CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 641 - 654

Online version at `https://semnul.com/carpathian/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.10

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Some Fixed Point Results in Spaces with Perturbed Metrics

ADRIAN NICOLAE BRANGA and ION MARIAN OLARU

ABSTRACT. In this paper, the concept of perturbed metric was introduced within the metric spaces and some fixed point results were established for self-mappings satisfying such contractive conditions, using Picard operators technique and generalized contractions. Moreover, some applications of the main result to continuous data dependence of the fixed points of Picard operators defined on these spaces were presented. Also, the main result of this paper was applied to study the existence and uniqueness of the solution for an integral equation which models an epidemiological problem.

1. INTRODUCTION

As many of the non-linear phenomena are mathematically modeled by various types of equations, an important aspect resides in the study of the existence, uniqueness and computation of the solutions of these problems. Numerous mathematical methods have been developed to study the problems raised by science and engineering, one of the most powerful being the Picard and weakly Picard operators technique. In 1993, Rus [9] introduced the concept of weakly Picard operator. More recently, Mureşan [5], Mureşan and Mureşan [6] applied Picard and weakly Picard operators method to study some abstract integro-differential equations. Using suitable Picard operators, Caliò et al. [2] established some results for Volterra-Fredholm integral equations and Rus [10] investigated some functional-differential equations of mixed type. Also, through Picard and weakly Picard operators theory, Wang et al. [12] studied some nonlocal Cauchy problems for differential equations in Banach spaces. In this paper, by using the Picard operators technique and generalized contractions we study the existence, uniqueness and continuous data dependence of fixed points for operators defined on spaces with perturbed metrics.

Throughout this paper we shall follow the standard terminologies and notations in nonlinear analysis. For the convenience of the reader we shall recall some of them.

Let X be a nonempty set and $T : X \to X$ an operator. We denote by

$$T^0 := 1_X,\ T^1 := T,\ T^{m+1} = T^m \circ T,\ m \in \mathbb{N},$$

the iterate operators of the operator T, and

$$F_T := \{x \in X \mid T(x) = x\}$$

the set of the fixed points of T.

Definition 1.1. ([12]) Let (X, d) be a metric space. An operator $T : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_T = \{x^*\}$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

Received: 25.01.2022. In revised form: 28.04.2022. Accepted: 05.05.2022
2020 *Mathematics Subject Classification.* 47H10, 47H09, 54E50.
Key words and phrases. *spaces with perturbed metrics, Picard operators, fixed point theorems, generalized contractions, data dependence, integral equations, epidemiological problems.*
Corresponding author: Ion Marian Olaru; marian.olaru@ulbsibiu.ro

Definition 1.2. ([9]) Let (X, d) be a metric space. An operator $T : X \to X$ is a weakly Picard operator if the sequence $T^m(x_0)$ converges as $m \to \infty$, for any arbitrary point $x_0 \in X$, and its limit (which may depend on x_0) is a fixed point of T.

If T is a weakly Picard operator, then we consider the operator

$$T^\infty : X \to X,\ T^\infty(x) = \lim_{m\to\infty} T^m(x).$$

Further, we denote by $\mathbb{R}_+$ the real interval $[0, \infty)$.

An important notion for our approach is the comparison function.

Definition 1.3. ([11]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if the following conditions are satisfied:

(i) φ is monotonically increasing;
(ii) the sequence $\varphi^m(t) \to 0$ as $m \to \infty$, for every $t > 0$.

Example 1.1. The maps $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \ln(t + 1)$, $\varphi(t) = \frac{t}{t+1}$ and $\varphi(t) = rt$, where $r \in (0, 1)$, are comparison functions.

A very fecund idea, which led to numerous results in mathematics, is the concept of φ-contraction, which generalizes the notion of contraction. We recall the definition of a φ-contraction.

Definition 1.4. ([11]) Let (X, d) be a metric space. An operator $T : X \to X$ is a φ-contraction if φ is a comparison function and the following condition is satisfied:

$$d(T(x), T(y)) \le \varphi(d(x, y)),\ (\forall)x, y \in X.$$

A generalization of the contraction principle (the Picard-Banach theorem) was established by J. Matkowski for φ-contractions in complete metric spaces.

Theorem 1.1. *([4]) Let (X, d) be a complete metric space and $T : X \to X$ a φ-contraction. Then T is a Picard operator.*

Next we recall the definitions of the right-continuous, upper semicontinuous and right upper semicontinuous functions.

Definition 1.5. ([8]) Let A be a subset of $\mathbb{R}$, $a \in A$ a point and $f : A \to \mathbb{R}$ a function. We say that:

1) f is right-continuous at a if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon \text{ for all } x \in (a, a + \delta(\varepsilon)) \cap A;$$

2) f is right-continuous if it is right-continuous at every point $a \in A$.

2. Results

Lemma 2.1. *Any comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties:*

(i) *$\varphi(t) < t$ for all $t > 0$;*
(ii) *$\varphi^m(0) \to 0$ as $m \to \infty$ if and only if $\varphi(0) = 0$;*
(iii) *if $\varphi(0) = 0$ then φ is right-continuous at the point 0.*

Proof. (i) We assume there exists $t > 0$ such that $\varphi(t) \ge t$. As φ is monotonically increasing, it follows that $\varphi(\varphi(t)) \ge \varphi(t) \ge t$, therefore $\varphi^2(t) \ge t$. Applying the induction method we deduce $\varphi^m(t) \ge t$ for all $m \in \mathbb{N}^*$. Passing to the limit as $m \to \infty$, in the previous inequality, we find $\lim\limits_{m\to\infty} \varphi^m(t) \ge t > 0$, which is in contradiction with the fact that the sequence $\varphi^m(t) \to 0$ as $m \to \infty$, for every $t > 0$. Hence, $\varphi(t) < t$ for all $t > 0$.

(ii) $\Rightarrow$ Suppose that $\varphi^m(0) \to 0$ as $m \to \infty$. We assume that $\varphi(0) > 0$. Since φ is monotonically increasing, we get $\varphi(\varphi(0)) \ge \varphi(0)$, i.e. $\varphi^2(0) \ge \varphi(0)$. Using the induction method we obtain $\varphi^m(0) \geq \varphi(0)$ for all $m \in \mathbb{N}^*$. Consequently, $\lim_{m\to\infty} \varphi^m(0) \geq \varphi(0) > 0$, which is in contradiction with the fact that $\varphi^m(0) \to 0$ as $p \to \infty$. Therefore, $\varphi(0) = 0$.

$\Leftarrow$ Assume that $\varphi(0) = 0$. We get $\varphi(\varphi(0)) = \varphi(0) = 0$, i.e. $\varphi^2(0) = 0$. Using the induction method we obtain $\varphi^m(0) = 0$ for all $m \in \mathbb{N}^*$. Consequently, $\varphi^m(0) \to 0$ as $m \to \infty$.

(iii) Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and using (i), we deduce $0 \leq \varphi(t) < t$ for all $t > 0$. Passing to the limit as $t \searrow 0$, we find $0 \leq \lim_{t\searrow 0} \varphi(t) \leq 0$, therefore $\lim_{t\searrow 0} \varphi(t) = 0$. Taking into account that $\varphi(0) = 0$, it follows that $\lim_{t\searrow 0} \varphi(t) = \varphi(0)$, hence φ is right-continuous at the point 0. □

Lemma 2.2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function that satisfies the condition (i) from Definition 1.3 and which is right-continuous. Then, the condition (ii) from Definition 1.3 is equivalent with:*

(ii′) *$\varphi(t) < t$ for all $t > 0$.*

Proof. $\Rightarrow$ Suppose that φ verifies the condition (ii) from Definition 1.3. As φ satisfies the condition (i) of Definition 1.3, it follows that φ is a comparison function. Apllying Lemma 2.1 (i) we deduce that $\varphi(t) < t$ for all $t > 0$, i.e. the condition (ii') is fulfilled.

$\Leftarrow$ Assume that φ satisfies the condition (ii'). Let $t > 0$ be an arbitrary number. As φ is monotonically increasing, it follows that $\varphi(\varphi(t)) \leq \varphi(t)$, therefore $\varphi^2(t) \leq \varphi(t)$. Applying the induction method we deduce $\varphi^{m+1}(t) \leq \varphi^m(t)$ for all $m \in \mathbb{N}^*$. On the other hand, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, hence $0 \leq \varphi(t)$. Since φ is monotonically increasing, we get $0 \leq \varphi(0) \leq \varphi(\varphi(t))$, i.e. $0 \leq \varphi^2(t)$. Applying the induction method we find $0 \leq \varphi^{m+1}(t)$ for all $m \in \mathbb{N}$. Consequently, we proved that $0 \leq \varphi^{m+1}(t) \leq \varphi^m(t)$ for all $m \in \mathbb{N}^*$. Therefore, $(\varphi^m(t))_{m\in\mathbb{N}^*}$ is a sequence of positive real numbers, monotonically decreasing and bounded from below by 0. We deduce that the sequence $(\varphi^m(t))_{m\in\mathbb{N}^*}$ converges to a unique limit $a(t) \geq 0$ and $\varphi^m(t) \geq a(t)$ for all $m \in \mathbb{N}^*$. Let suppose that $a(t) > 0$. Since φ satisfies the condition (ii') it follows that $\varphi(a(t)) < a(t)$. We have

$$a(t) = \lim_{m\to\infty} \varphi^m(t) = \lim_{m\to\infty} \varphi^{m+1}(t) = \lim_{m\to\infty} \varphi(\varphi^m(t)).$$

As the function φ is right-continuous we obtain

$$\lim_{m\to\infty} \varphi(\varphi^m(t)) = \lim_{x\searrow a(t)} \varphi(x) = \varphi(a(t)).$$

Consequently, $a(t) = \varphi(a(t))$, which is in contradiction with the fact that $\varphi(a(t)) < a(t)$. Therefore, $a(t) = 0$. As the number $t > 0$ was chosen arbitrarily, we find that the sequence $\varphi^m(t) \to 0$ as $m \to \infty$, for every $t > 0$, i.e. the condition (ii) from Definition 1.3 is verified. □

Definition 2.6. Let Ψ be the class of all the functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the following conditions:

(i) ψ is continuous;
(ii) ψ is monotonically increasing;
(iii) $\psi(t) = 0$ if and only if $t = 0$.

Let (X, d) be a metric space. If we modify the metric d by a function $\psi \in \Psi$ we remark that, in most cases, the application $\psi \circ d$ does not keep the metric properties.

Example 2.2. Let us consider (X, d) a metric space and the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+,\ \psi(t) = t^2$. Then:

1) ψ belongs to the class Ψ;
2) $\psi \circ d$ is not a metric on X.

Lemma 2.3. *Let us consider a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

1) *φ is right-continuous;*
2) *$\varphi(t) < t$ for all $t > 0$.*

Then:

$$\lim_{s \searrow t}(s - \varphi(s)) > 0 \text{ for every } t > 0.$$

Proof. Considering the properties of the limit from the right of a function and the hypotheses 1), 2), we deduce

$$\lim_{s \searrow t}(s - \varphi(s)) = t - \varphi(t) > 0 \text{ for every } t > 0.$$

□

Lemma 2.4. *Let us consider a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

1) *φ is monotonically increasing;*
2) *φ is right-continuous;*
3) *$\varphi(t) < t$ for all $t > 0$;*

and a function $\psi \in \Psi$. We define the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\phi(t) = \sup\{s \in \mathbb{R}_+ \mid \psi(s) \le \varphi(\psi(t))\}. \tag{2.1}$$

Then the following statements are true:

(i) *$\varphi(0) = 0$;*
(ii) *ϕ is well defined;*
(iii) *$\phi(0) = 0$;*
(iv) *$\phi(t) \le t$ for all $t \in \mathbb{R}_+$;*
(v) *$\psi(\phi(t)) \le \varphi(\psi(t))$ for all $t \in \mathbb{R}_+$;*
(vi) *$\phi(t) < t$ for all $t > 0$;*
(vii) *ϕ is monotonically increasing;*
(viii) *the sequence $\phi^m(t) \to 0$ as $m \to \infty$, for every $t > 0$;*
(ix) *ϕ is a comparison function.*

Proof. (i) Since the function φ is right-continuous at the point 0, we have $\lim\limits_{t \searrow 0} \varphi(t) = \varphi(0)$. On the other hand, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and considering the hypothesis 3) we deduce $0 \le \varphi(t) < t$ for all $t > 0$. Passing to the limit as $t \searrow 0$, we obtain $0 \le \lim\limits_{t \searrow 0} \varphi(t) \le 0$, hence $\lim\limits_{t \searrow 0} \varphi(t) = 0$. It follows that $\varphi(0) = 0$.

(ii) Let $t \in \mathbb{R}_+$ be an arbitrary number. We define the set

$$M_t := \{s \in \mathbb{R}_+ \mid \psi(s) \le \varphi(\psi(t))\}. \tag{2.2}$$

As $\psi(0) = 0$ (according to Definition 2.6 (iii)) and $\varphi(\psi(t)) \ge 0$ ($\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$), we obtain $\psi(0) \le \varphi(\psi(t))$, hence $0 \in M_t$, thus M_t is a non-empty set. We distinguish the following cases:

1. If $t = 0$. As $\psi(0) = 0$ (according to Definition 2.6 (iii)) and $\varphi(0) = 0$ (by (i)) we find $\varphi(\psi(0)) = 0$, hence $M_0 = \{s \in \mathbb{R}_+ \mid \psi(s) \le 0\}$. Considering Definition 2.6 (iii), we get $M_0 = \{0\}$. It follows that $\phi(0) = \sup M_0 = \sup\{0\} = 0$.
2. If $t > 0$. Choose $s \in M_t$ be an arbitrary element. We deduce $s \in \mathbb{R}_+$ and $\psi(s) \le \varphi(\psi(t))$. On the other hand, as $t > 0$, according to Definition 2.6 (iii), we have $\psi(t) > 0$. Using the hypothesis 3) we find $\varphi(\psi(t)) < \psi(t)$. It follows that $\psi(s) < \psi(t)$. Considering that ψ is monotonically increasing (by Definition 2.6 (ii)), we get $s < t$. Therefore, $s \in [0, t)$. Since $s \in M_t$ was choosen arbitrarily, we obtain $M_t \subseteq [0, t)$. Consequently, the set M_t is bounded from above by t. We deduce that,

there exists $\sup M_t \leq t$. Hence, $\phi(t) := \sup M_t \leq t$ is well defined and we have $\phi(t) \leq t$.

(iii), (iv) results from (ii).

(v) Let $t \in \mathbb{R}_+$ be an arbitrary element. According to (ii), the set M_t is bounded from above by t and $\phi(t) := \sup M_t$. It follows that, there exists a sequence $(s_n)_{n\in\mathbb{N}} \subseteq M_t$ such that $s_n \to \phi(t)$ as $n \to \infty$ and $s_n \leq \phi(t)$ for all $n \in \mathbb{N}$. Since $s_n \in M_t$ for all $n \in \mathbb{N}$, we deduce

$$\psi(s_n) \leq \varphi(\psi(t)) \text{ for all } n \in \mathbb{N}.$$

On the other hand, as ψ is continuous (by Definition 2.6 (i)), we obtain $\psi(s_n) \to \psi(\phi(t))$ as $n \to \infty$. Therefore, from the previous inequality we find $\psi(\phi(t)) \leq \varphi(\psi(t))$.

In other words, $\phi(t) \in M_t$ and $M_t \subseteq [0, \phi(t)]$. Choose $s \in [0, \phi(t)]$. We get $s \leq \phi(t)$ and considering that ψ is monotonically increasing (according to Definition 2.6 (ii)), it results $\psi(s) \leq \psi(\phi(t))$. Therefore, $\psi(s) \leq \varphi(\psi(t))$, i.e. $s \in M_t$. Consequently, $M_t = [0, \phi(t)]$.

(vi) From (iv) we have $\phi(t) \leq t$ for all $t \in \mathbb{R}_+$. Suppose that there exists $t > 0$ such that $\phi(t) = t$. By using (v) we deduce $\psi(t) \leq \varphi(\psi(t))$. On the other hand, $t > 0$ implies $\psi(t) > 0$ (according to Definition 2.6 (iii)) and using the hypothesis 3) we obtain $\varphi(\psi(t)) < \psi(t)$. It follows that, $\psi(t) < \psi(t)$, which is a contradiction. Consequently, $\phi(t) < t$ for all $t > 0$.

(vii) Let $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$ be arbitrary numbers. Taking into consideration that φ, ψ are monotonically increasing, we get $\varphi(\psi(t_1)) \leq \varphi(\psi(t_2))$, hence

$$\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(t_1))\} \subseteq \{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(t_2))\}.$$

It follows that

$$\phi(t_1) = \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(t_1))\} \leq \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(t_2))\} = \phi(t_2),$$

thus ϕ is monotonically increasing.

(viii) Let $t > 0$ be an arbitrary number. We consider the sequence $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}_+$ defined by

$$t_0 = t, t_m = \phi(t_{m-1}), m \geq 1. \tag{2.3}$$

Because $t_0 > 0$, from (vi) we get $\phi(t_0) < t_0$, hence $t_1 < t_0$. Applying the induction method and considering that ϕ is monotonically increasing (by (vii)), we deduce $t_m \leq t_{m-1}$ for all $m \geq 1$. It follows that hence $(t_m)_{m\in\mathbb{N}}$ is a monotonically decreasing sequence and bounded from below by 0. We deduce that the sequence $(t_m)_{m\in\mathbb{N}}$ converges to a limit $l \geq 0$ and $t_m \geq l$ for all $m \in \mathbb{N}$. Next, we show that $l = 0$. Let us suppose that $l > 0$. Applying (v) we find

$$\psi(\phi(t_{k-1})) \leq \varphi(\psi(t_{k-1})) \text{ for all } k \geq 1,$$

hence

$$\psi(t_k) \leq \varphi(\psi(t_{k-1})) \text{ for all } k \geq 1,$$

so

$$\psi(t_k) - \psi(t_{k-1}) \leq \varphi(\psi(t_{k-1})) - \psi(t_{k-1}) \text{ for all } k \geq 1. \tag{2.4}$$

Taking into account the inequality (2.4), we obtain

$$\psi(t_m) = \psi(t_0) + \sum_{k=1}^{m}(\psi(t_k) - \psi(t_{k-1})) \leq \psi(t_0) + \sum_{k=1}^{m}(\varphi(\psi(t_{k-1})) - \psi(t_{k-1}))$$

$$= \psi(t_0) - \sum_{k=1}^{m}(\psi(t_{k-1}) - \varphi(\psi(t_{k-1}))) \text{ for all } m \geq 1. \tag{2.5}$$

On the other hand, $(t_m)_{m\in\mathbb{N}}$ is a monotonically decreasing sequence which converges to $l > 0$ and ψ is continuous, monotonically increasing and $\psi(t) = 0$ if and only if $t = 0$

(by Definition 2.6). Therefore, $(\psi(t_m))_{m\in\mathbb{N}}$ is a monotonically decreasing sequence which converges to $\psi(l) > 0$. Since φ satisfies the hypotheses of Lemma 2.3, and using the properties of the limit, we find

$$\lim_{k\to\infty}(\psi(t_{k-1}) - \varphi(\psi(t_{k-1}))) > 0.$$

It follows that

$$\sum_{k=1}^{\infty}(\psi(t_{k-1}) - \varphi(\psi(t_{k-1}))) = \infty. \tag{2.6}$$

From the relation (2.5), (2.6) we get $\psi(t_m) \to \infty$ as $m \to \infty$, which is in contradiction with $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. Therefore, $l = 0$, i.e. $t_m \to 0$ as $m \to \infty$. Also, from the relation (2.3), we deduce $t_m = \phi^m(t)$ for all $m \in \mathbb{N}$. Consequently, $\phi^m(t) \to 0$ as $m \to \infty$.

(ix) Follows from (vii), (viii), according to Definition 1.3. □

Our purpose is to investigate the existence and uniqueness of fixed points for operators defined on spaces endowed with such perturbed metrics. Next we establish a fixed point result on spaces with modified metrics.

Theorem 2.2. *Let us consider a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

1) *φ is monotonically increasing;*
2) *φ is right-continuous;*
3) *$\varphi(t) < t$ for all $t > 0$;*

and a function $\psi \in \Psi$. If (X, d) is a complete metric space and $T : X \to X$ an operator such that:

$$\psi(d(T(x), T(y))) \leq \varphi(\psi(d(x, y))),\ (\forall)x, y \in X, \tag{2.7}$$

then the following statements are true:

(i) *T is a ϕ-contraction, where the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by the relation (2.1);*
(ii) *T is a Picard operator.*

Proof. (i) We remark that the functions φ, ψ fulfill the hypotheses of Lemma 2.4. It follows that, we can consider the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the relation (2.1), which is a comparison function (according to Lemma 2.4 (ix)).

Let $x, y \in X$ be arbitrary elements. Since the operator $T : X \to X$ satisfies the inequality (2.7), we obtain

$$d(T(x), T(y)) \in \{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(d(x, y)))\},$$

hence

$$d(T(x), T(y)) \leq \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(d(x, y)))\} = \phi(d(x, y)).$$

Therefore, ϕ is a comparison function and the operator $T : X \to X$ verifies the inequality

$$d(T(x), T(y)) \leq \phi(d(x, y)),\ (\forall)x, y \in X, \tag{2.8}$$

which means that $T : X \to X$ is a ϕ-contraction (according to Definition 1.4).

(ii) As (X, d) is a complete metric space and T is a ϕ-contraction, applying Theorem 1.1 we deduce that T is a Picard operator. □

Further, the above result will be applied to continuous data dependence of the fixed points of Picard operators defined on spaces with perturbed metrics.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function. If

$$s - \phi(s) \to \infty \text{ as } s \to \infty, \tag{2.9}$$

we can define the function

$$\theta_\phi : \mathbb{R}_+ \to \mathbb{R}_+,\ \theta_\phi(t) = \sup\{s \in \mathbb{R}_+ \mid s - \phi(s) \le t\}. \tag{2.10}$$

We remark that θ_ϕ is monotonically increasing and $\theta_\phi(t) \to 0$ as $t \to 0$. The function θ_ϕ appears when we study the data dependence of the fixed points.

Theorem 2.3. *Let us consider a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

1) *φ is monotonically increasing;*
2) *φ is right-continuous;*
3) *$\varphi(t) < t$ for all $t > 0$;*

and a function $\psi \in \Psi$. Suppose that the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the relation (2.1) fulfills the hypothesis (2.9). If (X, d) is a complete metric space and $T : X \to X$ an operator such that:

$$\psi(d(T(x), T(y))) \le \varphi(\psi(d(x, y))),\ (\forall)x, y \in X, \tag{2.11}$$

then the following statements are valid:

(i) *T has a unique fixed point $x^* \in X$;*
(ii) *$d(x, x^*) \le \theta_\phi(d(x, T(x)))$, $(\forall)x \in X$;*
(iii) *if $U : X \to X$ is an operator verifying the conditions:*
   a) *$F_U \neq \emptyset$,*
   b) *there exists $\eta > 0$ such that $d(U(x), T(x)) \le \eta$, $(\forall)x \in X$,*

   then $d(y^*, x^*) \le \theta_\phi(\eta)$, $(\forall)y^* \in F_U$.

Proof. We remark that the hypotheses of Theorem 2.2 are fulfilled.

(i) By using Theorem 2.2 (ii) we find that T is a Picard operator, thus T has a unique fixed point $x^* \in X$.

(ii) Applying Theorem 2.2 (i) we get that T is a ϕ-contraction, hence

$$d(T(x), T(y)) \le \phi(d(x, y)),\ (\forall)x, y \in X.$$

Let $x \in X$ be an arbitrary element. Considering the properties of the metric d and the previous inequality we obtain

$$\begin{aligned} d(x, x^*) &\le d(x, T(x)) + d(T(x), x^*) \\ &= d(x, T(x)) + d(T(x), T(x^*)) \le d(x, T(x)) + \phi(d(x, x^*)), \end{aligned}$$

hence

$$d(x, x^*) - \phi(d(x, x^*)) \le d(x, T(x)),$$

thus

$$d(x, x^*) \in \{s \in \mathbb{R}_+ \mid s - \phi(s) \le d(x, T(x))\}.$$

Taking into account the definition of the function θ_ϕ (by relation (2.10)), from the previous relation we deduce

$$d(x, x^*) \le \sup\{s \in \mathbb{R}_+ \mid s - \phi(s) \le d(x, T(x))\} = \theta_\phi(d(x, T(x))).$$

(iii) Let $y^* \in F_U$ be an arbitrary fixed point of the operator U. From (ii), using the condition b) and the fact that θ_ϕ is monotonically increasing, it follows that

$$d(y^*, x^*) \le \theta_\phi(d(y^*, T(y^*))) = \theta_\phi(d(U(y^*), T(y^*))) \le \theta_\phi(\eta).$$

□

Definition 2.7. We say that a sequence of functions $f_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$, fulfills the hypothesis (H_0) if, for any sequence $(d_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfying $d_n - f_n(d_n) \to 0$ as $n \to \infty$, we have $d_n \to 0$ as $n \to \infty$.

Theorem 2.4. *Let us consider a sequence of functions $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$, satisfying the conditions:*

1) *φ_n is monotonically increasing;*
2) *φ_n is right-continuous;*
3) *$\varphi_n(t) < t$ for all $t > 0$;*

and a function $\psi \in \Psi$. Let (X, d) be a complete metric space and a sequence of operators $T_n : X \to X$, $n \in \mathbb{N}$, verifying the condition:

$$\psi(d(T_n(x), T_n(y))) \leq \varphi_n(\psi(d(x, y))),\ (\forall)x, y \in X. \tag{2.12}$$

Suppose that:

a) *the sequence of functions φ_n, $n \in \mathbb{N}$, converges uniformly to a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(t) \neq t$ for all $t > 0$;*
b) *the sequence of operators T_n, $n \in \mathbb{N}$, converges pointwise to an operator $T : X \to X$.*

Then, the following statements are true:

(i) *$\phi_n : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi_n(t) = \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi_n(\psi(t))\}$ is a comparison function, for every $n \in \mathbb{N}$;*
(ii) *T_n is a ϕ_n-contraction, for every $n \in \mathbb{N}$;*
(iii) *T_n is a Picard operator, for every $n \in \mathbb{N}$;*
(iv) *$\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi(t) = \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi(\psi(t))\}$ is a comparison function;*
(v) *T is a ϕ-contraction;*
(vi) *T is a Picard operator;*
(vii) *in the hypothesis that the function ψ is subadditive and the sequence of functions φ_n, $n \in \mathbb{N}$, fulfills the hypothesis (H_0), if x_n^*, $n \in \mathbb{N}$, x^* are, respectively, the unique fixed points of the operators T_n, $n \in \mathbb{N}$, T, then $x_n^* \to x^*$ as $n \to \infty$;*
(viii) *in the hypothesis that the sequence of functions ϕ_n, $n \in \mathbb{N}$, fulfills the hypothesis (H_0), if x_n^*, $n \in \mathbb{N}$, x^* are, respectively, the unique fixed points of the operators T_n, $n \in \mathbb{N}$, T, then $x_n^* \to x^*$ as $n \to \infty$.*

Proof. (i) Let $n \in \mathbb{N}$ be an arbitrary number. We remark that the functions φ_n, ψ fulfill the hypotheses of Lemma 2.4. Therefore, the function $\phi_n : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi_n(t) = \sup\{s \in \mathbb{R}_+ \mid \psi(s) \leq \varphi_n(\psi(t))\}$ is a comparison function (according to Lemma 2.4 (ix)).

(ii), (iii) Consider $n \in \mathbb{N}$. We see that the hypotheses of Theorem 2.2 are satisfied by the functions φ_n, ψ, the complete metric space (X, d) and the operator $T_n : X \to X$. Applying Theorem 2.2 (i) we deduce that T_n is a ϕ_n-contraction and by Theorem 2.2 (ii) we obtain that T_n is a Picard operator.

(iv) First, we show that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing. Let $t_1 < t_2$ be arbitrary points in $\mathbb{R}_+$. As $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$, are monotonically increasing, it follows that $\varphi_n(t_1) \leq \varphi_n(t_2)$, $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, in the previous inequality, and taking into account that the sequence of functions φ_n, $n \in \mathbb{N}$, converges uniformly to the function φ (according to the hypothesis a)), it follows that $\varphi(t_1) \leq \varphi(t_2)$, which means that φ is monotonically increasing.

Next, we prove that φ is right-continuous on $\mathbb{R}_+$. We choose $t_0 \in \mathbb{R}_+$ and $\varepsilon > 0$ be arbitrary numbers. As the sequence $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$, uniformly on $\mathbb{R}_+$ (by the hypothesis a)), it follows that for $\frac{\varepsilon}{3} > 0$ there exists a number $n(\varepsilon) \in \mathbb{N}$ (which does not depend on t) such that

$$|\varphi_n(t) - \varphi(t)| < \frac{\varepsilon}{3},\ (\forall)n \geq n(\varepsilon),\ (\forall)t \in \mathbb{R}_+. \tag{2.13}$$

Since $\varphi_{n(\varepsilon)}$ is right-continuous at t_0, we find that for $\frac{\varepsilon}{3} > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\varphi_{n(\varepsilon)}(t_0) - \frac{\varepsilon}{3} < \varphi_{n(\varepsilon)}(t) < \varphi_{n(\varepsilon)}(t_0) + \frac{\varepsilon}{3},\ (\forall)t \in (t_0, t_0 + \delta(\varepsilon)) \cap \mathbb{R}_+. \tag{2.14}$$

Considering the inequalities (2.13) and (2.14) we deduce that

$$\begin{aligned}|\varphi(t)-\varphi(t_0)| &= |\varphi(t)-\varphi_{n(\varepsilon)}(t)+\varphi_{n(\varepsilon)}(t)-\varphi_{n(\varepsilon)}(t_0)+\varphi_{n(\varepsilon)}(t_0)-\varphi(t_0)|\\ &\leq |\varphi(t)-\varphi_{n(\varepsilon)}(t)|+|\varphi_{n(\varepsilon)}(t)-\varphi_{n(\varepsilon)}(t_0)|+|\varphi_{n(\varepsilon)}(t_0)-\varphi(t_0)|\\ &= |\varphi_{n(\varepsilon)}(t)-\varphi(t)|+|\varphi_{n(\varepsilon)}(t)-\varphi_{n(\varepsilon)}(t_0)|+|\varphi_{n(\varepsilon)}(t_0)-\varphi(t_0)|\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\ (\forall)t\in(t_0,t_0+\delta(\varepsilon))\cap\mathbb{R}_+,\end{aligned}$$

hence $\varphi(t_0)-\varepsilon<\varphi(t)<\varphi(t_0)+\varepsilon$, $(\forall)t\in(t_0,t_0+\delta(\varepsilon))\cap\mathbb{R}_+$. As the number $\varepsilon>0$ was chosen arbitrarily, we find that for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\varphi(t_0)-\varepsilon<\varphi(t)<\varphi(t_0)+\varepsilon,\ (\forall)t\in(t_0,t_0+\delta(\varepsilon))\cap\mathbb{R}_+,$$

i.e. φ is right-continuous at t_0. Since the number $t_0\in\mathbb{R}_+$ was arbitrarily selected, we deduce that φ is right-continuous on $\mathbb{R}_+$.

Further, we demonstrate that $\varphi(t)<t$ for all $t>0$. From the hypothesis 3) we have $\varphi_n(t)<t$ for all $t>0$, $n\in\mathbb{N}$. Passing to the limit as $n\to\infty$, in the previous inequality, and taking into account that the sequence of functions φ_n, $n\in\mathbb{N}$, converges uniformly to the function φ (according the hypothesis a)), it folows that $\varphi(t)\leq t$ for all $t>0$. Moreover, by the hypothesis a) we have $\varphi(t)\neq t$ for all $t>0$. Therefore, $\varphi(t)<t$ for all $t>0$.

Consequently, we proved that the function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ has the following properties: φ is monotonically increasing, φ is right-continuous and $\varphi(t)<t$ for all $t>0$. Also, we have $\psi\in\Psi$. It follows that the functions φ,ψ fulfill the hypotheses of Lemma 2.4. Therefore, the function $\phi:\mathbb{R}_+\to\mathbb{R}_+$, $\phi(t)=\sup\{s\in\mathbb{R}_+ \mid \psi(s)\leq\varphi(\psi(t))\}$ is a comparison function (according to Lemma 2.4 (ix)).

(v), (vi) Let $x,y\in X$ be arbitrary elements. According to the hypothesis b), the sequence of operators T_n, $n\in\mathbb{N}$, converges pointwise to an operator $T:X\to X$. It results that $T_n(x)\to T(x)$ as $n\to\infty$ and $T_n(y)\to T(y)$ as $n\to\infty$. Since the metric $d:X\times X\to\mathbb{R}_+$ is a continuous function, we deduce that $d(T_n(x),T_n(y))\to d(T(x),T(y))$ as $n\to\infty$. Also, the function ψ being a continuous function (according to Definition 2.6 (i)), the previous property implies

$$\psi(d(T_n(x),T_n(y)))\to\psi(d(T(x),T(y)))\text{ as }n\to\infty. \tag{2.15}$$

On the other hand, by the hypothesis a), the sequence of functions φ_n, $n\in\mathbb{N}$, converges uniformly to a function φ. It follows that

$$\varphi_n(\psi(d(x,y)))\to\varphi(\psi(d(x,y)))\text{ as }n\to\infty. \tag{2.16}$$

Passing to the limit as $n\to\infty$, in the inequality (2.12), and taking into account the relations (2.15), (2.16), we obtain $\psi(d(T(x),T(y)))\leq\varphi(\psi(d(x,y)))$. As the elements $x,y\in X$ were chosen arbitrarily, we find that

$$\psi(d(T(x),T(y)))\leq\varphi(\psi(d(x,y))),\ (\forall)x,y\in X. \tag{2.17}$$

From the proof of the statement (iv) we have that the function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ has the following properties: φ is monotonically increasing, φ is right-continuous and $\varphi(t)<t$ for all $t>0$. Also, we have $\psi\in\Psi$. It follows that the hypotheses of Theorem 2.2 are satisfied by the functions φ,ψ, the complete metric space (X,d) and the operator $T:X\to X$. Applying Theorem 2.2 (i) we deduce that T is a ϕ-contraction and by Theorem 2.2 (ii) we obtain that T is a Picard operator.

(vii) Let $n\in\mathbb{N}$ be an arbitrary number. As x_n^*, x^* are, respectively, the unique fixed points of the operators T_n, T, using the properties of the metric d we get

$$d(x_n^*,x^*)=d(T_n(x_n^*),T(x^*))\leq d(T_n(x_n^*),T_n(x^*))+d(T_n(x^*),T(x^*)). \tag{2.18}$$

Considering that ψ is monotonically increasing (according to Definition 2.6 (ii)), ψ is subadditive (by the hypothesis of the statement (vii)) and using the inequality (2.12) for $x := x_n^*$, $y := x^*$, from the relation (2.18) we deduce

$$\begin{aligned}\psi(d(x_n^*, x^*)) &\leq \psi(d(T_n(x_n^*), T_n(x^*)) + d(T_n(x^*), T(x^*)))\\ &\leq \psi(d(T_n(x_n^*), T_n(x^*))) + \psi(d(T_n(x^*), T(x^*)))\\ &\leq \varphi_n(\psi(d(x_n^*, x^*))) + \psi(d(T_n(x^*), T(x^*))),\end{aligned}$$

hence

$$\psi(d(x_n^*, x^*)) - \varphi_n(\psi(d(x_n^*, x^*))) \leq \psi(d(T_n(x^*), T(x^*))). \tag{2.19}$$

On the other hand, the function φ_n fulfills the hypotheses of Lemma 2.4, thus $\varphi_n(0) = 0$ (according to Lemma 2.4 (i)). Also, by the hypothesis 3), $\varphi_n(t) < t$ for all $t > 0$. Therefore, $\varphi_n(t) \leq t$ for all $t \in \mathbb{R}_+$. Considering that $d(x_n^*, x^*) \in \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we obtain $\psi(d(x_n^*, x^*)) \in \mathbb{R}_+$, hence $\varphi_n(\psi(d(x_n^*, x^*))) \leq \psi(d(x_n^*, x^*))$, thus

$$0 \leq \psi(d(x_n^*, x^*)) - \varphi_n(\psi(d(x_n^*, x^*))). \tag{2.20}$$

Combining the inequalities (2.19), (2.20) and taking into account that the number $n \in \mathbb{N}$ was arbitrarily selected, we find

$$0 \leq \psi(d(x_n^*, x^*)) - \varphi_n(\psi(d(x_n^*, x^*))) \leq \psi(d(T_n(x^*), T(x^*))),\ (\forall) n \in \mathbb{N}. \tag{2.21}$$

According to the hypothesis b), the sequence of operators T_n, $n \in \mathbb{N}$, converges pointwise to an operator $T : X \to X$. It results that $T_n(x^*) \to T(x^*)$ as $n \to \infty$, hence $d(T_n(x^*), T(x^*)) \to 0$ as $n \to \infty$. Since the function ψ is continuous (by Definition 2.6 (i)) and $\psi(0) = 0$ (by Definition 2.6 (iii)), we get

$$\psi(d(T_n(x^*), T(x^*))) \to \psi(0) = 0 \text{ as } n \to \infty. \tag{2.22}$$

Passing to the limit as $n \to \infty$, in the inequality (2.21), and considering the relation (2.22), we deduce

$$\psi(d(x_n^*, x^*)) - \varphi_n(\psi(d(x_n^*, x^*))) \to 0 \text{ as } n \to \infty. \tag{2.23}$$

As $(\psi(d(x_n^*, x^*))_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$ and the sequence of functions φ_n, $n \in \mathbb{N}$, fulfills the hypothesis (H_0) (according to the hypothesis of the statement (vii)), from the relation (2.23) we obtain

$$\psi(d(x_n^*, x^*)) \to 0 \text{ as } n \to \infty. \tag{2.24}$$

Taking into account that ψ is continuous (by Definition 2.6 (i)) and $\psi(t) = 0$ if and only if $t = 0$ (by Definition 2.6 (iii)), the relation (2.24) implies $d(x_n^*, x^*) \to 0$ as $n \to \infty$, hence $x_n^* \to x^*$ as $n \to \infty$.

(viii) Choose $n \in \mathbb{N}$. As x_n^*, x^* are, respectively, the unique fixed points of the operators T_n, T, using the properties of the metric d and the fact that T_n is a ϕ_n-contraction (according to (ii)), we get

$$\begin{aligned}d(x_n^*, x^*) = d(T_n(x_n^*), T(x^*)) &\leq d(T_n(x_n^*), T_n(x^*)) + d(T_n(x^*), T(x^*))\\ &\leq \phi_n(d(x_n^*, x^*)) + d(T_n(x^*), T(x^*)),\end{aligned}$$

hence

$$d(x_n^*, x^*) - \phi_n(d(x_n^*, x^*)) \leq d(T_n(x^*), T(x^*)). \tag{2.25}$$

Because the functions φ_n, ψ fulfill the hypotheses of Lemma 2.4, it follows that $\phi_n(t) \leq t$ for all $t \in \mathbb{R}_+$ (according to Lemma 2.4 (iv)). Considering that $d(x_n^*, x^*) \in \mathbb{R}_+$ we obtain $\phi_n(d(x_n^*, x^*)) \leq d(x_n^*, x^*)$, thus

$$0 \leq d(x_n^*, x^*) - \phi_n(d(x_n^*, x^*)). \tag{2.26}$$

Combining the inequalities (2.25), (2.26) and taking into account that the number $n \in \mathbb{N}$ was arbitrarily selected, we find

$$0 \le d(x_n^*, x^*) - \phi_n(d(x_n^*, x^*)) \le d(T_n(x^*), T(x^*)),\ (\forall) n \in \mathbb{N}. \tag{2.27}$$

According to the hypothesis b), the sequence of operators T_n, $n \in \mathbb{N}$, converges pointwise to an operator $T : X \to X$. It results that $T_n(x^*) \to T(x^*)$ as $n \to \infty$, hence

$$d(T_n(x^*), T(x^*)) \to 0 \text{ as } n \to \infty. \tag{2.28}$$

Passing to the limit as $n \to \infty$, in the inequality (2.27), and considering the relation (2.28), we get

$$d(x_n^*, x^*) - \phi_n(d(x_n^*, x^*)) \to 0 \text{ as } n \to \infty. \tag{2.29}$$

As the sequence of functions ϕ_n, $n \in \mathbb{N}$, fulfills the hypothesis (H_0) (according to the hypothesis of the statement (viii)), from relation (2.29) we deduce $d(x_n^*, x^*) \to 0$ as $n \to \infty$, hence $x_n^* \to x^*$ as $n \to \infty$. □

3. An application to epidemiological models

In the following, we will apply Theorem 2.2 to study the existence and uniqueness of the solution for the integral equation

$$x(t) = [g_1(t) + \int_a^t K_1(t, s, x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t, s, x(s))ds],\ t \in [a, b], \tag{3.30}$$

under certain hypotheses about functions g_i, K_i, $i = \overline{1, 2}$. We note that some particular cases of the equation (3.30) were considered by Griepenberg [3] and Brestovanska [1] to investigate the spread of an infectious disease.

Theorem 3.5. *We suppose that the functions g_i, K_i, $i = \overline{1, 2}$, fulfill the conditions:*

(i) *$g_i \in C([a, b], \mathbb{R})$, $K_i \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, $i = \overline{1, 2}$;*

(ii) *there are $M_{K_i} > 0$, $i = \overline{1, 2}$, such that*

$$|K_i(t, s, u)| \le M_{K_i},\ \text{for all}\ t, s \in [a, b],\ u \in \mathbb{R},\ i = \overline{1, 2};$$

(iii) *if we denote by $M_{g_i} = \max\limits_{t \in [a,b]} |g_i(t)|$, $i = \overline{1, 2}$, then*

$$(M_{g_1} + M_{K_1}(b - a))^2 + (M_{g_2} + M_{K_2}(b - a))^2 \le \frac{1}{2(b - a)^2};$$

(iv) *there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions: φ is monotonically increasing, φ is right-continuous, $\varphi(t) < t$ for all $t > 0$, such that*

$$|K_i(t, s, u) - K_i(t, s, v)|^2 \le \varphi(|u - v|^2),\ \text{for all}\ t, s \in [a, b],\ u, v \in \mathbb{R},\ i = \overline{1, 2}.$$

Then the equation (3.30) has a unique solution $x^* \in C([a, b], \mathbb{R})$.

Proof. We consider the linear space $C([a, b], \mathbb{R}) := \{x : [a, b] \to \mathbb{R} \mid \text{x is continuous on } [a, b]\}$, endowed with the maximum norm $\|\cdot\|_\infty : C([a, b], \mathbb{R}) \to \mathbb{R}_+$, $\|x\|_\infty = \max\limits_{t \in [a,b]} |x(t)|$, and the the derived metric $d_\infty : C([a, b], \mathbb{R}) \times C([a, b], \mathbb{R}) \to \mathbb{R}_+$, $d_\infty(x, y) = \|x - y\|_\infty$. It is well known that $(C([a, b], \mathbb{R}), d_\infty)$ is a complete metric space.

Since the functions g_i, K_i, $i = \overline{1, 2}$, satisfy the condition (i), we can define the operator $A : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$,

$$A(x)(t) = [g_1(t) + \int_a^t K_1(t, s, x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t, s, x(s))ds],\ t \in [a, b].$$

We remark that, a function $x \in C([a,b],\mathbb{R})$ is a solution of the equation (3.30) if and only if it is a fixed point of the operator A.

For every $x, y \in C([a,b],\mathbb{R})$, $t \in [a,b]$, we have successively:

$$\begin{aligned}
&|A(x)(t) - A(y)(t)| \\
&= |[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds] \\
&\quad -[g_1(t) + \int_a^t K_1(t,s,y(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds]| \\
&= |[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds] \\
&\quad -[g_1(t) + \int_a^t K_1(t,s,x(s)]ds) \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds] \\
&\quad +[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds] \\
&\quad -[g_1(t) + \int_a^t K_1(t,s,y(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds]| \\
&= |[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [\int_a^t (K_2(t,s,x(s)) - K_2(t,s,y(s)))ds] \\
&\quad +[g_2(t) + \int_a^t K_2(t,s,y(s))ds] \cdot [\int_a^t (K_1(t,s,x(s)) - K_1(t,s,y(s)))ds]| \\
&\leq [|g_1(t)| + \int_a^t |K_1(t,s,x(s))|ds] \cdot \int_a^t |K_2(t,s,x(s)) - K_2(t,s,y(s))|ds \\
&\quad +[|g_2(t)| + \int_a^t |K_2(t,s,y(s))|ds] \cdot \int_a^t |K_1(t,s,x(s)) - K_1(t,s,y(s))|ds.
\end{aligned} \tag{3.31}$$

Considering the condition (ii) we obtain, for all $x, y \in C([a,b],\mathbb{R})$, $t \in [a,b]$:

(3.32)

$$|g_1(t)| + \int_a^t |K_1(t,s,x(s))|ds \leq M_{g_1} + \int_a^t M_{K_1} ds = M_{g_1} + M_{K_1}(t-a) \leq M_{g_1} + M_{K_1}(b-a)$$

and

(3.33)

$$|g_2(t)| + \int_a^t |K_2(t,s,y(s))|ds \leq M_{g_2} + \int_a^t M_{K_2} ds = M_{g_2} + M_{K_2}(t-a) \leq M_{g_2} + M_{K_2}(b-a).$$

Combining the relations (3.31), (3.32) and (3.33), for every $x, y \in C([a,b],\mathbb{R})$, $t \in [a,b]$, we deduce:

$$\begin{aligned}
&|A(x)(t) - A(y)(t)| \\
&\leq (M_{g_1} + M_{K_1}(b-a)) \cdot \int_a^t |K_2(t,s,x(s)) - K_2(t,s,y(s))|ds \\
&\quad +(M_{g_2} + M_{K_2}(b-a)) \cdot \int_a^t |K_1(t,s,x(s)) - K_1(t,s,y(s))|ds.
\end{aligned}$$

Taking into account the inequality $(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ (for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$) and using the condition (iii), from the previous inequality, for all $x, y \in C([a,b],\mathbb{R})$,

$t \in [a,b]$, we find:

$$|A(x)(t) - A(y)(t)|^2$$

$$\leq [(M_{g_1} + M_{K_1}(b-a)) \cdot \int_a^t |K_2(t,s,x(s)) - K_2(t,s,y(s))|ds$$

$$+(M_{g_2} + M_{K_2}(b-a)) \cdot \int_a^t |K_1(t,s,x(s)) - K_1(t,s,y(s))|ds]^2$$

$$\leq [(M_{g_1} + M_{K_1}(b-a))^2 + (M_{g_2} + M_{K_2}(b-a))^2]$$

$$\cdot[(\int_a^t |K_1(t,s,x(s)) - K_1(t,s,y(s))|ds)^2 + (\int_a^t |K_2(t,s,x(s)) - K_2(t,s,y(s))|ds)^2]$$

(3.34)

$$\leq \frac{1}{2(b-a)^2} \cdot [(\int_a^t |K_1(t,s,x(s)) - K_1(t,s,y(s))|ds)^2 + (\int_a^t |K_2(t,s,x(s)) - K_2(t,s,y(s))|ds)^2].$$

Considering the Cauchy–Schwarz inequality, the condition (iv) and the fact that φ is a monotonically increasing function (being a comparison function), we get, for all $x, y \in C([a,b],\mathbb{R})$, $t \in [a,b]$, $i = \overline{1,2}$:

$$(\int_a^t |K_i(t,s,x(s)) - K_i(t,s,y(s))|ds)^2 = (\int_a^t |K_i(t,s,x(s)) - K_i(t,s,y(s))| \cdot 1 ds)^2$$

$$\leq \int_a^t |K_i(t,s,x(s)) - K_i(t,s,y(s))|^2 ds \cdot \int_a^t 1^2 ds$$

$$\leq \int_a^t \varphi(|x(s) - y(s)|^2) ds \cdot (t-a) \leq \int_a^t \varphi((\max_{s \in [a,b]} |x(s) - y(s)|)^2) ds \cdot (t-a)$$

$$= \int_a^t \varphi(\|x-y\|_\infty^2) ds \cdot (t-a) = \varphi(\|x-y\|_\infty^2) \cdot \int_a^t ds \cdot (t-a)$$

$$= \varphi(\|x-y\|_\infty^2) \cdot (t-a)^2 \leq \varphi(\|x-y\|_\infty^2) \cdot (b-a)^2 = \varphi(d_\infty^2(x,y)) \cdot (b-a)^2. \tag{3.35}$$

From the inequalities (3.34) and (3.35), for every $x, y \in C([a,b],\mathbb{R})$, $t \in [a,b]$, we obtain:

$$|A(x)(t) - A(y)(t)|^2$$

$$\leq \frac{1}{2(b-a)^2} \cdot (\varphi(d_\infty^2(x,y)) \cdot (b-a)^2 + \varphi(d_\infty^2(x,y)) \cdot (b-a)^2) = \varphi(d_\infty^2(x,y)),$$

hence

$$d_\infty^2(A(x),A(y)) = \|A(x) - A(y)\|_\infty^2 = (\max_{t \in [a,b]} |A(x)(t) - A(y)(t)|)^2 =$$

$$= \max_{t \in [a,b]} (|A(x)(t) - A(y)(t)|^2) \leq \varphi(d_\infty^2(x,y)).$$

Considering the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+,\ \psi(t) = t^2$, we remark that $\psi \in \Psi$ and the previous inequality can be written as

$$\psi(d_\infty(A(x),A(y))) \leq \varphi(\psi(d_\infty(x,y))),\ (\forall) x, y \in C([a,b],\mathbb{R}). \tag{3.36}$$

Consequently, $(C([a,b],\mathbb{R}),d_\infty)$ is a complete metric space, $\psi \in \Psi$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying the conditions: φ is monotonically increasing, φ is right-continuous, $\varphi(t) < t$ for all $t > 0$ and $A : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is an operator which satisfies the inequality (3.36). It follows that the hypotheses of Theorem 2.2 are fulfilled, therefore A is a Picard operator, hence the equation (3.30) has a unique solution $x^* \in C([a,b],\mathbb{R})$. $\square$

We note that the particular case for which there are $L_{K_i} > 0$, $i = \overline{1,2}$, such that

$$|K_i(t,s,u) - K_i(t,s,v)| \le L_{K_i}|u-v|, \text{ for all } t,s \in [a,b],\ u,v \in \mathbb{R},\ i = \overline{1,2},$$

was investigated by Olaru [7].

4. Conclusions

In this study, we define the concept of perturbed metric within the metric spaces and we obtained a fixed point result for self-mappings satisfying such contractive conditions. The established theorem generalize some results presented in the literature for φ-contractions. Further, the main theorem was applied to continuous data dependence of the fixed points of Picard operators defined on spaces with perturbed metrics. Finally, an application to the study of the existence and uniqueness of the solution for an integral equation, which models an epidemiological problem, was presented in the last part of the paper.

Acknowledgement. The authors thank the referees and the editors for their suggestions which improved the quality of this paper. Project financed by Lucian Blaga University of Sibiu & Hasso Plattner Foundation research grants LBUS-IRG-2021-07.

References

[1] Brestovanska, E. Qualitative behaviour of an integral equation related to some epidemic model. *Demonstr. Math.* **36** (2003), no. 3, 603–609.
[2] Caliò, F.; Marchetti, E.; Mureşan, V. On some Volterra-Fredholm integral equations. *Int. J. Pure Appl. Math.* **31** (2006), no. 2, 173–184.
[3] Gripenberg, G. On some epidemic models. *Quart. Appl. Math.* **39** (1981/82), no. 3, 317–327.
[4] Matkowski, J. Integrable solutions of functional equations. *Dissertationes Math.* **127** (1975), 1–68.
[5] Mureşan, V. A Volterra functional-integral equation, via weakly Picard operators' technique. *Carpathian J. Math.* **30** (2014), no. 3, 369–375.
[6] Mureşan, A. S.; Mureşan, V. Implicit functional differential equations with linear modification of the argument, via weakly Picard operator theory. *Carpathian J. Math.* **37** (2021) no. 2, 227–234.
[7] Olaru, I. M. An integral equation via weakly Picard operators. *Fixed Point Theory* **11** (2010), no. 1, 97–106.
[8] Royden, H. L.; Fitzpatrick, P. M. *Real analysis*. China Machine Press, Beijing, 2009.
[9] Rus, I. A. Weakly Picard mappings. *Comment. Math. Univ. Carolin.* **34** (1993), no. 4, 769–773.
[10] Rus I. A. Functional-differential equations of mixed type, via weakly Picard operators. *Semin. Fixed Point Theory* **3** (2002), 335–346.
[11] Rus, I. A. *Generalized contractions and applications*. Cluj University Press, Cluj-Napoca, 2001.
[12] Wang, J.; Zhoub, Y.; Medveď, M. Picard and weakly Picard operators technique for nonlinear differential equations in Banach spaces. *J. Math. Anal. Appl.* **389** (2012), no. 1, 261–274.

Department of Mathematics and Informatics
Lucian Blaga University of Sibiu
Dr. I. Raţiu Street no. 5-7, 550012 Sibiu, Romania
Email address: adrian.branga@ulbsibiu.ro
Email address: marian.olaru@ulbsibiu.ro