CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 769-776

Online version at `https://semnul.com/carpathian/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.20

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Fixed points and the stability of the linear functional equations in a single variable

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ABSTRACT. In this paper we prove that an interesting result concerning the generalized Hyers-Ulam stability of the linear functional equation $g(\varphi(x)) = a(x) \bullet g(x)$ on a complete metric group, given in 2014 by S.M. Jung, D. Popa and M.T. Rassias, can be obtained using the fixed point technique. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the linear equation mention above. Our results are also related to a recent result of G.H. Kim and Th.M. Rassias concerning the stability of Psi functional equation.

1. INTRODUCTION

"When a solution of an equation differing slightly from a given one must be somehow near to the solution of the given equation?" is the question formulated in 1940 by S.M. Ulam [33] while giving a lecture at the University of Wisconsin, on the stability group homomorphisms. In a more precise formulation, its problem of stability reads as follows:

Let $(G_1, \circ)$ be a group, $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$ and $\varepsilon > 0$. Does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(x \circ y), f(x) * f(y)) \leq \delta, \quad \text{for all } x, y \in G_1$$

there exists a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), h(x)) \leq \varepsilon, \quad \text{for all } x \in G_1?$$

A first answer to Ulam's question was given by D. H. Hyers [22] in 1941 concerning the Cauchy functional equation. Afterwards different generalizations of that initial answer of Hyers were obtained. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [31] for approximately linear mappings, by considering an unbounded Cauchy difference. See also [17], [30] and [32]. Nowadays we speak about the concept of *Hyers-Ulam stability*.

A further generalization was obtained by P. Găvruţa [18] in 1994. See also [19] and [21] for more generalizations. The papers mentioned above use the direct method (of Hyers), i.e., the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution.

For other results and generalizations, see the books [5],[15], [23],[25] and their references.

Received: 15.05.2022. In revised form: 10.07.2022. Accepted: 14.07.2022
1991 *Mathematics Subject Classification.* 39B62, 39B72, 39B82, 47H10.
Key words and phrases. *fixed points, generalized Hyers-Ulam stability, functional equations in a single variable.*

Among applications of the functional equations, we mention modeling in science and engineering (see [13]). An interesting application of the stability in the sense of Hyers-Ulam pointed out by D.H. Hyers, G. Isac and Th. M. Rassias is in the study of complementarity problems (see the book [23]). For other complementarity problems, see also the article of G. Isac [24].

On the other hand, J.A. Baker [2] used in 1991 the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation.

In 2003, V. Radu [29] proposed a new method, successively developed in [6], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative. For some other applications of the fixed point theorem in the generalized Hyers-Ulam stability see the papers [7], [8], [9], [11], [12], [14], [16], [20], [28].

Recently, J. Brzdęk, J. Chudziak & Z. Páles proved in [3] a general fixed point theorem for (not necessarily) linear operators and they used it to obtain Hyers-Ulam stability results for a class of functional equations in a single variable. A fixed point result of the same type was proved by J. Brzdęk & K. Ciepliński [4] in complete non-Archimedean metric spaces as well as in complete metric spaces. Also, they formulated an open problem concerning the uniqueness of the fixed point.

In the paper [10] we obtained a fixed point theorem for a class of operators with suitable properties, in very general conditions. Also, we showed that some recent results in [3] and [4] can be obtained as particular cases of our theorem. Moreover, by using our outcome, we gave affirmative answer to the open problem of J. Brzdęk & K. Ciepliński, formulated at the end of the paper [4]. We also showed that our main Theorem is an efficient tool for proving generalized Hyers-Ulam stability results of several functional equations in a single variable. To this end, we prove in this paper that an interesting result concerning generalized Hyers-Ulam-Rassias stability of a linear functional equation obtained in 2014 by S.M. Jung, D. Popa and M.T. Rassias in [26] is a particular case of a fixed point theorem given by us in [10]. Moreover, we give a characterization of the functions that can be approximated with a given error, by the solution of the previously mention linear equation.

We consider a nonempty set X, a complete metric space (Y, d) and the mappings

$$\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X \text{ and } \mathcal{T} : Y^X \to Y^X.$$

We recall that, for two sets M and N, N^M is the space of all mappings from M to N and if $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}_+^X$, we write

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{pointwise if} \quad \lim_{n\to\infty} \delta_n(x) = 0 \text{ for every } x \in X.$$

$\mathbb{R}_+$ stands for the set of all nonnegative numbers, i.e., $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^* = (0, \infty)$.

Definition 1.1. [10] *We say that $\mathcal{T}$ is $\Lambda-$ contractive if for all $u, v \in Y^X$ and $\delta \in \mathbb{R}_+^X$ with*

$$d(u(x), v(x)) \le \delta(x), \ \forall\, x \in X,$$

it follows

$$d((\mathcal{T}u)(x), (\mathcal{T}v)(x)) \le (\Lambda\delta)(x), \ \forall\, x \in X.$$

In the paper [10] we obtained the following fixed point theorem:

Theorem 1.1. *We suppose that the operator $\mathcal{T}$ is $\Lambda-$*contractive, *where Λ satisfies the condition:*
(C_1) *for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^X$ such that*

$$\lim_{n\to\infty} \delta_n = 0 \textit{ pointwise, it follows that } \lim_{n\to\infty} \Lambda\delta_n = 0 \textit{ pointwise.}$$

We suppose that $\varepsilon \in \mathbb{R}_+^X$ is a given function such that

$$(C_2) \qquad \varepsilon^*(x) := \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) < \infty, \ \forall x \in X.$$

We consider a mapping $f \in Y^X$ such that

$$(1.1) \qquad d((\mathcal{T}f)(x), f(x)) \leq \varepsilon(x), \ \forall x \in X.$$

Then, for every $x \in X$, the limit

$$(1.2) \qquad g(x) := \lim_{n\to\infty} (\mathcal{T}^n f)(x),$$

exists and the function g is the unique fixed point of $\mathcal{T}$ with the property

$$(1.3) \qquad d((\mathcal{T}^m f)(x), g(x)) \leq \sum_{k=m}^{\infty} \left(\Lambda^k \varepsilon\right)(t), \ x \in X, \ m \in \mathbb{N}.$$

Moreover, if we have

$$(C_3) \qquad \lim_{n\to\infty} \Lambda^n \varepsilon^* = 0 \textit{ pointwise},$$

then g is the unique fixed point of $\mathcal{T}$ with the property

$$(1.4) \qquad d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X.$$

Theorem 1.1 generalizes a result of J. Brzdęk and K. Ciepliński [4] concerning the existence of the fixed points. Moreover, our theorem provides a positive answer to the open question raised by these authors concerning the uniqueness of the fixed point.

2. Stability of the functional equation $g(\varphi(x)) = a(x) \bullet g(x)$

We take a nonempty set X and a complete metric group $(G, \bullet, d)$ with the metric d invariant to the left translation, i.e.,

$$d(x \bullet y, x \bullet z) = d(y, z), \text{ for all } x, y, z \in G.$$

We consider the given functions $\varphi : X \to X$ and $a : X \to G$.

We denote

$$A_n(x) := a\left(\varphi^{n-1}(x)\right) \bullet \ldots \bullet a(\varphi(x)) \bullet a(x), \quad x \in X, \ n \geq 1.$$

We have

$$A_n(\varphi(x)) = A_{n+1}(x) \bullet (a(x))^{-1}, \quad x \in X, n \geq 1,$$

and successive by

$$A_n\left(\varphi^m(x)\right) = A_{n+m}(x) \bullet \left(A_m(x)\right)^{-1}, \quad x \in X, \ m, n \geq 1.$$

In this section we discuss the generalized Hyers-Ulam-Rassias stability of the functional equation

$$(2.5) \qquad g(\varphi(x)) = a(x) \bullet g(x), x \in X,$$

where $g : X \to G$ is the unknown function.

The equation (2.5) is equivalent to

$$(2.6) \qquad (a(x))^{-1} \bullet g(\varphi(x)) = g(x), x \in X.$$

We remark also that

$$(2.7) \qquad g\left(\varphi^n(x)\right) = A_n(x) \bullet g(x), \quad x \in X, n \geq 1.$$

In what follows we will show that the main result of the paper [26] concerning the generalized Hyers-Ulam-Rassias stability of the equation (2.5) is a consequence of our Theorem 1.1. To this end, we start with the presentation of the main result from [26]:

Theorem 2.2. [26] *Let $\varepsilon : X \to \mathbb{R}_+$ be a given function with the property*

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \varepsilon\left(\varphi^k(x)\right) < \infty,\ \forall x \in X.$$

Then, for every function $f : X \to G$ satisfying the inequality

$$d(f(\varphi(x)), a(x) \bullet f(x)) \leq \varepsilon(x),\ \forall x \in X, \tag{2.8}$$

there exists a unique solution g of the equation (2.5) such that

$$d(f(x), g(x)) \leq \varepsilon^*(x), \forall x \in X. \tag{2.9}$$

This solution is given by the formula

$$g(x) := \lim_{n\to\infty} \left(A_n(x)\right)^{-1} \bullet f\left(\varphi^n(x)\right). \tag{2.10}$$

We can easily see that the above theorem is a particular case of our fixed point result emphasized in the first section.

Proof. We take in Theorem 1.1

$$(\mathcal{T}u)(x) = (a(x))^{-1} \bullet u(\varphi(x)) \quad \text{and} \quad (\Lambda\delta)(x) = \delta(\varphi(x)).$$

So, it follows

$$d\left((\mathcal{T}u)(x), (\mathcal{T}v)(x)\right) = d(u(\varphi(x)), v(\varphi(x))) \leq (\Lambda\delta)(x)$$

if

$$d(u(x), v(x)) \leq \delta(x),$$

hence the operator $\mathcal{T}$ is $\Lambda-$ contractive in the sense of the Definition 1.1.

On the other hand, by using the invariance property to the left translation of the metric d and the assumption (2.8), we obtain that (1.1) holds.

Uniqueness of g results also from Theorem 1.1. In fact, we prove that Λ satisfies the hypothesis (C_3) :

$$\begin{aligned} \Lambda^n(\varepsilon^*(x)) &= \Lambda^n\left(\sum_{k=0}^{\infty} \varepsilon\left(\varphi^k(x)\right)\right) = \\ &= \sum_{k=0}^{\infty} \varepsilon\left(\varphi^{n+k}(x)\right) = \sum_{m=n}^{\infty} \varepsilon\left(\varphi^m(x)\right). \end{aligned}$$

Thus

$$\lim_{n\to\infty} \Lambda^n(\varepsilon^*(x)) = 0,\ x \in X.$$

□

In the second Theorem of this section we will give a characterization of the functions $f : X \to G$ that can be approximated with a given error, by a solution of the equation (2.5).

We denote by

$$\mathcal{E}_\varphi = \left\{\varepsilon \in \mathbb{R}_+^X,\ \lim_{n\to\infty} \varepsilon\left(\varphi^n(x)\right) = 0, \forall x \in X\right\}.$$

Theorem 2.3. *The following statements are equivalent:*
(i) *There exists a unique solution g of (2.5) such that*

$$d(f(x), g(x)) \leq \varepsilon(x), \forall x \in X.$$

(ii) $\quad d\left(f\left(\varphi^n(x)\right), A_n(x) \bullet f(x)\right) \leq \varepsilon(x)+\varepsilon\left(\varphi^n(x)\right),\ x \in X, n \geq 1.$
(iii) *there exists* $\quad \delta \in \mathcal{E}_{\varphi}$ $\quad$ *such that*

$$d\left(f\left(\varphi^n(x)\right), A_n(x) \bullet f(x)\right) \leq \varepsilon(x)+\delta\left(\varphi^n(x)\right),\ x \in X, n \geq 1.$$

Proof. $(i) \Rightarrow (ii)$. We have, by using (2.7)

$$\begin{aligned} d\left(f\left(\varphi^n(x)\right), A_n(x) \bullet f(x)\right) & \leq d\left(f\left(\varphi^n(x)\right), g\left(\varphi^n(x)\right)\right)+d\left(g\left(\varphi^n(x)\right), A_n(x) \bullet f(x)\right) \\ & \leq \varepsilon\left(\varphi^n(x)\right)+d\left(A_n(x) \bullet g(x), A_n(x) \bullet f(x)\right) \\ & = \varepsilon\left(\varphi^n(x)\right)+\varepsilon(x). \end{aligned}$$

$(ii) \Rightarrow (iii)$. We take in (ii) $\delta=\varepsilon$.

$(iii) \Rightarrow (i)$. In (iii) with $\varphi^m(x)$ instead of x, we have

$$d\left(f\left(\varphi^{n+m}(x)\right), A_n\left(\varphi^m(x)\right) \bullet f\left(\varphi^m(x)\right)\right) \leq \varepsilon\left(\varphi^m(x)\right)+\delta\left(\varphi^{n+m}(x)\right),$$

which means

$$d\left(f\left(\varphi^{n+m}(x)\right), A_{n+m}(x) \bullet\left(A_m(x)\right)^{-1} \bullet f\left(\varphi^m(x)\right)\right) \leq \varepsilon\left(\varphi^m(x)\right)+\delta\left(\varphi^{n+m}(x)\right),$$

hence

$$d\left(\left(A_{n+m}(x)\right)^{-1} \bullet f\left(\varphi^{m+n}(x)\right),\left(A_m(x)\right)^{-1} \bullet f\left(\varphi^m(x)\right)\right) \leq \varepsilon\left(\varphi^m(x)\right)+\delta\left(\varphi^{n+m}(x)\right).$$

It follows that the sequence

$$\left\{\left(A_n(x)\right)^{-1} \bullet f\left(\varphi^n(x)\right)\right\}_{n \geq 1}$$

is a Cauchy sequence. Since $(G, \bullet, d)$ is complete, it results that there exists

$$g(x):=\lim_{n \rightarrow \infty}\left(A_n(x)\right)^{-1} \bullet f\left(\varphi^n(x)\right), x \in X.$$

We have

$$g(\varphi(x))=a(x) \bullet \lim_{n \rightarrow \infty}\left(A_{n+1}(x)\right)^{-1} \bullet f\left(\varphi^{n+1}(x)\right)=a(x) \bullet g(x), x \in X,$$

hence g is a solution of (2.5) and

$$d\left(g(x),\left(A_m(x)\right)^{-1} \bullet f\left(\varphi^m(x)\right)\right) \leq \varepsilon\left(\varphi^m(x)\right), x \in X, m \geq 1.$$

By (iii) it follows that

$$d\left(\left(A_n(x)\right)^{-1} \bullet f\left(\varphi^n(x)\right), f(x)\right) \leq \varepsilon(x)+\delta\left(\varphi^n(x)\right)$$

and by letting n go to infinity, we obtain

$$d(f(x), g(x)) \leq \varepsilon(x), \forall x \in X.$$

We prove now the uniqueness of g. To this end, let us consider a solution $h: X \rightarrow G$ of the equation (2.5), satisfying the relation

$$d(h(x), f(x)) \leq \varepsilon(x), \forall x \in X.$$

By replacing x by $\varphi^m(x)$, we have

$$d(h\left(\varphi^m(x)\right), f\left(\varphi^m(x)\right)) \leq \varepsilon\left(\varphi^m(x)\right), \forall x \in X.$$

Having in mind that $h\left(\varphi^m(x)\right)=A_m(x) \bullet h(x)$, it follows

$$d\left(h(x),\left(A_m(x)\right)^{-1} \bullet f\left(\varphi^m(x)\right)\right) \leq \varepsilon\left(\varphi^m(x)\right), x \in X.$$

Letting m go to infinity, we obtain

$$d(h(x), g(x)) = 0, \forall x \in X.$$

$\square$

As a direct application of the Theorem 2.3 we will obtain the following result concerning the characterization of the functions $f : \mathbb{R}_+^* \to \mathbb{R}$ that can be approximated with a given error, by the solutions of Digamma functional equation

$$g(x+1) = g(x) + \frac{1}{x},\ x \in \mathbb{R}_+^*. \tag{2.11}$$

Corollary 2.1. *The following statements are equivalent:*
(i) *There exists a unique solution* g *of (2.11) such that*

$$|f(x) - g(x)| \le \varepsilon(x), \forall x \in \mathbb{R}_+^*.$$

(ii)
$$\left| f(x+n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k} \right| \le \varepsilon(x) + \varepsilon(x+n),\ x \in \mathbb{R}_+^*, n \ge 1.$$

(iii) *There exists*

$$\delta \in \mathcal{E}_\varphi := \left\{ \varepsilon : X \to \mathbb{R}_+,\ \lim_{n\to\infty} \varepsilon\left(x+n\right) = 0, \forall x \in \mathbb{R}_+^* \right\}$$

so that

$$\left| f(x+n) - f(x) - \sum_{k=0}^{n-1} \frac{1}{x+k} \right| \le \varepsilon(x) + \delta(x+n),\ x \in \mathbb{R}_+^*, n \ge 1.$$

Proof. The result follows immediately by taking in Theorem 2.3, $X = \mathbb{R}_+^*$, $(G, \bullet) = (\mathbb{R}, +)$, d the Euclidean metric on $\mathbb{R}$, $\varphi(x) = x+1$, $a(x) = \dfrac{1}{x}, x \in \mathbb{R}_+^*$. $\square$

We give below a more general result obtained in the same way, which is in connection with the recent paper of G.H. Kim and Th. M. Rassias [27].

Corollary 2.2. *Let* p *be a positive real number. The following statements are equivalent:*
(i) *There exists a unique solution* g *of the functional equation*

$$g(x+p) = g(x) + a(x),\ x \in \mathbb{R}_+^*$$

such that

$$|f(x) - g(x)| \le \varepsilon(x), (\forall)\ x \in \mathbb{R}_+^*.$$

(ii)
$$\left| f(x+np) - f(x) - \sum_{k=0}^{n-1} a(x+kp) \right| \le \varepsilon(x) + \varepsilon(x+np),\ x \in \mathbb{R}_+^*, n \ge 1.$$

(iii) *There exists*

$$\delta \in \mathcal{E}_\varphi := \left\{ \varepsilon : X \to \mathbb{R}_+,\ \lim_{n\to\infty} \varepsilon\left(x+np\right) = 0, \forall x \in \mathbb{R}_+^* \right\}$$

so that

$$\left| f(x+np) - f(x) - \sum_{k=0}^{n-1} a(x+kp) \right| \le \varepsilon(x) + \delta(x+np),\ x \in \mathbb{R}_+^*, n \ge 1.$$

Funding

This research was funded by a grant of the Romanian Ministry of Research, Innovation and Digitalization, project number PFE 26/30.12.2021, PERFORM-CDI@UPT100 - *The increasing of the performance of the Polytechnic University of Timișoara by strengthening the research, development and technological transfer capacity in the field of "Energy, Environment and Climate Change" at the beginning of the second century of its existence*, within Program 1 - Development of the national system of Research and Development, Subprogram 1.2 - Institutional Performance - Institutional Development Projects - Excellence Funding Projects in RDI, PNCDI III".

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