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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

On certain boundary value problems associated to some fractional integro-differential inclusions

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ABSTRACT. Two classes of fractional integro-differential inclusions with certain boundary conditions are studied. The existence of solutions is established in the case when the set-valued map has nonconvex values.

1. INTRODUCTION

In this note we are considering the following boundary value problems. First we consider a fractional integro-differential inclusion defined by Caputo fractional derivative

$$D_C^{\beta_2}x(t) \in F(t, x(t), V(x)(t)) \quad a.e.\ ([0,1]) \tag{1.1}$$

with boundary conditions of the form

$$x(\nu) = x'(0) = ... = x^{(n-2)}(0) = 0, \quad I^{\beta_1}x(1) = 0, \tag{1.2}$$

where D_C^q is the Caputo fractional derivative of order q, $\beta_1 > 0$, $n-1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbf{N}$, $\nu \in (0,1)$, I^p is the Riemann-Liouville fractional integral of order p, $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $V : C([0,1], \mathbf{R}) \to C([0,1], \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function. We note that the fractional derivative introduced by Caputo in [3] and afterwards adopted in the theory of linear visco elasticity allows to use Cauchy conditions which have physical meanings.

Next we consider the problem

$$D_H^{\alpha,\beta}x(t) \in F(t, x(t), V(x)(t)) \quad a.e.\ ([0,1]) \tag{1.3}$$

with boundary conditions of the form

$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i), \tag{1.4}$$

where $D_H^{\alpha,\beta}$ is the Hilfer fractional derivative of order $\alpha \in (1,2)$ and type $\beta \in [0,1]$, $0 < \xi_i < 1$, $\delta_i \in \mathbf{R}$, $\varphi_i > 0$, $i = 1, 2, ..., m$, F and V are as above.

Our study is motivated by some recent papers. Namely, in [16] an existence result for problem (1.1)-(1.2) may be found in the case when F does not depends on the last variable and is upper semicontinuous with compact convex values. Also, in the case when F does not depends on the last variable several existence results for problem (1.3)-(1.4) are provided in [17]. All the results in [16, 17] are proved by using several suitable theorems from fixed point theory.

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Our goal is to obtain the existence of solutions for problems (1.1)-(1.2) and (1.3)-(1.4) in the case when the set-valued map F has nonconvex values but it is assumed to be Lipschitz in the second and third variable. Our results use Filippov's techniques ([9]); namely, the existence of solutions is obtained by starting from a given "quasi" solution. In addition, the result provides an estimate between the "quasi" solution and the solution obtained.

Our results extend or improve some existence theorems in [16, 17] in the case when the right-hand side is Lipschitz in the second variable as one can see later. Moreover, these results may be regarded as generalizations to the case when the right-hand side contains a nonlinear Volterra integral operator. Even if the method we use here is known in the theory of differential inclusions (e.g., [4, 5, 6, 7] etc.) it is largely ignored by the authors that are dealing with such problems in favor of fixed point approaches, most probably, because it is much easier to handle the applications of classical fixed point theorems.

Finally, we recall that the recent literature is full of motivations for considering systems defined by fractional order derivatives (see [2, 8, 12, 14, 15] etc.).

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I = [0,1]$, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$.

The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$.

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n = [\alpha]+1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)\mathrm{d}s,$$

where $n = [\alpha]+1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

A generalization of both Riemann-Liouville and Caputo derivatives was introduced by Hilfer in [10].

The Hilfer fractional derivative of order $\alpha \in (n-1, n)$ and type $\beta \in [0,1]$ of a function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$D_H^{\alpha,\beta} f(t) = I^{\beta(n-\alpha)} \frac{d^n}{dt^n} I^{(1-\beta)(n-\alpha)} f(t)$$

In fact, this derivative interpolates between Riemann-Liouville and Caputo derivatives. When $\beta = 0$ the Hilfer fractional derivative gives Riemann-Liouville fractional derivative $D_H^{\alpha,0} f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t)$ and when $\beta = 1$ the Hilfer fractional derivative gives Caputo fractional derivative $D_H^{\alpha,1} f(t) = I^{n-\alpha} \frac{d^n}{dt^n} f(t)$. Several properties and applications of Hilfer fractional derivative may be found in [11].

Lemma 2.1. ([13]) *Let $\nu \in (0,1)$ with $\nu^{n-1} \neq \Gamma(n)/(\beta_1 + n - 1)...(\beta_1 + 1)$, $\beta_1 > 0$, $n - 1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbf{N}$ and $h(.) \in C(I, \mathbf{R})$. Then, the solution of problem $D_C^{\beta_2} x(t) = h(t)$ with boundary conditions (1.2) is given by*

$$x(t) = \frac{1}{\Gamma(\beta_2)} \int_0^t (t-s)^{\beta_2 - 1} h(s) ds - \frac{1}{\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} h(s) ds + (\nu^{n-1} - t^{n-1}) Q [\frac{1}{\Gamma(\beta_1 + \beta_2)} \int_0^1 (1-s)^{\beta_1 + \beta_2 - 1} h(s) ds - \frac{1}{\Gamma(\beta_1 + 1)\Gamma(\beta_2)} \int_0^\nu (\nu - s)^{\beta_2 - 1} h(s) ds], \tag{2.5}$$

where $Q = \Gamma(\beta_1 + n)/[\Gamma(n) - \nu^{n-1}(\beta_1 + n - 1)...(\beta_1 + 1)]$.

By definition a function $x(.) \in C(I, \mathbf{R})$ is called a solution of a problem (1.1)-(1.2) if there exists $h(.) \in L^1(I, \mathbf{R})$ such that $h(t) \in F(t, x(t), V(x)(t))$ *a.e.* (I) and $x(.)$ is given by (2.5)

Remark 2.1. If we denote

$$G_1(t,s) = (\nu^{n-1} - t^{n-1}) Q \frac{1}{\Gamma(\beta_1 + \beta_2)} (1-s)^{\beta_1 + \beta_2 - 1} + \frac{1}{\Gamma(\beta_2)} (t-s)^{\beta_2 - 1} \chi_{[0,t]}(s) - \frac{1}{\Gamma(\beta_2)} (\nu - s)^{\beta_2 - 1} \chi_{[0,\nu]}(s) [1 + \frac{(\nu^{n-1} - t^{n-1}) Q}{\Gamma(\beta_1 + 1)}],$$

where $\chi_A(.)$ denotes the characteristic function of the set A, then the solution $x(.)$ in (2.5) may be put as $x(t) = \int_0^1 G_1(t,s) h(s) ds$.

Moreover, for any $t, s \in I$

$$|G_1(t,s)| \leq \frac{(1 + \nu^{n-1})|Q|}{\Gamma(\beta_1 + \beta_2)} + \frac{1}{\Gamma(\beta_2)} + \frac{\nu^{\beta_2 - 1}}{\Gamma(\beta_2)} [1 + \frac{(1 + \nu^{n-1})|Q|}{\Gamma(\beta_1 + 1)}] =: M_1$$

Lemma 2.2. ([17]) *Let $\alpha \in (1,2)$, $\beta \in [0,1]$, $\gamma = \alpha + 2\beta - \alpha\beta$, $0 < \xi_i < 1$, $\delta_i \in \mathbf{R}$, $\varphi_i > 0$, $i = 1, 2, ..., m$ with $\Lambda = \sum_{i=1}^m \frac{\delta_i \xi_i^{\gamma + \varphi_i - 1}}{\Gamma(\gamma + \varphi_i)} - \frac{1}{\Gamma(\gamma)} \neq 0$ and $h(.) \in C(I, \mathbf{R})$. Then, the solution of problem $D_H^{\alpha,\beta} x(t) = h(t)$ with boundary conditions (1.4) is given by*

$$x(t) = \frac{t^{\gamma - 1}}{\Lambda \Gamma(\gamma)} (I^\alpha h(1) - \sum_{i=1}^m \delta_i I^{\alpha + \varphi_1} h(\xi_i)) + I^\alpha h(t). \tag{2.6}$$

By definition a function $x(.) \in C(I, \mathbf{R})$ is called a solution of a problem (1.3)-(1.4) if there exists $h(.) \in L^1(I, \mathbf{R})$ such that $h(t) \in F(t, x(t), V(x)(t))$ *a.e.* (I) and $x(.)$ is given by (2.6)

Remark 2.2. If we denote

$$G_2(t,s) = \frac{t^{\gamma - 1}(1-s)^{\alpha - 1}}{\Lambda \Gamma(\gamma) \Gamma(\alpha)} - \sum_{i=1}^m \frac{t^{\gamma - 1}(\xi_i - s)^{\alpha + \varphi_i - 1} \delta_i \chi_{[0,\xi_i]}(s)}{\Lambda \Gamma(\gamma) \Gamma(\alpha + \varphi_i)} + \frac{(t-s)^{\alpha - 1} \chi_{[0,t]}(s)}{\Gamma(\alpha)}$$

then the solution $x(.)$ in (2.6) may be set as $x(t) = \int_0^1 G_2(t,s) h(s) ds$.

At the same time, for any $t, s \in I$

$$|G_2(t,s)| \leq \frac{1}{\Gamma(\alpha)}(\frac{1}{|\Lambda|\Gamma(\gamma)}+1)+\sum_{i=1}^{m}\frac{\xi_i^{\alpha+\varphi_i-1}|\delta_i|}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+\varphi_i)} =: M_2.$$

3. MAIN RESULTS

We need a variant of Kuratowski and Ryll-Nardzewski selection theorem concerning measurable set-valued maps.

Lemma 3.3. ([1]) *Consider X a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In order to prove our results we need the following hypotheses.

Hypothesis H1. i) *$F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}\times\mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I, F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leq L(t)(|x_1-x_2|+|y_1-y_2|) \quad \forall\ x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

iii) *$k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R},\ (t,s) \to k(t,s,x)$ is measurable.*

iv) *$|k(t,s,x) - k(t,s,y)| \leq L(t)|x-y| \quad a.e.\ (t,s) \in I \times I, \quad \forall x,y \in \mathbf{R}$.*

We use next the following notations

$$M(t) := L(t)(1+\int_0^t L(u)du), \quad t \in I, \quad K_0 = \int_0^T M(t)dt.$$

Theorem 3.1. *Let $\nu \in (0,1)$ with $\nu^{n-1} \neq \Gamma(n)/(\beta_1+n-1)...(\beta_1+1)$, $\beta_1 > 0, n-1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbf{N}$. Assume that Hypothesis H1 is satisfied and $M_1K_0 < 1$. Let $y(.) \in C(I,\mathbf{R})$ be such that $y(\nu) = y'(0) = ... = y^{(n-2)}(0) = 0$, $I^{\beta_1}y(1) = 0$ and there exists $p(.) \in L^1(I,\mathbf{R}_+)$ with $d(D_C^{\beta_2}y(t), F(t,y(t),V(y)(t))) \leq p(t)$ a.e. (I).*

Then there exists $x(.) : I \to \mathbf{R}$ a solution of problem (1.1)-(1.2) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M_1}{1-M_1K_0}||p(.)||_1.$$

Proof. The set-valued map $t \to F(t,y(t),V(y)(t))$ is measurable with closed values and

$$F(t,y(t),V(y)(t)) \cap \{D_C^{\beta_2}y(t) + p(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I).$$

From Lemma 3.3 there exists a measurable selection $h_1(t) \in F(t,y(t),V(y)(t))$ $a.e.$ (I) such that

$$|h_1(t) - D_C^{\beta_2}y(t)| \leq p(t) \quad a.e.\ (I) \tag{3.7}$$

Define $x_1(t) = \int_0^1 G_1(t,s)h_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq M_1\int_0^1 p(t)dt.$$

We construct two sequences $x_n(.) \in C(I,\mathbf{R})$, $h_n(.) \in L^1(I,\mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 G_1(t,s)h_n(s)ds, \quad t \in I, \tag{3.8}$$

$$h_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\,(I), \tag{3.9}$$

$$|h_{n+1}(t) - h_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|ds)\, a.e.\,(I) \tag{3.10}$$

If this is done, then from (3.7)-(3.10) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq M_1(M_1K_0)^n \int_0^1 p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$\begin{aligned}&|x_{n+1}(t) - x_n(t)| \leq \textstyle\int_0^1 |G_1(t,t_1)|.|h_{n+1}(t_1) - h_n(t_1)|dt_1 \leq \\ &M_1 \textstyle\int_0^1 L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} L(s)|x_n(s) - x_{n-1}(s)|ds]dt_1 \leq \\ &M_1 \textstyle\int_0^1 L(t_1)(1 + \int_0^{t_1} L(s)ds)dt_1.M_1^n K_0^{n-1} \int_0^1 p(t)dt = M_1(M_1K_0)^n \int_0^1 p(t)dt.\end{aligned}$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbf{R})$. Hence, by (3.10), for almost all $t \in I$, the sequence $\{h_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $h(.)$ be the pointwise limit of $h_n(.)$.

At the same time, one has

$$\begin{aligned}&|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq M_1 \int_0^1 p(t)dt + \\ &\textstyle\sum_{i=1}^{n-1} (M_1 \int_0^1 p(t)dt)(M_1K_0)^i \leq \frac{M_1 \int_0^1 p(t)dt}{1 - M_1K_0}.\end{aligned} \tag{3.11}$$

On the other hand, from (3.7), (3.10) and (3.11) we obtain for almost all $t \in I$

$$|h_n(t) - D_C^{\beta_2} y(t)| \leq \sum_{i=1}^{n-1} |h_{i+1}(t) - h_i(t)| + |h_1(t) - D_C^{\beta_2} y(t)| \leq L(t)\frac{M_1 \int_0^1 p(t)dt}{1 - M_1K_0} + p(t).$$

Hence the sequence $h_n(.)$ is integrably bounded and therefore $h(.) \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.8), (3.9) we deduce that $x(.)$ is a solution of (1.1)-(1.2). Finally, passing to the limit in (3.11) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), h_n(.)$ with the properties in (3.8)-(3.10). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,\mathbf{R})$ and $h_n(.) \in L^1(I,\mathbf{R})$, $n = 1, 2, ...N$ satisfying (3.8), (3.10) for $n = 1, 2, ...N$ and (3.9) for $n = 1, 2, ...N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$F(t, x_N(t), V(x_N)(t)) \cap \{h_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \textstyle\int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1,1]\} \neq \emptyset.$$

Lemma 3.3 yields that there exist a measurable selection $h_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|h_{N+1}(t) - h_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.8) with $n = N+1$. Thus $f_{N+1}(.)$ satisfies (3.9) and (3.10) and the proof is complete. □

Corollary 3.1. *Let $\nu \in (0,1)$ with $\nu^{n-1} \neq \Gamma(n)/(\beta_1+n-1)...(\beta_1+1)$, $\beta_1 > 0, n-1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbf{N}$. Assume that Hypothesis H1 is satisfied, $d(0, F(t,0,0) \leq L(t)$ a.e. (I) and $M_1 K_0 < 1$. Then there exists $x(.)$ a solution of problem (1.1)-(1.2) satisfying for all $t \in I$ $|x(t)| \leq \frac{M_1}{1-M_1K_0}||L(.)||_1$.*

Proof. It is enough to take $y(.) = 0$ and $p(.) = L(.)$ in Theorem 3.1. □

If F does not depend on the last variable, Hypothesis H1 becames

Hypothesis H2. i) *$F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I, (0,\infty))$ such that, for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1), F(t,x_2)) \leq L(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_0^T L(t)dt$.

Corollary 3.2. *Let $\nu \in (0,1)$ with $\nu^{n-1} \neq \Gamma(n)/(\beta_1+n-1)...(\beta_1+1)$, $\beta_1 > 0, n-1 < \beta_2 \leq n$, $n \geq 3$, $n \in \mathbf{N}$. Assume that Hypothesis H2 is satisfied, $d(0, F(t,0)) \leq L(t)$ a.e. (I) and $M_1 L_0 < 1$. Then there exists $x(.)$ a solution of the fractional differential inclusion*

$$D_C^{\beta_2} x(t) \in F(t, x(t)) \quad a.e.\ (I), \tag{3.12}$$

with boundary conditions (1.2) satisfying for all $t \in I$ $|x(t)| \leq \frac{M_1 L_0}{1-M_1 L_0}$.

Remark 3.3. An existence result for problem (3.12)-(1.2) is obtained in [16] under the hypothesis that $F(.,.)$ is upper semicontinuous with compact convex values. On one hand, our Corollary 3.2 provides an existence result for problem (3.12)-(1.2) under a hypothesis that avoids the convexity and, on the other hand, Theorem 3.1 above extends the study in [16] to the more general problem (1.1)-(1.2) where the right hand side contains a nonlinear Volterra integral operator.

The proof of the next theorem is similar to the proof of Theorem 3.1.

Theorem 3.2. *Let $\alpha \in (1,2)$, $\beta \in [0,1]$, $\gamma = \alpha + 2\beta - \alpha\beta$, $0 < \xi_i < 1$, $\delta_i \in \mathbf{R}$, $\varphi_i > 0$, $i = 1, 2, ..., m$ with $\Lambda = \sum_{i=1}^m \frac{\delta_i \xi_i^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)} - \frac{1}{\Gamma(\gamma)} \neq 0$. Assume that Hypothesis H1 is satisfied and $M_2 K_0 < 1$. Let $y(.) \in C(I, \mathbf{R})$ be such that $y(0) = 0$, $y(1) = \sum_{i=1}^m \delta_i I^{\varphi_i} y(\xi_i)$ and there exists $p(.) \in L^1(I, \mathbf{R})$ with $d(D_H^{\alpha,\beta} y(t), F(t, y(t), V(y)(t))) \leq p(t)$ a.e. (I).*

Then there exists $x(.) : I \to \mathbf{R}$ a solution of problem (1.3)-(1.4) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M_2}{1 - M_2 K_0} ||p(.)||_1. \tag{3.13}$$

Remark 3.4. If $F(.,.,.)$ does not depend on the last variable and $y(.) = 0$ a similar result to the one in Theorem 3.2 may be found in [17]; namely, Theorem 2. The proof of Theorem 2 in [17] is done by using the set-valued contraction principle. Our approach improves the hypothesis concerning the set-valued map in [17]. More exactly, we do not require for the values of F to be compact as in [17] and we do not require that the Lipschitz constant of F to be a mapping from $C(I, \mathbf{R})$ as in [17]. Moreover, Theorem 2 in [17] does not contains a priori bounds for solutions as in (3.13). As an example we consider the following problem

$$D_H^{\frac{3}{2},\frac{1}{2}} x(t) \in F(t, x(t)) \quad a.e.\ [0,1],$$

with boundary conditions

$$x(0) = 0, \quad x(1) = \frac{2}{3} I^{\frac{1}{2}} x(\frac{1}{2}) + \frac{3}{4} I^{\frac{3}{2}} x(\frac{3}{4}),$$

where

$$F(t,x) = \begin{cases} [-\frac{|x|}{a(1+|x|)}, 0] & \text{if} \quad t \in [0, \frac{1}{2}], \\ [0, \frac{|\cos(x)|}{(a+1)(1+|\cos(x)|)}] & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

Thus, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{7}{4}$, $m = 2$, $\delta_1 = \frac{2}{3}$, $\delta_2 = \frac{3}{4}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{3}{4}$, $\varphi_1 = \frac{1}{2}$, $\varphi_2 = \frac{3}{2}$, $\Lambda = \frac{1}{3 \cdot 2^{\frac{1}{4}} \Gamma(\frac{9}{4})} + \frac{3^{\frac{5}{4}}}{4^{\frac{13}{4}} \Gamma(\frac{13}{4})} - \frac{1}{\Gamma(\frac{7}{4})}$ and $M_2 = \frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{|\Lambda| \Gamma(\frac{3}{2}) \Gamma(\frac{7}{4})} + \frac{1}{3|\Lambda| \Gamma(2) \Gamma(\frac{7}{4})} + \frac{27}{64|\Lambda| \Gamma(3) \Gamma(\frac{7}{4})}$.
The Lipschitz constant of $F(t,.)$ is

$$L(t) = \begin{cases} \frac{1}{a}, & t \in [0, \frac{1}{2}], \\ \frac{1}{a+1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is enough to choose $a > M_2$ in order to deduce the existence of solutions for the problem considered.

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