilbert space. When \( A \geq 0 \) we write \( A^{1/2} \) for its square root. According to the terminology of [19] we say that an operator \( T \in B(\mathcal{H}) \) is an \( A \)-contraction for a positive operator \( A \in B(\mathcal{H}) \) if \( T^* AT \leq A \) and \( A \neq 0 \). In this case \( T' \mathcal{N}(A) \subset \mathcal{N}(A) \) and \( T' \overline{\mathcal{R}(A)} \subset \overline{\mathcal{R}(A)} \). Also we say that \( T \) is an \( A \)-isometry if \( T^* AT = A \). Clearly, \( T \) is a contraction if \( T^* T \leq I \) and \( T \) is an isometry if \( T^* T = I \). Also, \( T \) is unitary if \( T \) and \( T^* \) are isometries, and \( T \) is expansive if \( T^* T \geq I \), or in other words \( \Delta_T := T^* T - I \geq 0 \).
An operator $T \in B(H)$ is concave if it is a $\Delta_T$-contraction and $T$ is a 2-isometry if it is a $\Delta_T$-isometry. In both cases $T$ is expansive and $N(\Delta_T)$ is invariant for $T$, while $V = T|_{N(\Delta_T)}$ is an isometry.

A 2-isometry $T$ on $H$ is called Brownian unitary if $U = T^*|_{\mathcal{R}(\Delta_T)}$ is unitary, and $E = \delta^{-1} P_{N(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ is an isometry with $\mathcal{R}(E) = N(V^*)$, $V$ as above, while $\delta = \|\Delta_T^{1/2}\|$.

Obviously, the class of 2-isometries contains the isometries, while the unitary operators are considered to be Brownian unitaries with $\delta = 0$. These latter operators are essential in the dilation theory of contractions initiated by Bela Sz.-Nagy and Ciprian Foiaş and developed by many authors (see [11, 22]). On the other hand, different classes of operators close to 2-isometries and more general to $A$-contractions have been studied intensively lately. We are referring here only to some articles like [1, 2, 3, 4, 5, 9, 10, 12, 13, 16, 17, 18, 19].

In this paper we continue the study of operators $T$ with 2-isometric liftings, which was started and developed in [6, 7, 8, 14, 15, 20, 21]. So, in Section 2 we refer to general 2-isometric liftings and show that they can be obtained by some expansive (even concave) liftings. Also, we see that 2-isometric liftings for $T$ can be also induced by dilations of $T$ which have triangulations with contraction entries, which suggests a relationship with the isometric liftings of contractions. We characterize the operators $T$ with such a more particular triangulation, by 2-isometric liftings $S$ with $S^*SH \subset H$ and having the covariance operator $\Delta_S$ a scalar multiple of an orthogonal projection.

In Section 3 we study an extension $\tilde{T}$ of an operator $T$ that has a Brownian unitary dilation $B$. We show that $\tilde{T}^*$ is an $A$-contraction, where $A$ is related to $N(\Delta_B)$ and we describe the triangulation of $\tilde{T}$ under the decomposition $\mathcal{R}(A) \oplus N(A)$ in the terms of $B$. As an application we characterize the operators $T$ with 2-isometric liftings $S$ satisfying $S^*SH \subset H$ by using a Brownian unitary extension $B$ of $S$. Also, we prove that these operators have an extension with a more particular matrix structure, namely having as entries contractions and even coisometries. The cases when $\mathcal{R}(A)$ is closed and some compressions of $T$ are similar to contractions are also considered.

2. OPERATORS WITH LIFTINGS CLOSE TO 2-ISOMETRIES

We will further investigate the operators with 2-isometric liftings, by means of some intermediate liftings, extensions or dilations which lead to 2-isometries. In this regard we show first of all that the 2-isometric liftings can be obtained by intermediate expansive or $A$-contractive liftings.

**Theorem 2.1.** For $T \in B(H)$ non-contractive the following statements are equivalent:

(i) $T$ has a 2-isometric lifting;

(ii) $T$ has a lifting $\tilde{T} \in B(H)$ such that $\tilde{T}$ is an $A$-contraction for a positive operator $A$ on $\tilde{H}$ with $A \geq \Delta_{\tilde{T}}$;

(iii) $T$ has an expansive lifting $\tilde{T} \in B(\tilde{H})$ which under a decomposition $\tilde{H} = H_0 \oplus H_1$ has a triangulation of the form

$$\tilde{T} = \begin{pmatrix} V & X \\ 0 & Z \end{pmatrix},$$

where $V$ is an isometry on $H_0$ and $Z$ is an $A_1$-contraction on $H_1$ with $A_1 \geq X^*X + \Delta_Z$;

(iv) $T$ has a concave lifting.

**Proof.** The implication (i)$\Rightarrow$(iv) is trivial. Assume that $T$ has a concave lifting $\tilde{T}$ on a Hilbert space $\tilde{H} \supset H$. Then $\Delta_{\tilde{T}} = \tilde{T}^*\tilde{T} - I \geq 0$ i.e. $\tilde{T}$ is expansive, and $\tilde{T}^*\Delta_{\tilde{T}}\tilde{T} \leq \Delta_{\tilde{T}}$. 

$$\text{(2.1)}$$
i.e. $\tilde{T}$ is a $\Delta_{\tilde{T}}$-contraction. So $\mathcal{N}(\Delta_{\tilde{T}})$ is an invariant subspace for $\tilde{T}$, hence $\tilde{T}$ has a matrix representation (2.1) under a decomposition $\mathcal{H} = \mathcal{N}(\Delta_{\tilde{T}}) \oplus \mathcal{R}(\Delta_{\tilde{T}})$ with $V = \tilde{T}|_{\mathcal{N}(\Delta_{\tilde{T}})}$ an isometry. Also, since $\Delta_{\tilde{T}} \geq 0$, we have $V^*X = 0$, and using this fact we get that $\Delta_{\tilde{T}} = 0 \oplus \Delta_0$, where $\Delta_0 = X^*X + \Delta_Z \geq 0$ on $\mathcal{R}(\Delta_{\tilde{T}})$. In addition, the above inequality ensures that $Z^*\Delta_0Z \leq \Delta_0$ and $\Delta_0 \neq 0$ ($T$ being non-contractive) i.e. $Z$ is a $\Delta_0$-contraction. Thus the entries $V$, $X$, and $Z$ of $T$ have the required properties in (2.1), hence (iv) implies (iii).

Now suppose that $T$ has an expansive lifting $\tilde{T}$ of the form (2.1) under a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Since $V$ is an isometry on $\mathcal{H}_0$ and $\Delta_{\tilde{T}} \geq 0$ one has $V^*X = 0$ and $Z$ is an $A_1$-contraction on $\mathcal{H}_1$ with $A_1 \geq X^*X + \Delta_Z =: \Delta_1 \geq 0$, we obtain that $\tilde{T}$ is an $A$-contraction where $A = 0 \oplus A_1$ on $\mathcal{H}_0 \oplus \mathcal{H}_1$. Also, the previous inequality for $A_1$ leads to $A \geq 0 \oplus \Delta_1 = \Delta_{\tilde{T}}$, hence the lifting $\tilde{T}$ of $T$ has the required property in (ii). We conclude that (iii) implies (ii).

Finally, let’s assume that $\tilde{T}$ and $A$ on $\tilde{\mathcal{H}} \supset \mathcal{H}$ are as in the statement (ii). Let $\mathcal{H}' = \ell^2_+((\mathcal{R}(A - \Delta_{\tilde{T}})))$ and $\hat{S} \in \mathcal{B}(\mathcal{H}' \oplus \tilde{\mathcal{H}})$ be the operator with the block matrix

$$\hat{S} = \begin{pmatrix} S_+ & (A - \Delta_{\tilde{T}})^{1/2} \\ 0 & 0 \end{pmatrix},$$

where $S_+$ is the forward shift on $\mathcal{H}'$ with $\mathcal{N}(S_+^*) = \mathcal{R}(A - \Delta_{\tilde{T}})$. Then $\Delta_{\hat{S}} = 0 \oplus A$ on $\mathcal{K} = \mathcal{H}' \oplus \tilde{\mathcal{H}}$ and

$$\hat{S}^* \Delta_{\hat{S}} \hat{S} = 0 \oplus \tilde{T}^* A \tilde{T} \leq 0 \oplus A = \Delta_{\tilde{S}},$$

hence $\hat{S}$ is a concave operator. Since $\hat{S}$ has a 2-isometric lifting $S$ (see [7, 8]) and $\hat{S}$ is a lifting for $T$, it follows that $\tilde{S}$ is also a 2-isometric lifting for $T$. Thus (ii) implies (i).

\textbf{Remark 2.1.} In the implication (ii) $\Rightarrow$ (i) we can get by [8, Theorem 4.1] a 2-isometric lifting $\hat{S}$ for $\tilde{T}$ with $\hat{S}^* \hat{S} \mathcal{H} \subset \tilde{\mathcal{H}}$. Moreover, for the expansive lifting $\tilde{T}$ from (iii) of $T$ we get by [8, Theorem 3.7] a 2-isometric lifting $\tilde{S}$ on $\mathcal{K} \supset \tilde{\mathcal{H}}$ with $\mathcal{K} \ominus \tilde{\mathcal{H}} \subset \mathcal{N}(\Delta_{\tilde{S}})$. But $\mathcal{H}$ is neither invariant for $\hat{S} \hat{S}^*$, nor for $\hat{S}^* \hat{S}$ in general, when we consider $\hat{S}$ and $\tilde{S}$ as liftings for $T$.

However, if $\tilde{T}$ is a concave lifting for $T$ as in (iv) and $\tilde{T}$ is an extension for $T$ as in (2.1) with the properties from (iii), while $\hat{S}$ and $\tilde{S}$ are as above, then for $S_0 = \hat{S}|_{\mathcal{K} \ominus \tilde{\mathcal{H}}}$ and $S_1 = \tilde{S}|_{\tilde{\mathcal{H}} \ominus \mathcal{K}}$ we have $\mathcal{H} \ominus \mathcal{H} \subset \mathcal{N}(\Delta_{S_0})$, respectively $S_1^* S_1 \mathcal{H} \subset \mathcal{H}$. Obviously, $S_0$ and $S_1$ are liftings for $T$, and $S_0$ also satisfies the condition $S_0^* S_0 \mathcal{H} \subset \mathcal{H}$. Such 2-isometric liftings were studied in [7, 8, 14, 15, 20, 21]. But this special case can be now presented as a consequence of the above theorem.

\textbf{Corollary 2.1.} For $T \in \mathcal{B}(\mathcal{H})$ non-contractive the following statements are equivalent:

(i) $T$ has a 2-isometric lifting $S$ with $S^* S \mathcal{H} \subset \mathcal{H}$;
(ii) $T$ is an $A_0$-contraction for an operator $A_0 \geq \Delta_T$;
(iii) $T$ has an expansive lifting $\tilde{T}$ of the form (2.1) on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $\tilde{T}^* \tilde{T} \mathcal{H} \subset \mathcal{H}$, $V = \tilde{T}|_{\mathcal{H}_0}$ an isometry, $Z = P_{\mathcal{H}_1} \tilde{T}|_{\mathcal{H}_1}$ an $A_1$-contraction for an operator $A_1 \geq X^*X + \Delta_Z$, such that $\mathcal{H}_1 \subset \mathcal{H}$ where $\mathcal{A}_1 = 0 \oplus A_1$ on $\mathcal{H}_0 \oplus \mathcal{H}_1$ and $X = P_{\mathcal{H}_0} \tilde{T}|_{\mathcal{H}_1}$;
(iv) $T$ has a concave lifting $\tilde{T}$ with $\tilde{T}^* \tilde{T} \mathcal{H} \subset \mathcal{H}$.

\textbf{Proof.} The implications (i)$\Rightarrow$(iv) and (iv)$\Rightarrow$(iii) are obvious, if we take $\tilde{T} = S$, respectively $\tilde{T} = T$ and $A_1 = \Delta_{\tilde{T}}|_{\mathcal{H}_1} = X^*X + \Delta_Z$. 
Now let us assume that the assertion (iii) is true. We represent the lifting \( \tilde{T} \) of \( T \) and the operator \( \tilde{A}_1 \) (from (iii)) with \( \tilde{A}_1 \mathcal{H} \subset \mathcal{H} \) on \( \tilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H} \), in the form
\[
\tilde{T} = \begin{pmatrix} Y_0 & Y_1 \\ 0 & T \end{pmatrix}, \quad \tilde{A}_1 = A_2 \oplus A_0.
\]
Since \( Z \) is an \( A_1 \)-contraction in (2.1) and \( \tilde{A}_1 = 0 \oplus A_1 \) on \( \tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) we infer (using (2.1)) that \( \tilde{T}^* \tilde{A}_1 \tilde{T} \leq \tilde{A}_1 \). Expressing this relation in the terms of the above representations for \( \tilde{T} \) and \( \tilde{A}_1 \) on \( \tilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H} \) we get that \( \tilde{T}^* A_0 T \leq A_0 \) and \( A_0 \geq 0 \) because \( \tilde{A}_1 \geq 0 \). But \( A_0 \neq 0 \) as we will see below, so \( T \) is an \( A_0 \)-contraction.

Next we use that \( \tilde{T}^* \tilde{T} |_{\mathcal{H}} \subset \mathcal{H} \) (by (iii)), which means \( Y_0^* Y_1 = 0 \) in the above matrix of \( \tilde{T} \). Hence \( \Delta_{\tilde{T}^*} = \Delta_{Y_0} \oplus (Y_1^* Y_1 + \Delta_T) \) on \( \tilde{\mathcal{H}} = \mathcal{H}^\perp \oplus \mathcal{H} \), and \( \Delta_{\tilde{T}} \leq \tilde{A}_1 \) because \( \Delta_{\tilde{T}}|_{\mathcal{H}_1} = X^* X + \Delta_Z \leq A_1 \) (by (iii)). We obtain that
\[
\Delta_T \leq Y_1^* Y_1 + \Delta_T = \Delta_{\tilde{T}}|_{\mathcal{H}} \leq \tilde{A}_1|_{\mathcal{H}} = A_0,
\]
and as \( T \) is not a contraction we have \( A_0 \neq 0 \), which completes the assertion (ii). Thus (iii) implies (ii), while (ii) implies (i) by [8, Theorem 4.1]. \( \square \)

A direct consequence of Theorem 2.1 and of the last assertion in Remark 2.1 is the following

**Corollary 2.2.** Let \( T \in \mathcal{B}(\mathcal{H}) \) having an expansive lifting (or extension) \( \tilde{T} \) on \( \tilde{\mathcal{H}} \supset \mathcal{H} \), such that \( \tilde{T} \) has a triangulation (2.1) with \( V \) an isometry and \( Z \) similar to a contraction. Then \( T \) has a 2-isometric lifting (respectively, a 2-isometric lifting \( S \) with \( \mathcal{H}^\perp \subset \mathcal{N}(\Delta_S) \)).

Another characterization for the operators with 2-isometric liftings can be obtained using more general dilations than Brownian unitary dilations. Recall that by the famous result of Agler-Stankus from [2, Theorem 5.80] every 2-isometry has a Brownian unitary extension which retains the covariance. So each operator with 2-isometric lifting has a Brownian unitary dilation and the converse is also true. But an intermediate dilation appears in this setting, which can be easily used in applications and to provide examples.

**Theorem 2.2.** For \( T \in \mathcal{B}(\mathcal{H}) \) non-contractive the following statements are equivalent:

(i) \( T \) has a 2-isometric lifting;

(ii) \( T \) has a dilation \( \tilde{T} \) on \( \tilde{\mathcal{H}} \supset \mathcal{H} \) which under a decomposition \( \tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) has a triangulation of the form
\[
\tilde{T} = \begin{pmatrix} C_0 & \delta C_1 \\ 0 & C \end{pmatrix},
\]
where \( \delta > 0 \) is a scalar and \( C, C_j \) (\( j = 0,1 \)) are contractions, such that there exist a Hilbert space \( \mathcal{E} \), an isometry \( J_0 : \mathcal{D}_{C_0} \to \mathcal{E} \) and a contraction \( J_1 : \mathcal{D}_{C_1} \to \mathcal{E} \) satisfying the condition
\[
\mathcal{D}_{C_0} J_0^* J_1 \mathcal{D}_{C_1} + C_0^* C_1 = 0.
\]

**Proof.** Assume that \( T \) has a 2-isometric lifting \( S \) on \( \mathcal{K} = \mathcal{H}' \oplus \mathcal{H} \) and let \( B \) on \( \tilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}' \) be a Brownian unitary extension of \( S \). Then \( B \) has triangulations of the form
\[
B = \begin{pmatrix} S & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} W & * \\ 0 & T \end{pmatrix} = \begin{pmatrix} V & \delta \mathcal{E} \\ 0 & U \end{pmatrix}
\]
respectively under the decompositions
\[
\tilde{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}' = \mathcal{H}' \oplus (\mathcal{H} \oplus \mathcal{K}') = \mathcal{N}(\Delta_B) \oplus \mathcal{R}(\Delta_B),
\]
where the lifting $S$ of $T$ has on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$ the representation

$$S = \begin{pmatrix} W & \ast \\ 0 & T \end{pmatrix}.$$  

Clearly $W = S|_{\mathcal{H}'}$ is a 2-isometry, $\tilde{T} = P_M B|_M$ from (2.4) is an extension for $T$ on $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$, $V = B|_{\mathcal{N}(\Delta_B)}$ and $E : \mathcal{R}(\Delta_B) \to \mathcal{N}(\Delta_B)$ are isometries with $\mathcal{R}(E) = \mathcal{N}(V^*)$. $U$ is unitary on $\mathcal{R}(\Delta_B)$ and $\delta = \|\Delta_B\|^{1/2} = \|\Delta_S\|^{1/2} > 0$ ($T$ being non-contractive).

Since $B$ is a lifting for $\tilde{T}$ and $\tilde{T}$ is an extension for $T$ it follows that $B$ is a dilation for $T$, which has the form (2.2) by the last triangulation in (2.4). Here the condition (2.3) is given by $V^* E = 0$ (quoted above) and $\mathcal{E} = \{0\}$. So (i) implies (ii).

Conversely, we suppose that $T$ has a dilation $\tilde{T}$ as in (2.2) on $\hat{\mathcal{H}} \supset \mathcal{H}$, with $C, C_j$ contractions satisfying the condition (2.3) for $j = 0, 1$ (as in (ii)). Since $C$ is a contraction it has an isometric lifting. Then by [15, Theorem 2.5] (or by Theorem 2.3 below) it follows that $\tilde{T}$ has a 2-isometric lifting $\hat{S}$ on $\hat{\mathcal{K}} \supset \hat{\mathcal{H}}$. As $\hat{T}$ is a dilation for $T$, it has a matrix representation of the form

$$\hat{T} = \begin{pmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{pmatrix}$$

under a decomposition $\hat{\mathcal{H}} = \mathcal{K}_0 \oplus \mathcal{H} \oplus \mathcal{K}_1$. Since $\hat{S}$ is a lifting for $\hat{T}$ it follows that $\hat{S}$ is also a dilation for $T$, therefore $\hat{S}$ has relative to $T$ a similar representation as $\hat{T}$ of above, under the decomposition $\hat{\mathcal{K}} = [(\mathcal{K} \oplus \hat{\mathcal{H}}) \oplus \mathcal{K}_0] \oplus \mathcal{H} \oplus \mathcal{K}_1$. Hence $S_0 = \hat{S}|_{\mathcal{K} \oplus \mathcal{K}_1}$ will be a 2-isometric lifting for $\tilde{T}$, which proves that (ii) implies (i). \qed

Remark that the condition (2.3) is more general than $C_0 C_1 = 0$. In fact, this condition shows that there exist an isometric lifting $V_0 \in \mathcal{B}(\mathcal{E} \oplus \mathcal{H}_0)$ for $C_0$ and a contractive lifting $\tilde{C}_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{E} \oplus \mathcal{H}_0)$ for $C_1$ such that $V_0^* C_1 = 0$, for some Hilbert space $\mathcal{E}$.

Notice that Theorem 2.2.2 is an effective generalization of [21, Theorem 2.1] where we characterized the operators $T$ on $\mathcal{H}$ that have 2-isometric liftings $S$ with $S^* S \mathcal{H} \subset \mathcal{H}$, in terms of an extension for $T$ of the form (2.2). We retrieve this result in the Theorem 3.6 below.

In the general case, Theorem 2.2.2 shows that the operators $T$ with 2-isometric liftings are exactly the compressions of operators with triangulations (2.2), which satisfy the condition (2.3). But this means that one can get some extensions for $T$ that have liftings of the form (2.2), as we will see in the next section. We now describe by means of 2-isometric liftings the operators of the form (2.2).

**Theorem 2.3.** For $T \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:

(i) $T$ has a 2-isometric lifting $S$ with $S^* S \mathcal{H} \subset \mathcal{H}$ and $\Delta_S = \sigma^2 P$ with $P$ an orthogonal projection and a scalar $\sigma > 0$;

(ii) $T$ has a triangulation (2.2) under a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with $C_0 = T|_{\mathcal{H}_0}$, $C^* = T^*|_{\mathcal{H}_1}$ and $C_1 = \delta^{-1} P_{\mathcal{H}_1} T|_{\mathcal{H}_1}$ contractions for some scalar $\delta > 0$, such that $C_0$ and $C_1$ satisfy the condition (2.3).

**Proof.** Let $T, S, P$ and $\delta$ be as in (i). In what follows we may assume, without los of generality, that $T$ is not a contraction. Let $W = S|_{\mathcal{H}'}$ with $\mathcal{H}' = \mathcal{K} \ominus \mathcal{H}$. Then as $\Delta_S \mathcal{H} \subset \mathcal{H}$ (by (i)) it follows that $\Delta_S = 0 \oplus (\Delta_W|_{\mathcal{H}'}) \oplus (\Delta_S|_{\mathcal{H}})$ under the decomposition $\mathcal{K} = \mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W) \oplus \mathcal{H}$. We remark from this representation of $\Delta_S$ that

$$\mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_W) \oplus \mathcal{N}(\Delta_S|_{\mathcal{H}}), \quad \mathcal{R}(\Delta_S) = \Delta_W \mathcal{H}' \oplus \Delta_S \mathcal{H} = \mathcal{R}(\Delta_W) \oplus (\mathcal{H} \cap \mathcal{R}(\Delta_S)).$$

Since $\mathcal{R}(\Delta_S)$ is closed (by (i)) we obtain that $\mathcal{R}(\Delta_W)$ is closed, too.
Now we use the fact that $S$ is a $\Delta_S$-isometry i.e. $S^*\Delta_SS = \Delta_S$. This ensures that $\mathcal{N}(\Delta_S)$ is invariant for $S$, which implies that $\mathcal{H}_0 = \mathcal{N}(\Delta_S|_H)$ is invariant for $T$, because if $h \in \mathcal{H}$ and $\Delta_S h = 0$ then

$$T h = P_H S h \in P_H \mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_S|_{\mathcal{H}}),$$

therefore $T \mathcal{H}_0 \subset \mathcal{H}_0$. Thus it follows that on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with $\mathcal{H}_1 = \Delta_S \mathcal{H}_0$, $T$ has a triangulation of the form (2.2) with the entries $C_0 = T|_{\mathcal{H}_0}$, $C = P_{\mathcal{H}_1} T|_{\mathcal{H}_1}$ and $C_1 = P_{\mathcal{H}_0} S|_{\mathcal{H}_1}$. Putting $D = P_{\mathcal{N}(\Delta_S)} S|_{\mathcal{H}_1} = \delta C'$ for a contraction $C'$ and a scalar $\delta \geq \|D\|$, we have

$$D = \delta \begin{bmatrix} C_2 & C_1 \end{bmatrix}^T : \mathcal{H}_1 \rightarrow \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0,$$

with $C_j$ contractions ($j = 1, 2$) and $C_1 = \delta C_1$.

Notice that since $\mathcal{H}_0 \subset \mathcal{N}(\Delta_S)$ and $S|_{\mathcal{N}(\Delta_S)}$ is an isometry, we have that $C_0 = T|_{\mathcal{H}_0} = P_{\mathcal{H}_0} S|_{\mathcal{H}_0}$ is a contraction. On the other hand, as $S$ is a $\Delta_S$-isometry and $\Delta_S = \sigma^2 P$ with $P = P_{\mathcal{R}(\Delta_S)}$ (by (i)) it follows that $S^* P S = P$. Also, one has the relation $P S = P S P$, because $S \mathcal{N}(P) \subset \mathcal{N}(P) = \mathcal{N}(\Delta_S)$. But as $S$ is a $P$-isometry, there exists an isometry $V_1$ on $\mathcal{R}(P) = \mathcal{R}(\Delta_S)$ such that $P S = V_1 P$, which yields $S^*|_{\mathcal{R}(\Delta_S)} = P V_1^*$. Then for the operator $C$ from (2.2) we have $C^* = T^*|_{\mathcal{H}_1} = S^*|_{\mathcal{H}_1} = P V_1^*|_{\mathcal{H}_1}$, therefore $C = P_{\mathcal{H}_1} V_1|_{\mathcal{H}_1}$ is a contraction. This also implies that $\delta \neq 0$ (by our assumption that $T$ is not a contraction), and also that $\tilde{C}_1 \neq 0$, so $\delta > 0$ in (2.2). To end the proof of (ii) it remains to show the condition (2.3) for $C_0, C_1$.

For this (using the above notation) we represent the isometry $V = S|_{\mathcal{N}(\Delta_S)}$ on $\mathcal{N}(\Delta_S) = \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0$ and the operator $D : \mathcal{H}_1 \rightarrow \mathcal{N}(\Delta_W) \oplus \mathcal{H}_0$ in the form

$$V = \begin{pmatrix} V_0 & C'_0 \\ 0 & C_0 \end{pmatrix}, \quad D = \delta \begin{pmatrix} C_2 \\ C_1 \end{pmatrix},$$

where $V_0 = V|_{\mathcal{N}(\Delta_W)}$ is an isometry and $C'_0 : \mathcal{H}_0 \rightarrow \mathcal{N}(\Delta_W)$ is a contraction such that $C'_0 C_0 + C_0 C'_0 = I$ ($V$ being an isometry). So there exists an isometry $J_0 : D C_0 \rightarrow \mathcal{N}(\Delta_W)$ satisfying the relation $J_0 D C_0 = C'_0$. On the other hand, since $\delta^{-1} D = C'$ is a contraction we have $C_2^2 + C_1^2 C_1 \leq I$ i.e. $C_2^2 C_2 \leq D_2^2$. Hence there exists a contraction $J_1$ from $D C_1$ into $\mathcal{N}(\Delta_W)$ such that $C_2 = J_1 D C_1$. Finally, since $S$ is expansive and $V$ is an isometry we need to have $V^* P_{\mathcal{N}(\Delta_S)} S|_{\mathcal{R}(\Delta_S)} = 0$, which implies $V^* D = 0$ and later that

$$D C_0, J_0 J_1 D C_1 + C_0 C_1 = C_0^* C_2 + C_0^* C_1 = 0.$$

Therefore $C_0, C_1$ satisfy the condition (2.3), and we proved that (i) implies (ii).

Conversely, let us assume that $T$ has a triangulation as in (ii) on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Let $V_1$ be the (minimal) isometric lifting on $K_1 = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the contraction $C = P_{\mathcal{H}_1} T|_{\mathcal{H}_1}$, where $\mathcal{H}_2 = \ell_2^2(\mathcal{P}_C)$, while $D_C = \Delta_C H_1$ is the defect space of $C$. Consider the space $\mathcal{H}_{-1} = \ell_2^1(\mathcal{E} \oplus \mathcal{H}_2)$ where $\mathcal{E}$ is the Hilbert space quoted in (ii). Denote by $S_+$ the forward shift on $\mathcal{H}_{-1}$ and let $J : \mathcal{E} \rightarrow \mathcal{H}_{-1}$, $J_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_{-1}$ be the embedding mappings. We define the isometries $V_0$ on $K_0 = \mathcal{H}_{-1} \oplus \mathcal{H}_0$ and $V_2$ from $K_1 = \mathcal{H}_1 \oplus \mathcal{H}_2$ into $K_0 = \mathcal{H}_{-1} \oplus \mathcal{H}_0$ having, respectively, the block matrices

$$V_0 = \begin{pmatrix} S_+ & J J_0 D C_0 \\ 0 & C_0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} J J_1 D C_1 & J_2 \\ C_1 & 0 \end{pmatrix},$$

where the contractions $C_0, C_1$ and $J_0, J_1$ are from (2.2) and (2.3).

Now we define the operator $S$ on $\mathcal{K} = K_0 \oplus K_1$ with the block matrix

$$S = \begin{pmatrix} V_0 & \delta V_2 \\ 0 & V_1 \end{pmatrix},$$

where $\delta \delta^{-1} = 1.$
with the scalar $\delta > 0$ from (2.2). It is easy to see (by using the condition (2.3)) that $V_0^*V_2 = 0$, which leads to the fact that $\Delta_S = 0 \oplus \delta^2 I$ on $K_0 \oplus K_1$. Thus we have $\Delta_S = \delta^2 P_{R(\Delta_S)}$ and trivially $S^*\Delta_SS = \Delta_S$, that is $S$ is a 2-isometry. Also, with the matrices from (2.2), (2.5) and (2.6) we obtain for $S$ the representations

$$S = \begin{pmatrix} S_+ & C_2 & J_2 \\ 0 & T & 0 \\ 0 & C' & V' \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix} = \begin{pmatrix} S_+ & J_2 & C_2 \\ 0 & V' & C' \\ 0 & 0 & T \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H} \\ \mathcal{H}_2 \end{pmatrix} = \begin{pmatrix} W & E \end{pmatrix} \begin{pmatrix} K \oplus \mathcal{H} \end{pmatrix}.$$

Here $C_2 = [JJ_0DC_C, \delta JJ_1DC_C] : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{H}_1, C' = [0, J'DC_C] : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{H}_2$ with $J' : DC_C \rightarrow \mathcal{H}_2$ the embedding mapping, while $V'$ is the forward shift on $\mathcal{H}_2$. We firstly infer that $S$ is a lifting for $T$ and later, since $W^*E = 0$ (as $S_+^*JJ_1DC_C = 0, J_2^*JJ_1DC_C = 0$ for $i = 0, 1$ and $V'^*J'DC_C = 0$), we conclude that $S^*S\mathcal{H} \subset \mathcal{H}$. Thus $S$ has the required properties in (i), which proves that (ii) implies (i).

**Corollary 2.3.** If $T \in \mathcal{B}(\mathcal{H})$ satisfies the equivalent conditions of Theorem 2.3 then $T$ has a 2-isometric lifting $S$ with a triangulation of the form (2.6), where the entries $V_j$ ($j = 0, 1, 2$) are isometries and $\delta > 0$ is the scalar from the triangulation (2.2) of $T$.

Remark that the liftings from (2.6) are more special than those mentioned in Theorem 2.3 (i). Such 2-isometries were considered in [12, 13].

Notice finally that the two properties of $S$ from the assertion (i) before do not involve each other, in general. The condition $S^*S\mathcal{H} \subset \mathcal{H}$ ensures only that $T$ has an extension of the form (2.2) satisfying (2.3), by [21, Theorem 2.1]. So Theorem 2.3 refers to a more special class of operators than those mentioned in Corollary 2.1. Let us also mention that other characterizations for the operators $T$ from Theorem 2.3 were obtained in [21, Theorem 2.2].

3. EXTENSIONS OF OPERATORS WITH BROWNIAN UNITARY DILATIONS

We continue the study of operators described in Theorem 2.1 and Theorem 2.2. Each such operator $T$ has a Brownian unitary dilation obtained as an extension of a 2-isometric lifting of $T$. Using such dilations we describe some extensions for $T$, which lead to 2-isometric liftings for $T$. Let’s start with the following result.

**Theorem 3.4.** Let $T \in \mathcal{B}(\mathcal{H})$ be non-contractive and having a 2-isometric lifting $S$ on $\mathcal{K} = \mathcal{H'} \oplus \mathcal{H}$. Let $B \in \mathcal{K} = \mathcal{K} \oplus \mathcal{K'}$ be a Brownian unitary extension of $S$, and let $A = P_MP_N(\Delta_B)|_{\mathcal{M}}$, where $\mathcal{M} = \mathcal{H} \oplus \mathcal{K'}$. Then $A \neq 0$ and the following statements hold:

(i) $\tilde{T} = P_MB|_{\mathcal{M}}$ is an extension for $T$ which under the decomposition $\mathcal{M} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$ has the triangulation

$$\tilde{T} = \begin{pmatrix} B_0 & B_1 \\ 0 & V_1^* \end{pmatrix}, \quad B_0 = P_{\overline{\mathcal{R}(A)}}B|_{\overline{\mathcal{R}(A)}}, \quad B_1 = P_{\overline{\mathcal{R}(A)}}B|_{\mathcal{N}(A)}, \quad V_1 = B^*|_{\mathcal{N}(A)},$$

such that $\tilde{T}^*$ is an $A$-contraction, $B_0^*$ is an $A_0$-contraction with $A_0 = A|_{\overline{\mathcal{R}(A)}}$, and $V_1$ is an isometry.

(ii) $\mathcal{R}(A) = \overline{P_M\mathcal{N}(\Delta_B) \cap \mathcal{M} \cap \mathcal{N}(\Delta_B), \mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)}$ and

$$\mathcal{R}(\Delta_W) \oplus \overline{\mathcal{R}(A)} = \overline{[\mathcal{N}(\Delta_B) \oplus \mathcal{N}(\Delta_W)] \oplus [\mathcal{R}(\Delta_B) \oplus \mathcal{N}(A)]},$$

where $W = S|_{\mathcal{H'}}$. In addition, $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(\Delta_W)$ is closed; in this case $B_0$ is similar to a contraction.
Proof. Let $T, S$ and $B$ be as above. Then $B$ has triangulations of the form

$$(3.9) \quad B = \begin{pmatrix} S & \ast \\ 0 & \ast \end{pmatrix} = \begin{pmatrix} W & \ast \\ 0 & \ast \end{pmatrix} = \begin{pmatrix} V & \delta E' \\ 0 & U \end{pmatrix},$$

respectively under the decompositions

$$\tilde{K} = K \oplus K' = H' \oplus \mathcal{M} = \mathcal{N}(\Delta_B) \oplus \mathcal{R}(\Delta_B),$$

where the lifting $S$ of $T$ has on $K = H' \oplus H$ the triangulation

$$S = \begin{pmatrix} W & \ast \\ 0 & T \end{pmatrix}.$$  

Here $W = S|_{H'}$ is a 2-isometry, $\tilde{T} = P_{\mathcal{M}}B|_{\mathcal{M}}$ is an extension for $T$ on $\mathcal{M} = H \oplus K'$, $V = B|_{\mathcal{N}(\Delta_B)}$ and $E : \mathcal{R}(\Delta_B) \to \mathcal{N}(\Delta_B)$ are isometries with $\mathcal{R}(E) = \mathcal{N}(V^*)$, while $U$ is unitary on $\mathcal{R}(\Delta_B)$.

Using the last representation of $B$ in (3.9) as well as that $V^*E = 0$ we obtain

$$BP_{\mathcal{N}(\Delta_B)}B^* = VV^* \oplus 0 \leq P_{\mathcal{N}(\Delta_B)}.$$  

Since $P_{\mathcal{N}(\Delta_B)} \neq 0$ (as we see below) it follows that $B^*$ is a $P_{\mathcal{N}(\Delta_B)}$-contraction. Now representing $P_{\mathcal{N}(\Delta_B)}$ on $\tilde{K} = H' \oplus \mathcal{M}$ in the form

$$P_{\mathcal{N}(\Delta_B)} = \begin{pmatrix} \ast & \ast \\ \ast & A \end{pmatrix}, \quad A = P_{\mathcal{M}}P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}},$$

we get by the above inequality that

$$(3.10) \quad BP_{\mathcal{N}(\Delta_B)}B^* = \begin{pmatrix} \ast & \ast \\ \ast & \tilde{T}AT^* \end{pmatrix} \leq \begin{pmatrix} \ast & \ast \\ \ast & A \end{pmatrix}.$$  

Hence $\tilde{T}AT^* \leq A$.

Let us note that $A \neq 0$ (so $P_{\mathcal{N}(\Delta_B)} \neq 0$). Indeed, if $A = 0$ we have $P_{\mathcal{N}(\Delta_B)}\mathcal{M} = \{0\}$, so $\mathcal{H} \subset \mathcal{M} \subset \mathcal{R}(\Delta_B)$. This gives by (3.9) that $T = P_{\mathcal{H}}B|_{\mathcal{H}} = P_{\mathcal{H}}U|_{\mathcal{H}}$, therefore $T$ is a contraction, which contradicts the hypothesis. So $A \neq 0$ and as $A \geq 0$ from (3.10) it follows that $\tilde{T}^*$ is an $A$-contraction. Then $\overline{\mathcal{R}(A)}$ is an invariant subspace for $\tilde{T}$, hence $\tilde{T}$ has the triangulation (3.7) under $\mathcal{M} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$, with the entries $B_0, B_1$ and $V_1^*$ inferred from the second matrix of $B$ in (3.9).

Now from the definition of $A$ we see that $\mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)$ and

$$U^*\mathcal{N}(A) = B^*\mathcal{N}(A) = \tilde{T}^*\mathcal{N}(A) \subset \mathcal{N}(A),$$

so $V_1 = B^*|_{\mathcal{N}(A)} = U^*|_{\mathcal{N}(A)}$ is an isometry. Also, using the triangulation (3.7) of $\tilde{T}$ as well as the representation $A = A_0 \oplus 0$ with $A_0 = A|_{\overline{\mathcal{R}(A)}} \neq 0$, we infer from the inequality $\tilde{T}AT^* \leq A$ that $B_0A_0B_0^* \leq A_0$, that is $B_0$ is an $A_0$-contraction. The assertion (i) is proved.

Next we notice that because $W = S|_{H'} = \overline{\mathcal{R}(A)}$ is a 2-isometry, $\mathcal{N}(\Delta_W)$ is invariant for $W = S|_{H'} = B|_{H'}$ and also for $B$. So $B|_{\mathcal{N}(\Delta_W)}$ is an isometry, hence $\mathcal{N}(\Delta_W) \subset \mathcal{N}(\Delta_B)$. This and the fact that $\mathcal{N}(A) \subset \mathcal{R}(\Delta_B)$ give for $\tilde{K}$ the decompositions

$$\tilde{K} = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(\Delta_W)} \oplus \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A),$$

whence one obtains the relation (3.8).

Clearly, from the definition of $A$ we have $\mathcal{R}(A) \subset P_{\mathcal{M}}\mathcal{N}(\Delta_B)$. Conversely, if $k = P_{\mathcal{M}}k_0$ with $k_0 \in \mathcal{N}(\Delta_B)$ then for every $k_1 \in \mathcal{N}(A) = \mathcal{M} \cap \mathcal{R}(\Delta_B)$ we have $(k, k_1) = (k_0, k_1) = 0,$
therefore $k \in \overline{\mathcal{R}(A)}$. So $P_M\mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$ and with the converse inclusion of above we get finally $\overline{\mathcal{R}(A)} = P_M\mathcal{N}(\Delta_B)$. Obviously, $\mathcal{M} \cap \mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$.

For the last assertion in (ii) we infer from [10, Remark 2.12-2] that $\mathcal{R}(A)$ is closed (in $\overline{\mathcal{K}}$) if and only if $\mathcal{R}(\Delta_B) + \mathcal{M}$ is closed, or equivalently $\mathcal{N}(\Delta_B) + \mathcal{H}' = \mathcal{N}(\Delta_B) + \overline{\mathcal{R}(\Delta_W)}$ is closed. But for the 2-isometry $W = B|_{\mathcal{H}'}$ we have

$$\Delta_W = P_{\mathcal{H}'}\Delta_B|_{\mathcal{H}'} = \delta^2 P_{\mathcal{H}'} P_{\mathcal{R}(\Delta_B)}|_{\mathcal{H}'}$$

where $\delta = \|\Delta_B\|^{1/2}$. Then by the same remark in [10] we can assert that $\mathcal{R}(\Delta_W)$ is closed (in $\overline{\mathcal{K}}$) if and only if $\mathcal{R}(\Delta_B) + \mathcal{M}$ is closed. Thus we conclude that $\mathcal{R}(A)$ and $\mathcal{R}(\Delta_W)$ are simultaneously closed. Clearly, in this case the operator $A_0 = A|_{\mathcal{R}(A)}$ is invertible in $\mathcal{B}(\mathcal{R}(A))$, and as $B_0$ is an $A_0$-contraction it follows that $B_0$ is similar to a contraction. The assertion (ii) is proved.

Some arguments in this proof lead to the next improved version of the result mentioned in [21, Corollary 3.3]. Among other things, we will see that the inclusion $\mathcal{M} \cap \mathcal{N}(\Delta_B) \subset \overline{\mathcal{R}(A)}$ from Theorem 3.4 (ii) may be strict, in general.

**Theorem 3.5.** Let $T \in \mathcal{B}(\mathcal{H})$ having a 2-isometric lifting $S$ on $\mathcal{K} \supset \mathcal{H}$ and let $B$ be a Brownian unitary extension for $S$ on $\mathcal{K} \supset \mathcal{K}$ with $\|\Delta_B\| = \|\Delta_S\| > 0$. The following assertions are equivalent:

(i) $S^*S\mathcal{H} \subset \mathcal{H}$;

(ii) $\mathcal{R}(\Delta_B|_{\mathcal{K} \oplus \mathcal{H}}) \subset \mathcal{R}(\Delta_B)$;

(iii) $\mathcal{M} \cap \mathcal{N}(\Delta_B) = \overline{\mathcal{R}(A)}$, where $A = P_M P_{\mathcal{N}(\Delta_B)}|_{\mathcal{M}}$ and $\mathcal{M} = \mathcal{H} \oplus (\overline{\mathcal{K}} \ominus \mathcal{K})$.

If this is the case then $A$ is an orthogonal projection and

$$\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_B|_{\mathcal{K} \oplus \mathcal{H}}) \oplus \mathcal{R}(A), \quad \mathcal{R}(\Delta_B) = \mathcal{R}(\Delta_B|_{\mathcal{K} \oplus \mathcal{H}}) \oplus \mathcal{N}(A).$$

**Proof.** Preserving the notation from the previous proof we have $W := B|_{\mathcal{H}'} = S|_{\mathcal{H}'}$ where $\mathcal{H}' = \mathcal{K} \ominus \mathcal{H}$. Assume that the condition (iii) is verified. Then every $k \in \mathcal{N}(\Delta_B)$ can be written as $k = P_{\mathcal{H}'}k + P_Mk$, and $P_Mk \in \overline{\mathcal{R}(A)} \subset \mathcal{N}(\Delta_B)$. So $P_{\mathcal{H}'}k \in \mathcal{N}(\Delta_B)$ which gives $\Delta_W P_{\mathcal{H}'}k = P_{\mathcal{H}'} \Delta_B P_{\mathcal{H}'}k = 0$ i.e. $P_{\mathcal{H}'}k \in \mathcal{N}(\Delta_W)$. Thus it follows that $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$ and this implies $\mathcal{R}(\Delta_B|_{\mathcal{H}'}) \subset \mathcal{R}(\Delta_B)$ i.e. the condition of (ii). We conclude that (iii) implies (ii).

Next we assume the condition from (ii) to be satisfied. This firstly yields $\mathcal{R}(\Delta_W) = \mathcal{H}' \cap \mathcal{R}(\Delta_B)$, so $\mathcal{R}(\Delta_W)$ is closed. Now by (3.8) we have $\mathcal{R}(\Delta_W) \oplus \mathcal{N}(A) \subset \mathcal{R}(\Delta_B)$. Let $k \in \mathcal{R}(\Delta_B)$ such that $k$ is orthogonal on $\mathcal{R}(\Delta_W) \oplus \mathcal{N}(A)$. So by (3.8) we get $k \in \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$. Since $k$ is orthogonal on $\mathcal{N}(\Delta_B)$, $k$ is also orthogonal on $\mathcal{N}(\Delta_B) \subset \mathcal{N}(\Delta_W)$, hence $k \in \mathcal{R}(A)$. Then $Ak = P_M P_{\mathcal{N}(\Delta_B)}k = 0$, so $k = 0$ because $A$ is injective on $\overline{\mathcal{R}(A)}$. Given the choice of $k$ we conclude that $\mathcal{R}(\Delta_B) = \mathcal{R}(\Delta_W) \oplus \mathcal{N}(A)$ and $\mathcal{N}(\Delta_B) = \mathcal{N}(\Delta_W) \oplus \overline{\mathcal{R}(A)}$.

Now $S^*S|_{\mathcal{H}'} = P_K B^* B|_{\mathcal{H}'}$, $\mathcal{H}'$ being invariant for $S$ and $B$, and because $B$ is Brownian unitary we have $\Delta_B = \delta^2 P_{\mathcal{R}(\Delta_B)}$, where $\delta^2 = \|\Delta_B\| = \|\Delta_S\| > 0$. Thus we obtain

$$S^*S\mathcal{H}' = S^*S(\mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W)) \subset \mathcal{N}(\Delta_W) + P_K B^* B \mathcal{R}(\Delta_W) \subset \mathcal{N}(\Delta_W) \oplus \mathcal{R}(\Delta_W) + P_K \Delta_B \mathcal{R}(\Delta_W) = \mathcal{H}' + \delta^2 P_K \mathcal{R}(\Delta_B) = \mathcal{H}'$$

taking into account that $\mathcal{R}(\Delta_W) \subset \mathcal{H}' \cap \mathcal{R}(\Delta_B)$ and $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}$. Hence $S^*S\mathcal{H} \subset \mathcal{H}$ i.e. the condition (i). In addition, we obtain that $A = I \oplus 0$ on $\mathcal{M} = [\mathcal{N}(\Delta_B) \oplus \mathcal{N}(\Delta_W)] \oplus [\mathcal{R}(\Delta_B) \oplus \mathcal{R}(\Delta_W)]$, that is $A$ is an orthogonal projection. We have shown that (ii) implies (i), while (i) implies (ii) by the proof of [21, Theorem 2.1], because $\|\Delta_B\| = \|\Delta_S\|$. 


Finally, we saw above that in hypothesis (ii) we have $N(\Delta_B) = N(\Delta_W) \oplus R(A)$, therefore $R(A) \subseteq M \cap N(\Delta_B)$. Since the converse inclusion is also valid (see Theorem 3.4 (ii)), we obtain the condition of (iii). Hence (ii) implies (iii).

The special 2-isometric liftings discussed in this theorem are expressed by their Brownian unitary extensions. But they can be also described in terms of triangulation (3.7), which has a particular shape in this case. Thus we add another statement equivalent to those of Corollary 2.1.

**Theorem 3.6.** An operator $T \in B(H)$ has a 2-isometric lifting $S$ on $K \supset H$ with $S^*SH \subset H$ if and only if $T$ has an extension $\tilde{T}$ on $M \supset H$ which under a decomposition $M = M_0 \oplus M_1$ has a triangulation of the form (2.2) with $C_0 = \tilde{T}|_{M_0}$ and $C_1 = \delta^{-1}P_{M_0}T|_{M_1}$ contractions for a scalar $\delta > 0$ which satisfy the condition (2.3), and with $C = P_{M_1}\tilde{T}|_{M_1}$ a coisometry.

**Proof.** Assume that $T$ on $H$ and $S$ on $K = H' \oplus H$ are as above such that $S^*SH \subset H$. Then $T$ has the extension $\tilde{T}$ of the form (3.7) on $M = R(A) \oplus N(A)\delta$, induced by a Brownian unitary extension $B$ of $S$ on a space $\tilde{K} = K \oplus K' = H' \oplus M$. Since $R(A) \subset N(\Delta_B)$ by Theorem 3.5, in the matrix (3.7) we obtain that $B_0$ is a contraction, $B_1 = \delta C_1$ with a contraction $C_1$ and $\delta = \parallel \Delta_B \parallel^{1/2} > 0$, while $C = V_T^*$ is a coisometry.

Now by Theorem 3.5 we have $N(\Delta_B) = N(\Delta_W) \oplus R(A)$ and $R(\Delta_B) = R(\Delta_W) \oplus N(A)$ where $W = B|_{H'}$, while $R(A)$ and $R(\Delta_W)$ are closed. So the isometries $V = B|_{N(\Delta_B)}$ and $E = \delta^{-1}P_N(\Delta_B)B|_{R(\Delta_B)}$ from the canonical triangulation (3.9) of $B$ have the block matrices of the form

$$V = \begin{pmatrix} V_0 & 0 \\ 0 & B_0 \end{pmatrix} \begin{pmatrix} N(\Delta_W) \\ R(A) \end{pmatrix}, \quad E = \begin{pmatrix} W_0 & 0 \\ 0 & C_1 \end{pmatrix} : \begin{pmatrix} R(\Delta_W) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} N(\Delta_W) \\ R(A) \end{pmatrix}$$

with $V_0, W_0, J_0 : D_{B_0} \rightarrow N(\Delta_W)$ and $J_1 : D_{C_1} \rightarrow N(\Delta_W)$ isometries. Since $V^*E = 0$ and using these representations for $V$ and $E$ it follows that $D_{B_0}J_0^*J_1D_{C_1} + B_0^*C_1 = 0$, that is the condition (2.3) for the contractions $B_0$ and $C_1$ from the triangulation (3.7) of the extension $\tilde{T}$ for $T$. An implication of the proposition is proved.

Conversely, let us assume that $T$ has an extension $\tilde{T}$ on $M \supset H$ as above. Then $\tilde{T}$ has a 2-isometric lifting $\tilde{S}$ on $\tilde{K} = M^{\perp} \oplus M$ such that $\tilde{S}^*\tilde{S}M \subset M$ (by Theorem 2.3). But $\tilde{K}_0 = M^{\perp} \oplus H$ is invariant for $\tilde{S}$, so $\tilde{S}_0|_{\tilde{K}_0}$ is a 2-isometric lifting for $T$. Also, since $\tilde{S}^*\tilde{S}M^{\perp} \subset M^{\perp}$ we get $S_0^*S_0M^{\perp} = \tilde{S}^*\tilde{S}M^{\perp} \subset M^{\perp}$, that is $S_0^*S_0H \subset H$. The converse assertion is proved.

From the last part of this proof we see that $\Delta_{\tilde{S}} = \delta^2P_{R(\Delta_{\tilde{S}})}$ with $\delta > 0$ (by Theorem 2.3), but $\Delta_{\tilde{S}_0}$ has not this form, in general. However, as $R(\Delta_{\tilde{S}})|_{M^{\perp}} \subset R(S_{\tilde{S}})$ and $\Delta_{\tilde{S}}|_{M^{\perp}} = \Delta_{S_0|_{M^{\perp}}}$ we have $\Delta_{S_0|_{M^{\perp}}} = \delta^2P$ with an orthogonal projection $P$. This leads to the following

**Corollary 3.4.** If $T \in B(H)$ satisfies the equivalent assertions of Theorem 3.6 then $T$ has a 2-isometric lifting $S$ on $K \supset H$ such that $S^*SH \subset H$ and $\Delta_S|_{K \oplus H} = \delta^2P$ for an orthogonal projection $P$ and a scalar $\delta > 0$.

Regarding the operators $\tilde{T}$ and $A$ from Theorem 3.4 we give some additional properties.

**Proposition 3.1.** Let $T \in B(H)$ having a Brownian unitary dilation $B$ on $\tilde{K} = H' \oplus H \oplus K'$ with $\parallel \Delta_B \parallel > 0$, and let $\tilde{T} = P_MB|_M$ and $A = P_MP_N(\Delta_B)|_M$ where $M = H \oplus K'$. The following statements hold.
(i) If $\mathcal{N}(A) \neq \{0\}$ then $\bar{T}$ is a $P_{\mathcal{N}(A)}$-contraction. In this case, either $\mathcal{H} \subset \overline{\mathcal{R}(A)}$ and then $\bar{T}|_{\overline{\mathcal{R}(A)}}$ is an extension for $T$, or $T$ is an $A_1$-contraction with $A_1 = P_{\mathcal{H}}P_{\mathcal{N}(A)}|_{\mathcal{H}}$ and $\mathcal{N}(A_1) = \mathcal{H} \cap \overline{\mathcal{R}(A)}$.

(ii) If $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \neq \{0\}$ then $T$ (respectively $T|_{\mathcal{N}(A_1)}$) is similar to a contraction if $A_1 = 0$ (respectively if $\mathcal{N}(A_1) \neq \{0\}$).

Proof. (i). Assume that $\mathcal{N}(A) \neq \{0\}$. Then using the block matrix (3.7) we get $\bar{T}^* P_{\mathcal{N}(A)} \bar{T} \leq P_{\mathcal{N}(A)}$, that is $\bar{T}$ is a $P_{\mathcal{N}(A)}$-contraction. Let $A_1 = P_{\mathcal{H}}P_{\mathcal{N}(A)}|_{\mathcal{H}}$. Clearly, $A_1 = 0$ if and only if $P_{\mathcal{N}(A)} \mathcal{H} = \{0\}$ i.e. $\mathcal{H} \subset \overline{\mathcal{R}(A)}$. In this case, as $\bar{T}$ is an extension of $T$ and $\overline{\mathcal{R}(A)}$ is invariant for $\bar{T}$, it follows that $\bar{T}|_{\overline{\mathcal{R}(A)}}$ is an extension for $T$. If $A_1 \neq 0$ then using the triangulations of $\bar{T}$ and $P_{\mathcal{N}(A)}$ under the decomposition $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}'$ we get relations of the form

$$
\begin{pmatrix}
T^* A_1 T & * \\
* & *
\end{pmatrix}
= \bar{T}^* P_{\mathcal{N}(A)} \bar{T} \leq P_{\mathcal{N}(A)} = \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix},
$$

whence one infers that $T^* A_1 T \leq A_1$, that is $T$ is an $A_1$-contraction. In this case it is obvious that $\mathcal{N}(A_1) = \mathcal{H} \cap \overline{\mathcal{R}(A)}$.

(ii). Assume that $\mathcal{R}(A)$ is closed and $\mathcal{N}(A) \neq \{0\}$. If $A_1 \neq 0$ then $T$ is an $A_1$-contraction (by (i)), so $\mathcal{N}(A_1)$ is invariant for $T$. In this case we have that $\mathcal{N}(A_1) = \mathcal{H} \cap \overline{\mathcal{R}(A)}$, therefore $T|_{\mathcal{N}(A_1)} = \bar{T}|_{\mathcal{N}(A_1)} = B_0|_{\mathcal{N}(A_1)}$, where $B_0 = \bar{T}|_{\mathcal{R}(A)}$ as in (3.7). Since $B_0$ is similar to a contraction (by Theorem 3.4 (ii)) it follows that $T|_{\mathcal{N}(A_1)}$ is similar to a contraction.

In the case when $A_1 = 0$ we have $\mathcal{H} \subset \mathcal{R}(A)$, so $T = B_0|_{\mathcal{H}}$ and (as above) $T$ will be similar to a contraction. $\square$

Remark 3.2. If $T$, $\bar{T}$ and $A$ are as in Theorem 3.4 then the $A$-contraction $\bar{T}^*$ is a lifting for $T^*$ having a triangulation

$$
\bar{T}^* = \begin{pmatrix} V_1 & B_1^* \\ 0 & B_0^* \end{pmatrix}
$$

under $\mathcal{M} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, where $V_1$ is an isometry. But when $\mathcal{N}(A) \neq \{0\}$ it is not contained in $\mathcal{N}(\Delta_{\bar{T}^*})$, where $\Delta_{\bar{T}^*} = \bar{T} T^* - I$ has the decomposition

$$
\Delta_{\bar{T}^*} = \begin{pmatrix} 0 & V_1 B_1^* \\ B_1 V_1 & B_1 B_1^* + B_0 B_0^* - I \end{pmatrix}.
$$

In fact we have $B_1 V_1 k \neq 0$ for $0 \neq k \in \mathcal{N}(A)$. Indeed, for such $k$ we obtain from the proof of Theorem 3.4 the relations

$$
B_1 V_1 k = P_{\overline{\mathcal{R}(A)}} B B^* k = P_{\overline{\mathcal{R}(A)}} \Delta_{\bar{T}^*} k = \delta E U^* k.
$$

Here for the last equality we used the triangulation of the Brownian unitary $B$ from (3.9) with $E$ an isometry and $U$ unitary. Thus $B_1 V_1 k \neq 0$ for $k \neq 0$, which shows that $\mathcal{N}(A) \nsubseteq \mathcal{N}(\Delta_{\bar{T}^*})$.

Remark 3.3. Even under the condition $S^* SH \subset \mathcal{H}$ (as in Theorem 3.6) it can be seen that $B_0^* B_1 \neq 0$ in (3.11), considering that $A \neq 0$ (by Theorem 3.4). In this case $B_0$ is a contraction (as we noted earlier), so $\bar{T}$ has a triangulation of the form (2.2), where $B_0$ and $B_1$ satisfy the condition (2.3), more general than $B_0^* B_1 = 0$.

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REFERENCES


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