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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75th anniversary

On Polynomial Dichotomies of Discrete Nonautonomous Systems on the Half-Line

DAVOR DRAGIČEVIĆ[1], ADINA LUMINIŢA SASU[2,3] and BOGDAN SASU[2,3]

ABSTRACT. The aim of this paper is to provide new characterizations for polynomial dichotomies of discrete nonautonomous systems on the half-line. First, we establish equivalent structures for the ranges of projections for a polynomial dichotomy with respect to a sequence of norms. Next, we establish the connections between polynomial dichotomies and other dichotomic behaviors. We obtain for the first time a characterization of polynomial dichotomy with respect to a sequence of norms in terms of ordinary dichotomy and exponential dichotomy of suitable systems with respect to well-chosen sequences of norms. The results are obtained in the most general case, without any additional assumptions regarding the coefficients of the underlying systems.

1. INTRODUCTION

The notion of exponential dichotomy introduced in the late 1920s (see Perron [47, 48]) plays an important role in the asymptotic theory of nonautonomous dynamical systems. It essentially corresponds to assuming the existence of an exponential contraction and expansion along complementary directions. For detailed expositions on various aspects of this theory and related studies we refer chronologically to monographs that substantially marked the foundations of this area - see Massera and Schäffer [35], Daleckĭi and Kreĭn [23], Coppel [22], Henry [33] and also to some remarkable books from the last decades - see Chicone and Latushkin [18], Palmer [46], Elaydi [28], Pötzsche [51], Kloeden and Rasmussen [34], Barreira, Dragičević and Valls [6]. We emphasize that ever since the nineties and especially during the recent years, the studies on exponential dichotomies significantly advanced. A large variety of methods were progressively brought into attention, starting on the one hand from diverse characterizations of exponential dichotomies and their related applications and arriving on the other hand at thorough analyzes regarding certain properties of systems that admit exponential dichotomies (see [2–5, 9, 12–14, 19–21, 26, 29, 30, 36–39, 41–45, 49, 50, 52, 55–60, 62–68, 70, 72–75]).

Despite its importance, in some situations, the exponential dichotomy might be regarded as being restrictive, since it requires that the rates of contraction and expansion along the stable and unstable directions are exponential. On the other hand, it is relatively easy to construct broad classes of nonautonomous dynamics which admit a splitting into stable and unstable directions, but with non-exponential rates in describing stability and instability. Among many meaningful and useful concepts used to describe this type of asymptotic behaviors, we particularly mention those of polynomial type (see [7,8,10,11,15–17,24,25,31,32,40,53,54,69] and the references therein). The polynomial

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Corresponding author: Bogdan Sasu; bogdan.sasu@e-uvt.ro

stability was studied in the autonomous case by Bátkai, Engel, Prüss and Schnaubelt in [8] and later, from a different perspective, by Hai for evolution families in [31] and for skew-evolution semiflows in [32]. The nonuniform polynomial dichotomy was introduced in slightly different forms, independently, by Barreira and Valls in [7] and respectively by Bento and Silva in [10, 11]. In contrast to the exponential dichotomy, the rates of contraction and expansion in the concept of polynomial dichotomy are of polynomial type. Moreover, in the general case of nonuniform polynomial dichotomies, these rates are also allowed to depend on the initial time.

In [24], Dragičević introduced (for nonautonomous dynamics in the discrete-time case) the concept of polynomial dichotomy with respect to a sequence of norms and obtained a characterization of this asymptotic property by means of a so-called admissibility condition. The relevance of these results stem from the fact that the notion of polynomial dichotomy with respect to a sequence of norms includes both the notions of polynomial dichotomy and respectively of nonuniform polynomial dichotomy as particular cases. For similar results in the case of continuous time dynamics, we refer to Dragičević [25]. Moreover, the polynomial dichotomies and their properties are also relevant in the context of generalized dichotomies (see Silva [71]). In this framework, over the last decade, many studies initiated and coordinated by Megan were devoted to polynomial behaviors (among which we mention polynomial stabilities, polynomial instabilities, polynomial dichotomies). Most of these studies were focused on the connections between various concepts (of stability, instability, dichotomy) and also on Datko or Barbashin type criteria for diverse asymptotic properties of polynomial type (see [15–17, 40, 53, 54, 69] and the references therein).

Some important technical aspects that occur when one explores an asymptotic property such as dichotomy, trichotomy or exponential (forward) splitting is related to the families of projections with respect to which one describes that behavior. We recall that on the one hand, it is of interest to establish the structure of the potential candidates for projections and, on the other hand, one should investigate whether the families of projections are unique or not. The studies on these matters were developed from various perspectives (see Aulbach and Kalkbrenner [1], Aulbach, Minh and Zabreiko [2], Barreira, Dragičević and Valls [5], Battelli, Franca and Palmer [9], Chow and Yi [19], Chow and Leiva [20, 21], Dragičević, Sasu and Sasu [26], Kloeden and Rasmussen [34], Megan, Sasu and Sasu [36–38], Minh, Räbiger and Schnaubelt [41], Minh and Huy [42], Palmer [44, 45], Pötzsche [51], Pliss and Sell [50], Sacker and Sell [55], Sasu [56], Sasu, Babuţia and Sasu [63], Sasu and Sasu [57–62, 64–68], Zhou, Lu and Zhang [72], Zhou and Zhang [73, 75]). We stress that depending on the method, sometimes it is important to establish the structure of the (initial) stable subspace(s) (see Sasu [56], Sasu and Sasu [57, 62, 65], Sasu, Babuţia and Sasu [63]), in some cases it is necessary to study the projections uniqueness (see Battelli, Franca and Palmer [9], Megan, Sasu and Sasu [37,38], Sasu and Sasu [58, 61, 65, 66, 68]) and, if so, in certain situations it is natural to give suitable descriptions of the ranges and/or of the kernels of the dichotomy (or trichotomy) projections in every moment by means of trajectories (see Battelli, Franca and Palmer [9], Megan, Sasu and Sasu [37,38], Sasu and Sasu [58–60,64–68], Zhou and Zhang [75] and the references therein).

Thus, in the case of the uniform asymptotic behaviors the studies advanced significantly, while in the nonuniform setting there still are open questions to be answered. For instance, it turned out that the projections for uniform exponential dichotomy or uniform

exponential trichotomy of nonautonomous systems on the whole line are uniquely determined and so they are in the case of variational systems over flows (see e.g. Megan, Sasu and Sasu [37, 38], Sasu and Sasu [58–61, 66–68] and the details provided therein). But, when exploring a uniform exponential dichotomy of nonautonomous systems on the (positive) half-line, one can determine only the expression of the ranges of the projections, because in this case, in general, the dichotomy projections are not unique (see for instance Minh, Räbiger and Schnaubelt [41], Sasu and Sasu [62, 65]). For all that, it is worth mentioning that the situation is different in the autonomous case, in which, even on the half-line, both the range and the kernel of the dichotomy projection for exponential dichotomy can be determined and so the projection is unique (see Sasu and Sasu [64]).

In this framework, it should be emphasized that even when a family of projections (for dichotomy or trichotomy) is unique, their ranges and kernels can be expressed in diverse equivalent forms (see Sasu and Sasu [59, 60, 66, 68]). A new and interesting study regarding the dichotomy projections that completely treated both the case of the exponential dichotomy on the half-line(s) and also the exponential dichotomies on the whole line was presented by Battelli, Franca and Palmer in [9]. In the case of the uniform exponential dichotomy with respect to a sequence (and respectively to a family) of norms, on the whole line, the structure of the dichotomy projections was obtained by Zhou and Zhang in [75]. An important contribution in exploring the structure of the ranges and kernels of projections in the case of nonuniform exponential dichotomies was brought by Zhou and Zhang in [73]. In this context - of the progress made in this direction so far - a natural question arises how the projections for a polynomial dichotomy with respect to a sequence of norms can be represented, which comes along with another question that concerns their uniqueness.

As mentioned above, the majority of the studies regarding the polynomial behaviors were devoted to Datko and Barbashin type criteria (see [15–17, 32, 40, 53, 54] and the references therein), while the admissibility type methods have proved to be a challenging topic that required distinct approaches and completely different techniques (see [24, 25, 31] and the presentations therein). In order to further develop studies in this direction, based on new admissibility properties among other topics, it would be necessary to establish new connections between polynomial dichotomies and other dichotomy concepts for which the admissibility methods are more advanced. We recall for instance that (especially in the uniform case) exponential dichotomies (or trichotomies) of nonautonomous systems for continuous time can be recovered from the homologous behaviors of some associated discrete-time systems (see Henry [33], Megan, Sasu and Sasu [36], Palmer [46], Sasu and Sasu [65–67], Zhou and Zhang [75]). Furthermore, these equivalences were also shown in the variational case for both dichotomy and trichotomy (see Chow and Leiva [21], Megan, Sasu and Sasu [37], Sasu and Sasu [61]). These kind of connections between asymptotic behaviors - by considering information only at certain moments of time (that are sufficient for making complete conclusions in describing a property) - lead to important applications as it was pointed out in [37, 61, 65, 66, 75], being particularly useful in admissibility methods. Thus, coming back to the aim described above, a natural question is whether a polynomial dichotomy of a system with respect to a family (or a sequence) of norms could be "recovered" from the dichotomic behavior of an associated system whose time variable is "rescaled". In what follows one of our purposes will be to answer the questions presented above.

The central aim of the present paper is to study the notion of polynomial dichotomy with respect to a sequence of norms in the case of discrete nonautonomous systems and

to provide new criteria for this asymptotic property. More precisely, we first explore the structure of the dichotomy projections for a polynomial dichotomy on the half-line. We prove that the associated stable directions are uniquely determined and we obtain their various descriptions (see Theorem 2.1). Secondly, we study the connections between the dichotomy notions - ordinary dichotomy, polynomial dichotomy, exponential dichotomy with respect to sequences of norms. We show for the first time that the concept of polynomial dichotomy with respect to a sequence of norms can be completely characterized in terms of the notion of ordinary dichotomy with respect to a sequence of norms and the notion of exponential dichotomy with respect to a sequence of norms, provided that this is admitted by a suitable system and relative to a well-chosen sequence of norms (see Theorem 3.1). This result is achieved by proving that the polynomial dichotomy of a nonautonomous system can be (roughly speaking) related with an exponential dichotomy of an associated rescaled system (subjected to a suitable "renormalization" of time variable). The results in this paper are obtained in the most general case, without making any additional assumptions on the systems coefficients and without assuming any growth type property for the propagators of the systems whose dichotomic properties are studied or connected.

2. Dichotomies with Respect to Sequences of Norms

In this section we present the dichotomy notions that will be considered in our study, discuss some connections between them and explore the structure of the dichotomy projections for polynomial dichotomy with respect to a sequence of norms.

Denote by $\mathbb{N} = \{1, 2, \ldots\}$ and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\Gamma = \{(m, k) \in \mathbb{N} \times \mathbb{N} : m \geq k\}$.

Let $(X, \|\cdot\|)$ be an arbitrary Banach space. By $\mathcal{B}(X)$ we will denote the Banach algebra of all bounded linear operators on X, equipped with the operator norm that we will also denote by $\|\cdot\|$. By I_d we denote the identity operator on X.

Let $\{A(n)\}_{n\in\mathbb{N}} \subset \mathcal{B}(X)$. Consider the nonautonomous system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{N}$$

and the evolution family $\Phi_A = \{\Phi_A(m, k)\}_{(m,k)\in\Gamma}$ associated to (A) defined by

$$\Phi_A(m, k) = \begin{cases} A(m-1)\cdots A(k), & m > k \\ I_d, & m = k \end{cases}.$$

We recall first the dichotomy notions. For more details and motivation we refer to [4, 24, 65].

Let $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ be an arbitrary but fixed sequence of norms on X such that $\|\cdot\|_k$ is equivalent to $\|\cdot\|$, for each $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ and every linear operator $T : X \to X$ we denote by

$$\|T\|_k = \inf\{\, M \in (0, \infty] \,:\, \|Tx\|_k \leq M\|x\|_k, \quad \forall x \in X \,\}. \tag{2.1}$$

Remark 2.1. Let $k \in \mathbb{N}$. We note that:

(i) $T \in \mathcal{B}(X)$ if and only if $\|T\|_k < \infty$;

(ii) $\|\cdot\|_k$ defined by (2.1) is a norm on $\mathcal{B}(X)$, which is equivalent to the operator norm $\|\cdot\|$.

Definition 2.1. We say that (A) admits an *ordinary dichotomy with respect to the sequence of norms* $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ if there are a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and $K \geq 1$ such that:

(d_1) $A(k)P(k) = P(k+1)A(k)$, for all $k \in \mathbb{N}$;

(d_2) $\sup\limits_{k\in\mathbb{N}} \|P(k)\|_k < \infty$;

(d_3) for every $k \in \mathbb{N}$, $A(k)_| : KerP(k) \to KerP(k+1)$ is an isomorphism;

(d_4) $\|\Phi_A(m,k)x\|_m \leq K\,\|x\|_k$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;

(d_5) $\|\Phi_A(m,k)y\|_m \geq \dfrac{1}{K}\,\|y\|_k$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Remark 2.2. In the particular case $\|\cdot\|_k = \|\cdot\|$, for all $k \in \mathbb{N}$, in Definition 2.1 we obtain the notion of *ordinary dichotomy* (see [65] and the references therein).

Remark 2.3. We note that

(i) property (d_1) is equivalent to

$$\Phi_A(m,k)P(k) = P(m)\Phi_A(m,k), \quad \forall (m,k) \in \Gamma;$$

(ii) property (d_3) is equivalent to $\Phi_A(m,k)_| : KerP(k) \to KerP(m)$ invertible, for all $(m,k) \in \Gamma$.

Definition 2.2. We say that (A) admits an *exponential dichotomy with respect to the sequence of norms* $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ if there are a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and $N \geq 1, \nu > 0$ such that properties $(d_1) - (d_3)$ from Definition 2.1 are satisfied and furthermore:

(d_4') $\|\Phi_A(m,k)x\|_m \leq N\,e^{-\nu(m-k)}\,\|x\|_k$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;

(d_5') $\|\Phi_A(m,k)y\|_m \geq \dfrac{1}{N}\,e^{\nu(m-k)}\,\|y\|_k$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Remark 2.4. In the particular case $\|\cdot\|_k = \|\cdot\|$, for all $k \in \mathbb{N}$, in Definition 2.2 we obtain the concept of *exponential dichotomy* (see [33] and the references therein).

Definition 2.3. We say that (A) admits a *polynomial dichotomy with respect to the sequence of norms* $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ if there are projections $\{P(k)\}_{k\in\mathbb{N}}$ and $L \geq 1, \delta > 0$ such that properties $(d_1) - (d_3)$ from Definition 2.1 are satisfied and in addition:

(d_4'') $\|\Phi_A(m,k)x\|_m \leq L\left(\dfrac{m}{k}\right)^{-\delta}\|x\|_k$, for all $x \in RangeP(k)$ and all $(m,k) \in \Gamma$;

(d_5'') $\|\Phi_A(m,k)y\|_m \geq \dfrac{1}{L}\left(\dfrac{m}{k}\right)^{\delta}\|y\|_k$, for all $y \in KerP(k)$ and all $(m,k) \in \Gamma$.

Remark 2.5. In the particular case $\|\cdot\|_k = \|\cdot\|$, for all $k \in \mathbb{N}$, in Definition 2.3 we obtain the notion of *polynomial dichotomy*.

Remark 2.6. We note that in Definition 2.2 the assembly of relations (d_2), (d_4') and (d_5') can be expressed in equivalent form by

$(\tilde{d}_4')$ $\|\Phi_A(m,k)P(k)x\|_m \leq K\,e^{-\nu(m-k)}\|x\|_k$, for all $x \in X$ and all $(m,k) \in \Gamma$;

$(\tilde{d}_5')$ $\|\Phi_A(m,k)_|^{-1}(I_d - P(m))y\|_k \leq K\,e^{-\nu(m-k)}\,\|y\|_m$, for all $y \in X$ and all $(m,k) \in \Gamma$

for some constants $K \geq 1, \nu > 0$.

Similarly, in Definition 2.3 the assembly of relations (d_2), (d_4'') and (d_5'') can be expressed in equivalent form by

$(\tilde{d}_4'')$ $\|\Phi_A(m,k)P(k)x\|_m \leq K \left(\frac{m}{k}\right)^{-\delta} \|x\|_k$, for all $x \in X$ and all $(m,k) \in \Gamma$;

$(\tilde{d}_5'')$ $\|\Phi_A(m,k)_|^{-1}(I_d - P(m))y\|_k \leq K \left(\frac{m}{k}\right)^{-\delta} \|y\|_m$, for all $y \in X$ and all $(m,k) \in \Gamma$

for some constants $K \geq 1, \delta > 0$.

Here, for each $(m,k) \in \Gamma$, $\Phi_A(m,k)_|^{-1}$ denotes the inverse of $\Phi_A(m,k)_| : KerP(k) \to KerP(m)$.

Remark 2.7. The concept of polynomial dichotomy with respect to a sequence of norms was introduced by Dragičević in [24]. The notion of ordinary dichotomy with respect to a sequence of norms is defined for the first time in the present paper and the notion of exponential dichotomy with respect to a sequence of norms was first presented and studied by Barreira, Dragičević and Valls in [4].

One of the aims in what follows will be to establish characterizations for a polynomial dichotomy based on the other two dichotomy notions (in the general case, i.e. with respect to sequences of norms).

Remark 2.8. From Definitions 2.1-2.3 we deduce the following:

(i) if the system (A) admits a polynomial dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$, then (A) admits an ordinary dichotomy with respect to $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ (and the same sequence of projections);

(ii) since
$$\frac{m}{k} \leq e^{m-k}, \quad \forall (m,k) \in \Gamma$$
if (A) admits an exponential dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$, then (A) admits a polynomial dichotomy with respect to $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ (and the same sequence of projections).

In particular:

(i') if (A) admits a polynomial dichotomy, then (A) admits an ordinary dichotomy (with the same projections);

(ii') if (A) admits an exponential dichotomy, then (A) admits a polynomial dichotomy (with the same projections).

The first aim is to explore the structure of the projections for polynomial dichotomy with respect to sequences of norms. The following result provides equivalent expressions of the stable subspaces for a polynomial dichotomy.

Theorem 2.1. *If (A) admits a polynomial dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, then there exists $r \in (1,\infty)$ such that for every $k \in \mathbb{N}$ we have*

$$\begin{aligned} RangeP(k) &= \{x \in X : \sup_{j\geq k} \|\Phi_A(j,k)x\|_j < \infty\} \\ &= \{x \in X : \lim_{j\to\infty} \|\Phi_A(j,k)x\|_j = 0\} \\ &= \{x \in X : \sum_{j=k}^{\infty} \|\Phi_A(j,k)x\|_j^p < \infty\}, \quad \forall p \in (r,\infty). \end{aligned}$$

Proof. Let $L \geq 1$ and $\delta > 0$ be given by Definition 2.3. Then

$$\|\Phi_A(m,k)x\|_m \leq L\left(\frac{m}{k}\right)^{-\delta}\|x\|_k, \quad \forall x \in RangeP(k), \forall (m,k) \in \Gamma \tag{2.2}$$

and

$$\|\Phi_A(m,k)y\|_m \geq \frac{1}{L}\left(\frac{m}{k}\right)^{\delta}\|y\|_k, \quad \forall y \in KerP(k), \forall (m,k) \in \Gamma. \tag{2.3}$$

Let $r = 1/\delta$ and let $p \in (r, \infty)$.

Let $k \in \mathbb{N}$ be arbitrary but fixed. We denote by

$$X_1(k) = \{x \in X : \sup_{j \geq k} \|\Phi_A(j,k)x\|_j < \infty\}$$

$$Y_1(k) = \{x \in X : \lim_{j \to \infty} \|\Phi_A(j,k)x\|_j = 0\}$$

and

$$Z_1^p(k) = \{x \in X : \sum_{j=k}^{\infty} \|\Phi_A(j,k)x\|_j^p < \infty\}.$$

Let $x \in RangeP(k)$. Using (2.2) we observe that

$$\|\Phi_A(j,k)x\|_j^p \leq (Lk^{\delta}\|x\|_k)^p \frac{1}{j^{p\delta}}, \quad \forall j \geq k.$$

This shows that $x \in Z_1^p(k)$, so $Range\ P(k) \subset Z_1^p(k)$. It follows that

$$Range\ P(k) \subset Z_1^p(k) \subset Y_1(k) \subset X_1(k). \tag{2.4}$$

Let now $x \in X_1(k)$. Then

$$\lambda := \sup_{j \geq k} \|\Phi_A(j,k)x\|_j < \infty.$$

Setting $u = P(k)x$ and $v = (I_d - P(k))x$, from (2.2) and (2.3) we deduce that

$$\frac{j^{\delta}}{Lk^{\delta}}\|v\|_k \leq \|\Phi_A(j,k)v\|_j \leq \|\Phi_A(j,k)x\|_j + \|\Phi_A(j,k)u\|_j \leq \lambda + L\|u\|_k, \quad \forall j \geq k$$

which implies that

$$\|v\|_k \leq Lk^{\delta}(\lambda + L\|u\|_k)j^{-\delta}, \quad \forall j \geq k. \tag{2.5}$$

Letting $j \to \infty$ in (2.5) we obtain that $v = 0$. So $x = u \in RangeP(k)$. This shows that

$$X_1(k) \subset RangeP(k). \tag{2.6}$$

From (2.4) and (2.6) it yields that

$$Range\ P(k) = Z_1^p(k) = Y_1(k) = X_1(k)$$

and the proof is complete. □

Corollary 2.1. *If (A) admits a polynomial dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$, then there exists $r \in (1,\infty)$ such that for every $k \in \mathbb{N}$ we have*

$$\begin{aligned} RangeP(k) &= \{x \in X : \sup_{j\geq k} \|\Phi_A(j,k)x\| < \infty\} \\ &= \{x \in X : \lim_{j\to\infty} \Phi_A(j,k)x = 0\} \\ &= \{x \in X : \sum_{j=k}^{\infty} \|\Phi_A(j,k)x\|^p < \infty\}, \quad \forall p \in (r,\infty). \end{aligned}$$

Proof. This immediately follows from Theorem 2.1, for $\|\cdot\|_k = \|\cdot\|$, for all $k \in \mathbb{N}$. □

Remark 2.9. The representations of the ranges of projections given by Theorem 2.1 (and respectively by Corollary 2.1) hold true also for the weaker concepts of polynomial dichotomies for which the property (d_2) is dropped (i.e. without being necessary to assume the uniform boundedness of projections).

Based on the previous theorem we can show that, in general, the converse implications in the statements from Remark 2.8 (i), (ii) and respectively (i'), (ii') do not hold true.

Example 2.1. Let $X = \mathbb{R}^2$ with $||(x_1,x_2)|| = \max\{|x_1|,|x_2|\}$.

For $k \in \mathbb{N}$ and $(x_1,x_2) \in X$, let

$$A(k)(x_1,x_2) = (e^{\sin(k+1)-\sin k}x_1, e^{\cos k-\cos(k+1)}x_2)$$

and

$$P(k)(x_1,x_2) = (x_1,0).$$

Then we note that

$$\Phi_A(m,k)(x_1,x_2) = (e^{\sin m-\sin k}x_1, e^{\cos k-\cos m}x_2), \quad \forall (m,k) \in \Gamma. \tag{2.7}$$

We observe that (A) admits an ordinary dichotomy with respect to the sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$.

We show that (A) does not admit a polynomial dichotomy.

Suppose to the contrary that (A) admits a polynomial dichotomy with respect to a sequence of projections $\{\tilde{P}(k)\}_{k\in\mathbb{N}}$. By Corollary 2.1 we obtain that

$$Range\tilde{P}(1) = \{x \in X : \sup_{j\geq 1} \|\Phi_A(j,1)x\| < \infty\}. \tag{2.8}$$

Let now $L \geq 1, \delta > 0$ be given by Definition 2.3. Then, in particular, we have

$$\|\Phi_A(m,1)x\| \leq \frac{L}{m^\delta}\|x\|, \quad \forall x \in Range\tilde{P}(1), \forall m \in \mathbb{N}. \tag{2.9}$$

From (2.7) and (2.8) we have that $u = (1,0) \in Range\tilde{P}(1)$. Then, from (2.7) and (2.9) we obtain

$$e^{\sin m-\sin 1} = \|\Phi_A(m,1)u\| \leq \frac{L}{m^\delta}, \quad \forall m \in \mathbb{N}$$

which yields a contradiction.

In conclusion, (A) does not admit a polynomial dichotomy.

Example 2.2. Let $X = \mathbb{R}^2$ with $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$.

For $k \in \mathbb{N}$ and $(x_1, x_2) \in X$, let

$$A(k)(x_1, x_2) = \left(\frac{k}{k+1}x_1, \frac{k+1}{k}x_2\right)$$

and

$$P(k)(x_1, x_2) = (x_1, 0).$$

Then

$$\Phi_A(m, k)(x_1, x_2) = (\frac{k}{m}x_1, \frac{m}{k}x_2), \quad \forall (m, k) \in \Gamma. \tag{2.10}$$

We observe that (A) admits a polynomial dichotomy with the projections $\{P(k)\}_{k\in\mathbb{N}}$.

We show that (A) does not admit an exponential dichotomy.

Suppose to the contrary that (A) admits an exponential dichotomy with respect to a sequence of projections $\{\tilde{P}(k)\}_{k\in\mathbb{N}}$. By Remark 2.8 (ii') and Corollary 2.1 we obtain that

$$Range\tilde{P}(1) = \{x \in X : \sup_{j\geq 1} \|\Phi_A(j, 1)x\| < \infty\}. \tag{2.11}$$

Let now $N \geq 1, \nu > 0$ be given by Definition 2.2. Then, in particular, we have that

$$\|\Phi_A(m, 1)x\| \leq Ne^{-\nu(m-1)}\|x\|, \quad \forall x \in Range\tilde{P}(1), \forall m \in \mathbb{N}. \tag{2.12}$$

From (2.10) and (2.11) we deduce that $u = (1, 0) \in Range\tilde{P}(1)$. Then, from (2.10) and (2.12) we obtain

$$\frac{1}{m} = \|\Phi_A(m, 1)u\| \leq Ne^{-\nu(m-1)}, \quad \forall m \in \mathbb{N}$$

which yields a contradiction.

In conclusion, (A) does not admit an exponential dichotomy.

Remark 2.10. Example 2.1 shows that in general the ordinary dichotomy does not imply the polynomial dichotomy and Example 2.2 shows that in general the polynomial dichotomy does not imply the exponential dichotomy. In conclusion, the converse implications in Remark 2.8 are not valid, not even in the classic case, when all the norms $\|\cdot\|_k$ coincide with $\|\cdot\|$.

Even if the structures of the ranges of eligible projections are known via Theorem 2.1, we emphasize that the sequence of projections for a polynomial dichotomy on the half-line is not uniquely determined, as the next example shows:

Example 2.3. Let $X = \mathbb{R}^2$ with $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$.

For $k \in \mathbb{N}$ and $(x_1, x_2) \in X$, let

$$A(k)(x_1, x_2) = \left(\frac{1}{e}x_1, e\,x_2\right).$$

Then

$$\Phi_A(m, k)(x_1, x_2) = (e^{-(m-k)}x_1, e^{m-k}x_2), \quad \forall (m, k) \in \Gamma. \tag{2.13}$$

Let

$$X(k) = \mathbb{R} \times \{0\}, \quad \forall k \in \mathbb{N}.$$

Let $\alpha \in [1, \infty)$ be arbitrary. Let

$$Y_\alpha(k) = \{(e^{-(k-1)}\xi, \alpha e^{k-1}\xi) : \xi \in \mathbb{R}\}, \quad \forall k \in \mathbb{N}.$$

Then, we observe that

$$X = X(k) \oplus Y_\alpha(k), \quad \forall k \in \mathbb{N}.$$

Then there exists a sequence of projections $\{P_\alpha(k)\}_{k\in\mathbb{N}}$ such that

$$RangeP_\alpha(k) = X(k) \quad \text{and} \quad KerP_\alpha(k) = Y_\alpha(k), \quad \forall k \in \mathbb{N}.$$

It is easy to see that

$$\|\Phi_A(m,k)x\| = e^{-(m-k)}\|x\|, \quad \forall x \in RangeP_\alpha(k), \forall (m,k) \in \Gamma \tag{2.14}$$

and

$$\|\Phi_A(m,k)y\| = e^{m-k}\|y\|, \quad \forall y \in KerP_\alpha(k), \forall (m,k) \in \Gamma. \tag{2.15}$$

In addition, we note that

$$\|P_\alpha(k)\| = 1, \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Using (2.14)-(2.16) we deduce that (A) admits an exponential dichotomy with respect to $\{P_\alpha(k)\}_{k\in\mathbb{N}}$. Thus, it admits a polynomial dichotomy and also an ordinary dichotomy with respect to $\{P_\alpha(k)\}_{k\in\mathbb{N}}$. Furthermore, these dichotomy properties are satisfied for every $\alpha \in [1, \infty)$.

In conclusion, the sequence of projections for (ordinary, exponential or polynomial) dichotomy on the half-line is not unique.

Remark 2.11. We stress that Example 2.3 shows even more that there is an infinite number of sequences of projections through which one can describe a dichotomic behavior on the half-line. Thus, while one knows a priori the candidates for stable subspaces (as shown in Theorem 2.1), the adequate choice of the initial unstable subspace plays an important role in the studies devoted to dichotomies on the half-line (see [36,56,57,65] and the references therein).

3. Polynomial Dichotomy in terms of Ordinary and Exponential Dichotomy

In this section, our aim is establish further the connections between the three dichotomy notions introduced in Section 2, pointing out the way how the polynomial dichotomy with respect to a sequence of norms can be expressed in terms of ordinary dichotomy and exponential dichotomy of certain systems with respect to (well-chosen) sequences of norms.

We maintain in what follows the framework (and all the notations) from Section 2.

Let now $h \in \mathbb{N}, h \geq 2$. We associate with the system (A) the following operators

$$Q^h(n) = \Phi_A(h^n, h^{n-1}), \quad \forall n \in \mathbb{N} \tag{3.1}$$

and we consider the nonautonomous system

$$y(n+1) = Q^h(n)y(n), \quad n \in \mathbb{N}. \tag{Q^h}$$

Remark 3.1. If $\Phi_{Q^h} = \{\Phi_{Q^h}(m,k)\}_{(m,k)\in\Gamma}$ is the evolution family associated to (Q^h), then

$$\Phi_{Q^h}(m,k) = \Phi_A(h^{m-1}, h^{k-1}), \quad \forall (m,k) \in \Gamma. \tag{3.2}$$

We associate with the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ a new sequence of norms $\{\|\cdot\|_k^h\}_{k\in\mathbb{N}}$ (that depend on h), given by

$$\|\cdot\|_k^h := \|\cdot\|_{h^{k-1}}, \quad \forall k \in \mathbb{N}. \tag{3.3}$$

For every $k \in \mathbb{N}$ and every operator $T \in \mathcal{B}(X)$ we consider

$$\|T\|_k^h := \|T\|_{h^{k-1}} \tag{3.4}$$

where $\|\cdot\|_{h^{k-1}}$ is the operator norm introduced in Section 2 (see relation (2.1)).

The central result of this section is:

Theorem 3.1. *The following assertions are equivalent:*

- (i) *the system (A) admits a polynomial dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$;*
- (ii) *(A) admits an ordinary dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and for every $h \in \mathbb{N}$, $h \geq 2$, the system (Q^h) admits an exponential dichotomy with respect to the norms $\{\|\cdot\|_k^h\}_{k\in\mathbb{N}}$ relative to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$, where*

$$P^h(k) = P(h^{k-1}), \quad \forall k \in \mathbb{N}; \tag{3.5}$$

- (iii) *(A) admits an ordinary dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and there is $h \in \mathbb{N}$, $h \geq 2$ such that the system (Q^h) admits an exponential dichotomy with respect to the norms $\{\|\cdot\|_k^h\}_{k\in\mathbb{N}}$ relative to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$ that satisfy* (3.5).

Proof. $(i) \Longrightarrow (ii)$ Let $\{P(k)\}_{k\in\mathbb{N}}$ be a sequence of projections and let $L \geq 1, \delta > 0$ be given by Definition 2.3. Then

$$\|\Phi_A(m,k)x\|_m \leq L\left(\frac{m}{k}\right)^{-\delta} \|x\|_k, \quad \forall x \in RangeP(k),\ \forall (m,k) \in \Gamma \tag{3.6}$$

and

$$\|\Phi_A(m,k)y\|_m \geq \frac{1}{L}\left(\frac{m}{k}\right)^{\delta} \|y\|_k, \quad \forall y \in KerP(k),\ \forall (m,k) \in \Gamma. \tag{3.7}$$

As already noted in Remark 2.8 (i), the system admits an ordinary dichotomy with the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ and the same projections $\{P(k)\}_{k\in\mathbb{N}}$.

Let $h \in \mathbb{N}, h \geq 2$ and let $\{Q^h(n)\}_{n\in\mathbb{N}}$ be defined as in (3.1). Consider the corresponding system (Q^h) and respectively the evolution family Φ_{Q^h} (see Remark 3.1).

Let now $\{P^h(k)\}_{k\in\mathbb{N}}$ be defined as in relation (3.5).

Then, using (3.1) and Remark 2.3 (i) we obtain that

$$\begin{aligned} Q^h(k)P^h(k) &= \Phi_A(h^k, h^{k-1})P(h^{k-1}) \\ &= P(h^k)\Phi_A(h^k, h^{k-1}) \\ &= P^h(k+1)Q^h(k),\ \forall k \in \mathbb{N} \end{aligned} \tag{3.8}$$

and respectively from (3.4) we have that

$$\sup_{k\in\mathbb{N}} \|P^h(k)\|_k^h = \sup_{k\in\mathbb{N}} \|P(h^{k-1})\|_{h^{k-1}} \leq \sup_{j\in\mathbb{N}} \|P(j)\|_j < \infty. \tag{3.9}$$

In addition, from Remark 2.3 (ii) it follows that for every $k \in \mathbb{N}$, $\Phi_A(h^k, h^{k-1})_| : KerP(h^{k-1}) \to KerP(h^k)$ is invertible, so $Q^h(k)_| : KerP^h(k) \to KerP^h(k+1)$ is invertible.

We set

$$\nu := \delta \ln h \tag{3.10}$$

and we note that $\nu > 0$.

Let $k \in \mathbb{N}$ and $x \in RangeP^h(k) = RangeP(h^{k-1})$. Then, using (3.2), (3.3), (3.6) and (3.10) we deduce that

$$\begin{aligned} \|\Phi_{Q^h}(m,k)x\|_m^h &= \|\Phi_A(h^{m-1}, h^{k-1})x\|_{h^{m-1}} \\ &\leq Lh^{-\delta(m-k)}\|x\|_{h^{k-1}} \\ &= Le^{-\nu(m-k)}\|x\|_k^h, \quad \forall m \geq k. \end{aligned} \tag{3.11}$$

Let $k \in \mathbb{N}$ and $y \in KerP^h(k) = KerP(h^{k-1})$. Then, using (3.2), (3.3), (3.7) and (3.10) we obtain that

$$\begin{aligned} \|\Phi_{Q^h}(m,k)y\|_m^h &= \|\Phi_A(h^{m-1}, h^{k-1})y\|_{h^{m-1}} \\ &\geq \frac{1}{L}h^{\delta(m-k)}\|y\|_{h^{k-1}} \\ &= \frac{1}{L}e^{\nu(m-k)}\|y\|_k^h, \quad \forall m \geq k. \end{aligned} \tag{3.12}$$

From relations (3.8), (3.9), (3.11) and (3.12) it follows that (Q^h) admits an exponential dichotomy with respect to the sequence of norms $\{\|\cdot\|_k^h\}_{k\in\mathbb{N}}$ relative to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$.

$(ii) \Longrightarrow (iii)$ This implication is obvious.

$(iii) \Longrightarrow (i)$ Assume that (A) admits an ordinary dichotomy with respect to a sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to the sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and let $K \geq 1$ be given by Definition 2.1. Then

$$\|\Phi_A(m,k)x\|_m \leq K\,\|x\|_k, \quad \forall x \in RangeP(k),\, \forall (m,k) \in \Gamma \tag{3.13}$$

and

$$\|\Phi_A(m,k)y\|_m \geq \frac{1}{K}\,\|y\|_k, \quad \forall y \in KerP(k),\, \forall (m,k) \in \Gamma. \tag{3.14}$$

Let $h \geq 2$ be such that the system (Q^h) admits an exponential dichotomy with respect to the sequence of norms $\{\|\cdot\|_k^h\}_{k\in\mathbb{N}}$ relative to the projections $\{P^h(k)\}_{k\in\mathbb{N}}$ given by relation (3.5).

Then, via Definition 2.2 there are $N \geq 1, \nu > 0$ such that

$$\|\Phi_{Q^h}(m,k)x\|_m^h \leq N\,e^{-\nu(m-k)}\,\|x\|_k^h, \quad \forall x \in RangeP^h(k), \forall (m,k) \in \Gamma \tag{3.15}$$

and

$$\|\Phi_{Q^h}(m,k)y\|_m^h \geq \frac{1}{N}\,e^{\nu(m-k)}\,\|y\|_k^h, \quad \forall y \in KerP^h(k), \forall (m,k) \in \Gamma. \tag{3.16}$$

We denote by

$$\delta := \frac{\nu}{\ln h} \quad \text{and} \quad L_1 := K^2 N e^{2\nu}. \tag{3.17}$$

In addition, we set

$$L := \max\{L_1, Kh^{\delta}\}. \tag{3.18}$$

We prove that (A) admits a polynomial dichotomy with respect to the sequence of norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to the projections $\{P(k)\}_{k\in\mathbb{N}}$ and with the constants L and δ.

Let $k \in \mathbb{N}$ and $x \in RangeP(k)$. Let $m \in \mathbb{N}, m \geq k$. We have two cases:

Case 1. If $m \geq kh$, then we take

$$j := \left[\frac{\ln k}{\ln h}\right] \quad \text{and} \quad n := \left[\frac{\ln m}{\ln h}\right]. \tag{3.19}$$

From (3.19) we observe that

$$k \leq h^{j+1} \quad \text{and} \quad h^n \leq m.$$

In addition, since $m \geq kh$, from (3.19) we deduce that $n \geq j+1$.

From Remark 2.3 (i) and from (3.5) it is clear that $\Phi_A(h^{j+1}, k)x \in RangeP^h(j+2)$. Hence, from (3.2)-(3.4), (3.13) and (3.15) we have that

$$\begin{aligned}
\|\Phi_A(m,k)x\|_m &= \|\Phi_A(m,h^n)\Phi_A(h^n,k)x\|_m \\
&\leq K\|\Phi_A(h^n,h^{j+1})\Phi_A(h^{j+1},k)x\|_{h^n} \\
&= K\|\Phi_{Q^h}(n+1,j+2)\Phi_A(h^{j+1},k)x\|^h_{n+1} \\
&\leq KNe^{-\nu(n-j-1)}\|\Phi_A(h^{j+1},k)x\|^h_{j+2} \\
&= KNe^{-\nu(n-j-1)}\|\Phi_A(h^{j+1},k)x\|_{h^{j+1}} \\
&\leq K^2Ne^{2\nu}e^{-\nu(n+1-j)}\|x\|_k.
\end{aligned} \tag{3.20}$$

We note that

$$n+1-j \geq \frac{\ln m}{\ln h} - \frac{\ln k}{\ln h} = \frac{\ln \frac{m}{k}}{\ln h}. \tag{3.21}$$

From (3.20), (3.21) and (3.17) it follows that

$$\|\Phi_A(m,k)x\|_m \leq L_1\left(\frac{m}{k}\right)^{-\delta}\|x\|_k. \tag{3.22}$$

Case 2. If $m \in \{k, \ldots, kh-1\}$, then, using (3.13), we get that

$$\|\Phi_A(m,k)x\|_m \leq K\|x\|_k = K\left(\frac{m}{k}\right)^{\delta}\left(\frac{m}{k}\right)^{-\delta}\|x\|_k \leq Kh^{\delta}\left(\frac{m}{k}\right)^{-\delta}\|x\|_k. \tag{3.23}$$

Then from (3.22), (3.23) and (3.18) we deduce that

$$\|\Phi_A(m,k)x\|_m \leq L\left(\frac{m}{k}\right)^{-\delta}\|x\|_k, \quad \forall x \in RangeP(k), \forall (m,k) \in \Gamma. \tag{3.24}$$

Let $k \in \mathbb{N}$ and $y \in KerP(k)$. Let $m \in \mathbb{N}, m \geq k$. We have two cases:

Case 1. If $m \geq kh$, then we take j and n as in relation (3.19).

Using Remark 2.3 (i) and (3.5) we have that $\Phi_A(h^{j+1}, k)y \in KerP^h(j+2)$. Similarly as above, from (3.2)-(3.4), (3.14), (3.16) and (3.17) we deduce that

$$\begin{aligned}
\|\Phi_A(m,k)y\|_m &= \|\Phi_A(m,h^n)\Phi_A(h^n,k)y\|_m \\
&\geq \frac{1}{K}\|\Phi_A(h^n,h^{j+1})\Phi_A(h^{j+1},k)y\|_{h^n} \\
&= \frac{1}{K}\|\Phi_{Q^h}(n+1,j+2)\Phi_A(h^{j+1},k)y\|^h_{n+1} \\
&\geq \frac{1}{KN}e^{\nu(n-j-1)}\|\Phi_A(h^{j+1},k)y\|^h_{j+2} \\
&= \frac{1}{KN}e^{\nu(n-j-1)}\|\Phi_A(h^{j+1},k)y\|_{h^{j+1}} \\
&\geq \frac{1}{K^2N}e^{-2\nu}e^{\nu(n+1-j)}\|y\|_k \\
&= \frac{1}{L_1}e^{\nu(n+1-j)}\|y\|_k.
\end{aligned} \tag{3.25}$$

Since

$$\nu(n+1-j) \geq \delta \ln \frac{m}{k}$$

from (3.17), (3.18) and (3.25) it follows that

$$\|\Phi_A(m,k)y\|_m \geq \frac{1}{L_1}\left(\frac{m}{k}\right)^\delta \|y\|_k \geq \frac{1}{L}\left(\frac{m}{k}\right)^\delta \|y\|_k. \tag{3.26}$$

Case 2. If $m \in \{k, \ldots, kh-1\}$, then, using (3.14) and (3.18), we get that

$$\begin{aligned}
\|\Phi_A(m,k)y\|_m &\geq \frac{1}{K}\|y\|_k = \frac{1}{K}\left(\frac{m}{k}\right)^{-\delta}\left(\frac{m}{k}\right)^\delta \|y\|_k \\
&\geq \frac{1}{Kh^\delta}\left(\frac{m}{k}\right)^\delta \|y\|_k \geq \frac{1}{L}\left(\frac{m}{k}\right)^\delta \|y\|_k.
\end{aligned} \tag{3.27}$$

From (3.26) and (3.27) we deduce that

$$\|\Phi_A(m,k)y\|_m \geq \frac{1}{L}\left(\frac{m}{k}\right)^\delta \|y\|_k, \quad \forall y \in KerP(k), \forall (m,k) \in \Gamma. \tag{3.28}$$

In conclusion, from (3.24) and (3.28) we obtain that (A) admits a polynomial dichotomy with respect to the norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ relative to the projections $\{P(k)\}_{k\in\mathbb{N}}$. □

In the case of the classic dichotomy notions, we obtain the following consequence:

Theorem 3.2. *The following assertions are equivalent:*

(i) *the system (A) admits a polynomial dichotomy;*

(ii) *(A) admits an ordinary dichotomy with respect to a sequence of projections $\{P(k)\}_{k\in\mathbb{N}}$ and for every $h \in \mathbb{N}$, $h \geq 2$, the system (Q^h) admits an exponential dichotomy with respect to the sequence of projections $\{P^h(k)\}_{k\in\mathbb{N}}$, where*

$$P^h(k) = P(h^{k-1}), \quad \forall k \in \mathbb{N}; \tag{3.29}$$

(iii) (A) *admits an ordinary dichotomy with respect to a sequence of projections* $\{P(k)\}_{k\in\mathbb{N}}$ *and there is* $h \in \mathbb{N}$, $h \geq 2$ *such that the system* (Q^h) *admits an exponential dichotomy with respect to the sequence of projections* $\{P^h(k)\}_{k\in\mathbb{N}}$ *that satisfy* (3.29).

Proof. This immediately yields from Theorem 3.1 for $\|\cdot\|_k = \|\cdot\|$, for all $k \in \mathbb{N}$. □

Remark 3.2. The characterizations for polynomial dichotomies given in Theorem 3.1 and Theorem 3.2 remain valid even if all the dichotomy notions are of weaker nature in the sense that one removes the property (d_2) for all of the dichotomies therein. More specific, the criteria hold true inclusive for dichotomy notions in which the property regarding the boundedness of the sequences of projections is dropped.

Remark 3.3. The central results in this paper have been obtained in the most general case, without making any additional assumptions on the system(s) coefficients. Furthermore, we haven't assumed any kind of (polynomial or exponential) growths for the propagators of the systems considered herein.

4. Conclusions

In this paper we have shown that the ranges of projections for polynomial dichotomies of discrete nonautonomous systems can be represented in terms of the belonging of the associated trajectories to some spaces of bounded sequences or to certain ℓ^p-spaces. This representation plays a key-role both in establishing connections between the dichotomy notions as well as in the future studies of polynomial dichotomies by means of various admissibility properties (among other topics).

We have proved that the polynomial dichotomy (relative to a sequence of norms) can be characterized via ordinary dichotomy and exponential dichotomy (relative to a sequence of norms) provided that these behaviors hold for suitable systems, with respect to well-chosen sequences of norms and relative to appropriate sequences of projections. This characterization (and its consequence in the classic case) is important in the methods that will be employed to explore the polynomial behaviors (among we mention those via input-output techniques or those devoted to their robustness). In addition, it opens new perspectives in this area, as the study begun here can be extended to other classes of dynamical systems.

A first class of methods in which we will apply the central results of the present paper will be devoted to admissibility criteria for polynomial dichotomies and will be provided in [27], based on a new and different approach compared with the one considered in [24].

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[1]Faculty of Mathematics
University of Rijeka, Croatia
Email address: ddragicevic@math.uniri.hr

[2]Department of Mathematics
West University of Timişoara, Romania
Email address: adina.sasu@e-uvt.ro
Email address: bogdan.sasu@e-uvt.ro

[3]Academy of Romanian Scientists, Bucharest, Romania
Email address: adina.sasu@e-uvt.ro
Email address: bogdan.sasu@e-uvt.ro