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Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75^{th} anniversary

Lyapunov Conditions for One-Sided Discrete-Time Random Dynamical Systems

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ABSTRACT. This paper considers nonuniform exponential stability and nonuniform exponential instability concepts for one-sided discrete-time random dynamical systems. These concepts are generalizations from the deterministic case. Using this, characterizations in terms of Lyapunov functions respectively Lyapunov norms are presented. Also, an approach in terms of considered concepts for the inverse and adjoint random discrete-time systems is derived.

1. INTRODUCTION AND PRELIMINARY RESULTS

For time-varying linear systems the stability analysis is one of the key issues. Important results were derived in the framework of exponential stability for infinite dimensional systems (see [4], [17], [16], [20]). For an overview on their history and important accommmplishments in the variational case see [19] and the reference therein. In [9], [14], respectively [13] we have extended these results for both exponential stability and exponential instability. It is worth mentioning, in this line the paper of Aitken, i.e. [1] for various applications. Moreover, [7], [10], [11] and [12] deal with the exponential stability concept for linear cocycles. For exponential instability concept see for example [18]. Interesting studies regarding nonuniform exponential behaviors with constant exponent have been reported in [6]. In the stochastic framework we refer the reader to [2].

In this line, in the present work we consider random discrete-time systems which are defined only on the semi-axes, the so-called one-sided systems. Key arguments and motivation in this direction for considering such systems, that is non-invertible systems have been presented in [21] and [15].

The purpose of the present paper is to extend various results from the deterministic case of linear discrete-time skew-product over semiflows to the stochastic one-sided discrete-time random dynamical systems. In Section 2 we introduce the concept of nonuniform exponential stability for a random discrete-time system and we give some necessary and sufficient conditions for this property. Thus, we extend some results obtained in [14] for nonuniform exponential stability of random one-sided discrete-time system. Also, there is proved a characterization of nonuniform exponential stability in terms of Lyapunov functions. In Section 3 we establish some results for nonuniform exponential instability of a random discrete-time system in the spirit on Datko's approach (see [13] for deterministic aproach). The characterization of this property in terms of Lyapunov functions is also obtained. In Section 4 we extend our study to the case of Lyapunov

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norms and finally, in Section 5, characterizations for nonuniform exponential stability and nonuniform exponential instability of a random one-sided discrete-time system using the inverse and the adjoint random discrete-time system are obtained.

1.1. **Notations and preliminary results.** Let $\mathbb{Z}_+$ denotes the set of positive integers. $(X, \|\cdot\|)$ denotes a Banach space. $\mathcal{B}(X)$ is the Banach algebra of all bounded linear operators acting from X into X. $(\Omega, \mathfrak{F}, \mathbb{P})$ denotes a probability space. $\theta : \Omega \to \Omega$ is a measurable map preserving the probability measure $\mathbb{P}$, that is $\mathbb{P} \circ \theta = \mathbb{P}$, hence $\mathbb{P}(\theta B) = \mathbb{P}(B)$, for any $B \in \mathfrak{F}$.

A random variable $\varphi : \Omega \to (0, +\infty)$ is $\theta-$invariant if $\varphi \circ \theta = \varphi$, that is $\varphi(\theta\omega) = \varphi(\omega)$, for all $\omega \in \Omega$. By convention we have $\theta^0 = I_\Omega$, where I denotes the identity. As a fast property we have that $\theta^n \circ \theta^m = \theta^{n+m} = \theta^{m+n} = \theta^m \circ \theta^n$, for all $m, n \in \mathbb{Z}_+$.

The application $\mathbb{Z}_+ \times \Omega \ni (n, z) \to \theta^n\omega \in \Omega$ is measurable for all $n \in \mathbb{Z}_+$. By induction we obtain that $\mathbb{P} \circ \theta^n = \mathbb{P}$, i.e. $\mathbb{P}(\theta^n B) = \mathbb{P}(B)$, for all $n \in \mathbb{Z}_+$ and all $B \in \mathfrak{F}$. Also, by induction we have that $\varphi(\theta^n\omega) = \varphi(\omega)$, for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

We consider the metric semi-dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, which is a probability space, with $\theta : \Omega \to \Omega$, measurable. A linear random one-sided discrete-time system on X over a measurable semi-dynamical system θ, is a measurable application $\phi : \mathbb{Z}_+ \times \Omega \in \mathcal{B}(X)$. For a more detailed presentation regarding these notions we can point out the references [5] and [3]. In the following, we shall review only the properties which will be of interest in this paper:

(a) $\phi(0, \omega) = I_X$, for all $\omega \in \Omega$;
(b) $\phi(n + m, \omega) = \phi(n, \theta^m\omega)\phi(m, \omega)$, for all $n, m \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Obvious from relation (b) we have that

$$\phi(n + m, \omega) = \phi(m, \theta^n\omega)\phi(n, \omega), \quad \text{for all } \ n, m \in \mathbb{Z}_+ \ \text{ and } \ \omega \in \Omega.$$

Throughout this work the notation (θ, ϕ) will be used for a linear random one-sided discrete-time system (RDTS).

2. Nonuniform exponential stability

Definition 2.1. *We say that a RDTS (θ, ϕ) is nonuniformly exponentially stable (NES) if there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$, and $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n, \omega)x\| \leq N(\omega)e^{-\alpha(\omega)n}\|x\|, \tag{2.1}$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

We observe that previous definition provides us the following equivalent relations.

Proposition 2.1. *The following are equivalent:*

(a) *the RDTS (θ, ϕ) is NES;*
(b) *there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$ and $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n + m, \omega)x\| \leq N(\omega)e^{-\alpha(\omega)(m+n)}\|x\|, \tag{2.2}$$

for all $m, n \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$.

(c) *there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$ and $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n + m, \omega)x\| \leq N(\theta^m\omega)e^{-\alpha(\omega)n}\|\phi(m, \omega)x\|, \tag{2.3}$$

for all $m, n \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$.

(d) *there exists a* $\theta-$*invariant random variable* $\alpha : \Omega \to (0, +\infty)$ *and* $N : \Omega \to [1, +\infty)$ *such that*

$$\|\phi(n+m,\omega)x\| \leq N(\theta^n\omega)e^{-\alpha(\omega)m}\|\phi(n,\omega)x\|, \tag{2.4}$$

for all $m, n \in \mathbb{Z}_+$, $\omega \in \Omega$ *and* $x \in X$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is obvious. The equivalences $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (d)$, respectively, are consequences of

$$\alpha(\theta^m\omega) = \alpha(\theta^n\omega) = \alpha(\omega),$$

for all $m, n \in \mathbb{Z}_+$ and all $\omega \in \Omega$. Indeed

$$\begin{aligned}\|\phi(n+m,\omega)x\| &= \|\phi(n,\theta^m\omega)\phi(m,\omega)x\| \\ &\leq N(\theta^m\omega)e^{-\alpha(\theta^m\omega)n}\|\phi(m,\omega)x\| \\ &= N(\theta^m\omega)e^{-\alpha(\omega)n}\|\phi(m,\omega)x\|\end{aligned}$$

for all $m, n \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$. □

Theorem 2.1. *The RDTS* (θ, ϕ) *is NES if and only if there exists a* $\theta-$*invariant random variable* $\beta : \Omega \to (0, +\infty)$ *and* $D, N : \Omega \to [1, +\infty)$ *such that*

$$\sum_{k=n}^{+\infty} e^{\beta(\omega)(k-n)}N(\theta^n\omega)^{-1}\|\phi(k,\omega)x\| \leq D(\omega)\|\phi(n,\omega)x\| \tag{2.5}$$

for all $\omega \in \Omega$, *and all* $(n, x) \in \mathbb{Z}_+ \times X$.

Proof. Necessity. Let α and N as in Definition 2.1. We consider $0 < \beta < \alpha$. Further, using relation (2.4) from Proposition 2.1 we obtain

$$\begin{aligned}\sum_{k=n}^{\infty} e^{\beta(\omega)(k-n)}N(\theta^n\omega)^{-1}\|\phi(k,\omega)x\| &= \sum_{m=0}^{\infty} e^{\beta(\omega)m}N(\theta^n\omega)^{-1}\|\phi(n+m,\omega)x\| \\ &\leq \sum_{m=0}^{\infty} e^{-(\alpha(\omega)-\beta(\omega))m}\|\phi(n,\omega)x\| \\ &= \frac{1}{1-e^{-\alpha(\omega)+\beta(\omega)}}\|\phi(n,\omega)x\| \\ &= \frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)}-e^{\beta(\omega)}}\|\phi(n,\omega)x\|.\end{aligned}$$

This will conclude that (2.5) is verified.
Sufficiency. For $n = 0$ relation (2.5) became

$$\sum_{k=0}^{\infty} e^{\beta(\omega)k}\|\phi(k,\omega)x\| \leq N(\omega)D(\omega)\|x\|,$$

from where

$$\|\phi(k,\omega)x\| \leq N(\omega)D(\omega)e^{-\beta(\omega)k}\|x\|,$$

which conclude that the RDTS (θ, ϕ) is NES. □

Theorem 2.2. *The RDTS* (θ, ϕ) *is NES if and only if there exists a* $\theta-$*invariant random variable* $\beta : \Omega \to (0, +\infty)$ *and a function* $D : \Omega \to [1, +\infty)$ *such that*

$$\sum_{k=0}^{+\infty} e^{\beta(\omega)k}\|\phi(k,\omega)x\| \leq D(\omega)\|x\| \tag{2.6}$$

for all $(\omega, x) \in \Omega \times X$.

Proof. *Necessity.* We consider $\beta : \Omega \to (0, \infty)$ satisfying $0 < \beta < \alpha$. We obtain

$$\begin{aligned}\sum_{k=0}^{\infty} e^{\beta(\omega)k}\|\phi(k,\omega)x\| &\leq \sum_{k=0}^{\infty} e^{\beta(\omega)k} N(\omega) e^{-\alpha(\omega)k}\|x\| \\ &= N(\omega)\|x\| \sum_{k=0}^{\infty} e^{-(\alpha(\omega)-\beta(\omega))k} \\ &= N(\omega)\|x\| \frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)-e^{\beta(\omega)}}}.\end{aligned}$$

Hence, for $D(\omega) = N(\omega)\frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)}-e^{\beta(\omega)}}$ we have that (2.6) is verified.
Sufficiency. Obviously, for all $(k, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$ we have that

$$\|\phi(k,\omega)x\| \leq D(\omega)e^{-\beta(\omega)k}\|x\|,$$

which concludes that the RDTS (θ, ϕ) is NES. $\square$

We now introduce the function $L : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}_+$ satisfying

$$L(n,\omega,x) \leq K(\omega)\|x\|, \tag{2.7}$$

with the function $K : \Omega \to [1, +\infty)$, where $\omega \in \Omega$ and $(n, x) \in \mathbb{Z}_+ \times X$.

Theorem 2.3. *The RDTS (θ, ϕ) is NES if and only if there exists a function $L : \mathbb{Z}_+\times\Omega\times X \to \mathbb{R}_+$ satisfying (2.7), a θ-invariant random variable $\eta : \Omega \to (0, +\infty)$ such that*

$$L(0,\omega,x) - L(n,\omega,x)e^{\eta(\omega)n} \geq \sum_{k=0}^{n-1} e^{\eta(\omega)k}\|\phi(k,\omega)x\| \tag{2.8}$$

for all $(n, \omega, x) \in \mathbb{Z}_+^* \times \Omega \times X$.

Proof. *Necessity.* Let α and N as in Definition 2.1. We set

$$L(n,\omega,x) := \sum_{k=n}^{\infty} e^{\beta(\omega)(k-n)}\|\phi(k,\omega)x\|,$$

where we have considered $\eta = \beta$ as is Theorem 2.1. First we prove that relation (2.7) is satisfied. Based on (b) from Proposition 2.1 we obtain

$$\begin{aligned}L(n,\omega,x) &= \sum_{m=0}^{\infty} e^{\beta(\omega)m}\|\phi(m+n,\omega)x\| \\ &\leq \sum_{m=0}^{\infty} e^{\beta(\omega)m} N(\omega)e^{-\alpha(\omega)(m+n)}\|x\| \\ &= N(\omega)e^{-\alpha(\omega)n}\|x\| \frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)} - e^{\beta(\omega)}}.\end{aligned}$$

We have that $e^{\alpha(\omega)} > e^{\alpha(\omega)} - e^{\beta(\omega)}$ which is equivalent with $e^{\beta(\omega)} \geq e^0 = 1$. Thus we have obtained that $K(\omega) = \frac{e^{\alpha(\omega)}N(\omega)}{e^{\alpha(\omega)}-e^{\beta(\omega)}} \geq 1$. Hence

$$L(n,\omega,x) \leq K(\omega)\|x\|,$$

This will conclude that (2.7) is verified. Further, we have that

$$\begin{aligned}L(0,\omega,x) &= \sum_{k=0}^{\infty} e^{\beta(\omega)k}\|\phi(k,\omega)x\| \\ &= \|x\| + e^{\beta(\omega)}\|\phi(1,\omega)x\| + \ldots + e^{\beta(\omega)n}\|\phi(n,\omega)x\| + e^{\beta(\omega)(n+1)}\|\phi(n+1,\omega)x\| + \ldots\end{aligned}$$

respectively

$$L(n,\omega,x)=\|\phi(n,\omega)x\|+e^{\beta(\omega)}\|\phi(n+1,\omega)x\|+\dots$$

It follows that

$$L(n,\omega,x)e^{\beta(\omega)n}=L(0,\omega,x)-\sum_{k=0}^{n-1}e^{\beta(\omega)k}\|\phi(k,\omega)x\|,$$

from where

$$L(0,\omega,x)-L(n,\omega,x)e^{\beta(\omega)n}=\sum_{k=0}^{n-1}e^{\beta(\omega)k}\|\phi(k,\omega)x\|$$

which proves that (2.8) is verified.
Sufficiency. From (2.8) we have

$$K(\omega)\|x\|\geq L(0,\omega,x)\geq\sum_{k=0}^{n-1}e^{\beta(\omega)k}\|\phi(k,\omega)x\|.$$

Now, taking the limit for $n\to\infty$ we obtain

$$\sum_{k=0}^{\infty}e^{\beta(\omega)k}\|\phi(k,\omega)x\|\leq K(\omega)\|x\|,$$

from where we have

$$\|\phi(k,\omega)x\|\leq K(\omega)e^{-\beta(\omega)k}\|x\|.$$

thus, we have that RDTS (θ,ϕ) is NES. □

Definition 2.2. *We say that $L_1:\mathbb{Z}_+\times\Omega\times X\to\mathbb{R}_+$ is a Lyapunov function for a RDTS (θ,ϕ) if there exists a $\theta-$invariant random variable $\eta:\Omega\to(0,+\infty)$ such that*

$$L_1(n,\omega,x)+\sum_{k=0}^{n-1}e^{\eta(\omega)k}\|\phi(k,\omega)x\|\leq L_1(0,\omega,x)\tag{2.9}$$

for all $(n,\omega,x)\in\mathbb{Z}_+^*\times\Omega\times X$.

Theorem 2.4. *The RDTS (θ,ϕ) is NES if and only if there exists a Lyapunov function $L_1:\mathbb{Z}_+\times\Omega\times X\to\mathbb{R}_+$ and a function $K:\Omega\to[1,+\infty)$ such that*

$$L_1(0,\omega,x)\leq K(\omega)\|x\|,\tag{2.10}$$

for all $(\omega,x)\in\Omega\times X$.

Proof. Necessity. We suppose that the RDTS (θ,ϕ) is NES. Let α and N as in Definition 2.1 and β as in Theorem 2.1. Let $\eta=\beta$ and $L_1:\mathbb{Z}_+\times\Omega\times X\to\mathbb{R}$ defined for all $(n,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$ by

$$L_1(n,\omega,x):=\sum_{k=n}^{\infty}e^{\eta(\omega)k}\|\phi(k,\omega)x\|.$$

Let $(n,\omega,x)\in\mathbb{Z}_+^*\times\Omega\times X$, then

$$\begin{aligned}L_1(n,\omega,x)+\sum_{k=0}^{n-1}e^{\eta(\omega)k}\|\phi(k,\omega)x\|&=\sum_{k=n}^{\infty}e^{\eta(\omega)k}\|\phi(k,\omega)x\|+\sum_{k=0}^{n-1}e^{\eta(\omega)k}\|\phi(k,\omega)x\|\\&=\sum_{k=0}^{\infty}e^{\eta(\omega)k}\|\phi(k,\omega)x\|=L_1(0,\omega,x).\end{aligned}$$

Moreover

$$\begin{aligned} L_1(0,\omega,x) &= \sum_{k=0}^{\infty} e^{\eta(\omega)k}\|\phi(k,\omega)x\| \\ &\le \sum_{k=0}^{\infty} e^{\eta(\omega)k} N(\omega) e^{-\alpha(\omega)k}\|x\| \\ &= N(\omega)\sum_{k=0}^{\infty} e^{-(\alpha(\omega)-\eta(\omega))k}\|x\|. \end{aligned}$$

Hence, relation (2.10) is verified.

Sufficiency. Let $(n,\omega,x)\in\mathbb{Z}_+^*\times\Omega\times X$. We have that

$$\sum_{k=0}^{n-1} e^{\eta(\omega)k}\|\phi(k,\omega)x\| \le L_1(0,\omega,x) \le K(\omega)\|x\|,$$

respectively

$$\sum_{k=0}^{\infty} e^{\eta(\omega)k}\|\phi(k,\omega)x\| \le K(\omega)\|x\|.$$

Hence

$$\|\phi(k,\omega)x\| \le K(\omega)e^{-\eta(\omega)k}\|x\|$$

for all $(k,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$. Thus, we obtain that RDTS (θ,ϕ) is NES, which conclude the proof. $\square$

3. Nonuniform exponential instability

Definition 3.3. *We say that a RDTS (θ,ϕ) is nonuniformly exponentially instable (NEI) if there exists a $\theta-$invariant random variable $\alpha:\Omega\to(0,+\infty)$, and $N:\Omega\to[1,+\infty)$ such that*

$$N(\omega)\|\phi(n,\omega)x\| \ge e^{\alpha(\omega)n}\|x\|, \tag{3.11}$$

for all $(n,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$.

We observe that previous definition can be stated as follows:

Remark 3.1. *The following are equivalent:*

(a) *the RDTS (θ,ϕ) is NEI;*

(b) *there exists a $\theta-$invariant random variable $\alpha:\Omega\to(0,+\infty)$ and $N:\Omega\to[1,+\infty)$ such that*

$$N(\theta^m\omega)\|\phi(n+m,\omega)x\| \ge e^{\alpha(\omega)n}\|\phi(m,\omega)x\|, \tag{3.12}$$

for all $m,n\in\mathbb{Z}_+$, $\omega\in\Omega$ and $x\in X$.

(c) *there exists a $\theta-$invariant random variable $\alpha:\Omega\to(0,+\infty)$ and $N:\Omega\to[1,+\infty)$ such that*

$$N(\theta^n\omega)\|\phi(n+m,\omega)x\| \ge e^{\alpha(\omega)m}\|\phi(n,\omega)x\|, \tag{3.13}$$

for all $m,n\in\mathbb{Z}_+$, $\omega\in\Omega$ and $x\in X$.

(d) *there exists a $\theta-$invariant random variable $\alpha:\Omega\to(0,+\infty)$ and $N:\Omega\to[1,+\infty)$ such that*

$$N(\omega)\|\phi(n+m,\omega)x\| \ge e^{\alpha(\omega)(n+m)}\|x\|,$$

for all $m,n\in\mathbb{Z}_+$, $\omega\in\Omega$ and $x\in X$.

Theorem 3.5. *The RDTS (θ, ϕ) is NEI if and only if there exists a $\theta-$invariant random variable $\delta : \Omega \to (0, +\infty)$ and $M, N : \Omega \to [1, +\infty)$ such that*

$$\sum_{k=0}^{n} e^{\delta(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\| \leq M(\omega)\|\phi(n, \omega)x\| \tag{3.14}$$

for all $\omega \in \Omega$, and all $(n, x) \in \mathbb{Z}_+ \times X$.

Proof. First we prove the implication (3.11)⇒(3.14). From

$$\|\phi(n, \omega)x\| = \|\phi(n - k + k, \omega)x\| \geq e^{\alpha(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\|$$

it follows that

$$\begin{aligned}\sum_{k=0}^{n} e^{\delta(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\| &\leq \sum_{k=0}^{n} e^{\delta(\omega)(n-k)} e^{-\alpha(\omega)(n-k)} \|\phi(n, \omega)x\| \\ &= \|\phi(n, \omega)x\| \sum_{k=0}^{n} e^{(\delta(\omega)-\alpha(\omega))(n-k)}\end{aligned}$$

If $\delta : \Omega \to (0, +\infty)$ verifies the property $0 < \delta(\omega) < \alpha(\omega)$ one can check that

$$\sum_{k=0}^{n} e^{\delta(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\| \leq \|\phi(n, \omega)x\| \frac{1 - e^{-(\alpha(\omega)-\delta(\omega))(n+1)}}{1 - e^{-\alpha(\omega)+\delta(\omega)}}$$

$$\leq \|\phi(n, \omega)x\| \frac{1}{1 - e^{-\alpha(\omega)+\delta(\omega)}} = \frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)} - e^{\delta(\omega)}} \|\phi(n, \omega)x\| = M(\omega)\|\phi(n, \omega)x\|.$$

To show that the converse implication holds, we consider $k = 0$ and we have that

$$e^{\delta(\omega)n} \|x\| \leq N(\omega)M(\omega)\|\phi(n, \omega)x\|,$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. Thus the proof ends. □

Definition 3.4. *We say that $L_2 : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}_+$ is a Lyapunov function for a RDTS (θ, ϕ) if there exists a $\theta-$invariant random variable $\xi : \Omega \to (0, +\infty)$ and a function $N : \Omega \to [1, +\infty)$ such that*

$$L_2(0, \omega, x) + \sum_{k=0}^{n-1} e^{\xi(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\| \leq L_2(n, \omega, x) \tag{3.15}$$

for all $(n, \omega, x) \in \mathbb{Z}_+^* \times \Omega \times X$.

Theorem 3.6. *The RDTS (θ, ϕ) is NEI if and only if there exists a Lyapunov function $L_2 : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}_+$ and a function $K : \Omega \to [1, +\infty)$ such that*

$$L_2(n, \omega, x) \leq K(\omega)\|\phi(n, \omega)x\|, \tag{3.16}$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Necessity. Suppose that the RDTS (θ, ϕ) is NEI. Let α and N as in Definition 3.3 and δ as in Theorem 3.5. Let $\xi = \delta$ and $L_2 : \mathbb{Z}_+ \times \Omega \times X \to \mathbb{R}$ defined for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$ by

$$L_2(n, \omega, x) := \begin{cases} \sum_{k=0}^{n} e^{\xi(\omega)(n-k)} N(\theta^k \omega)^{-1} \|\phi(k, \omega)x\|, & \text{if } n > 0 \\ 0, & \text{if } n = 0. \end{cases}$$

Further, we have to take into account two cases. First, we consider $n = 0$. In this situation it is obvious that

$$L_2(0, \omega, x) = 0 \leq M(\omega)\|\phi(0, \omega)x\| = M(\omega)\|x\|.$$

Second, we consider $(n,\omega,x)\in\mathbb{Z}_+^*\times\Omega\times X$. Then

$$\begin{aligned}L_2(0,\omega,x)+\sum_{k=0}^{n-1}e^{\xi(\omega)(n-k)}N(\theta^k\omega)^{-1}\|\phi(k,\omega)x\| &= \sum_{k=0}^{n}e^{\xi(\omega)(n-k)}N(\theta^k\omega)^{-1}\|\phi(k,\omega)x\|\\ &= L_2(n,\omega,x).\end{aligned}$$

Further, using Theorem 3.5 we have that

$$\begin{aligned}L_2(n,\omega,x) &= \sum_{k=0}^{n}e^{\xi(\omega)(n-k)}N(\theta^k\omega)^{-1}\|\phi(k,\omega)x\|\\ &\le M(\omega)\|\phi(n,\omega)x\|.\end{aligned}$$

Hence (3.16) holds.
Sufficiency. Let $(n,\omega,x)\in\mathbb{Z}_*^+\times\Omega\times X$. Then we have

$$\begin{aligned}&\sum_{k=0}^{n}e^{\xi(\omega)(n-k)}N(\theta^k\omega)^{-1}\|\phi(k,\omega)x\|\\ &=\sum_{k=0}^{n-1}e^{\xi(\omega)(n-k)}N(\theta^k\omega)^{-1}\|\phi(k,\omega)x\|+N(\theta^n\omega)^{-1}\|\phi(n,\omega)x\|\\ &\le L_2(n,\omega,x)+\|\phi(n,\omega)x\|\\ &\le K(\omega)\|\phi(n,\omega)x\|+\|\phi(n,\omega)x\|\\ &=(K(\omega)+1)\|\phi(n,\omega)x\|.\end{aligned}$$

This will conclude that the RDTS (θ,ϕ) is NEI. Thus the proof is complete. □

4. Lyapunov norms

Definition 4.5. *We say that a RDTS (θ,ϕ) has exponential growth if there exists a $\theta-$invariant random variable $\beta:\Omega\to(0,+\infty)$ and a function $M:\Omega\to[1,+\infty)$ such that*

$$\|\phi(n,\omega)x\|\le M(\omega)e^{\beta(\omega)n}\|x\|,$$

for all $(n,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$.

Definition 4.6. *Let (θ,ϕ) be a RDTS with exponentially growth. For any $\omega\in\Omega$ the application $\|\cdot\|_\omega:X\to\mathbb{R}_+$ defined for any $x\in X$ by*

$$\|x\|_\omega=\sup_{n\ge0}e^{-\beta(\omega)n}\|\phi(n,\omega)x\| \tag{4.17}$$

it is called Lyapunov norm generated by the RDTS (θ,ϕ).

Remark 4.2. (a) *According to the Definition 4.5, one can easily seen that any RDTS (θ,ϕ) which is NES has exponentially growth.*
(b) *Because $\|\phi(0,\omega)x\|=\|x\|$ we have that*

$$\|x\|\le\|x\|_\omega\le M(\omega)\|x\|. \tag{4.18}$$

(c) *If the RDTS (θ,ϕ) is NES then we may consider $M=N$ and $\beta=\alpha$, in this case the inequalities from (4.17) became*

$$\|x\|\le\|x\|_\omega\le N(\omega)\|x\|.$$

(d) *The topology generated by family of norms* $\{\|\cdot\|_\omega \ : \ \omega\in\Omega\}$ *does not depend on the* $\theta-$*invariant random variable* $\beta:\Omega\to(0,+\infty)$. *Indeed, let* $x\in X$ *and*

$$\|x\|_{\omega,\beta}=\sup_{n\geq 0} e^{-\beta(\omega)n}\|\phi(n,\omega)x\|.$$

If we consider $\beta_1,\beta_2:\Omega\to(0,+\infty)$ *two* $\theta-$*invariant random variables such that* $\beta_1\leq\beta_2$, *then*

$$\|x\|_{\omega,\beta_2}\leq\|x\|_{\omega,\beta_1}.$$

Based on Banach's theorem (see for example [8]*) we have that the norms* $\|\cdot\|_{\omega,\beta_1}$ *and* $\|\cdot\|_{\omega,\beta_2}$ *are equivalent.*

If we consider $\beta_1,\beta_2:\Omega\to(0,+\infty)$ *two arbitrary* $\theta-$*invariant random variables, then we can consider* $\beta_3=\max\{\beta_1,\beta_2\}$. *From the above, both* $\|\cdot\|_{\omega,\beta_3}$ *and* $\|\cdot\|_{\omega,\beta_1}$, *respectively,* $\|\cdot\|_{\omega,\beta_3}$ *and* $\|\cdot\|_{\omega,\beta_2}$ *are equivalent, from where it results that the norms* $\|\cdot\|_{\omega,\beta_1}$ *and* $\|\cdot\|_{\omega,\beta_2}$ *are equivalent. This proves that the norms* $\|\cdot\|_{\omega,\beta}$ *are equivalent no matter of the* $\theta-$*invariant random variable* $\beta:\Omega\to(0,+\infty)$.

Lemma 4.1. *If the RDTS* (θ,ϕ) *has exponentially growth then*

$$\|\phi(m,\omega)x\|_{\theta^m\omega}\leq e^{\beta(\omega)m}\|x\|_\omega, \tag{4.19}$$

for all $(m,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$.

Proof. By direct computation one obtains

$$\begin{aligned}\|\phi(m,\omega)x\|_{\theta^m\omega}&=\sup_{k\geq 0} e^{-\beta(\theta^m\omega)k}\|\phi(k,\theta^m\omega)\phi(m,\omega)x\|\\&=\sup_{k\geq 0} e^{-\beta(\omega)k}\|\phi(k+m,\omega)x\|=\sup_{j\geq m} e^{-\beta(\omega)(j-m)}\|\phi(j,\omega)x\|\\&\leq e^{\beta(\omega)m}\sup_{j\geq 0} e^{-\beta(\omega)j}\|\phi(j,\omega)x\|=e^{\beta(\omega)m}\|x\|_\omega.\end{aligned}$$

□

Theorem 4.7. *Let* (θ,ϕ) *be a RDTS with exponentially growth. Then* (θ,ϕ) *is NES if and only if there exists a* $\theta-$*invariant random variable* $\xi:\Omega\to(0,+\infty)$ *and a function* $D:\Omega\to[1,+\infty)$ *such that*

$$\sum_{m=0}^{\infty} e^{\xi(\omega)m}\|\phi(m,\omega)x\|_{\theta^m\omega}\leq D(\omega)\|x\|_\omega, \tag{4.20}$$

for all $(\omega,x)\in\Omega\times X$.

Proof. Necessity. Based on Definition 2.1 there exists a $\theta-$invariant random variable $\alpha:\Omega\to(0,+\infty)$ and a function $N:\Omega\to[1,+\infty)$ such that

$$\|\phi(n,\omega)x\|\leq N(\omega)e^{-\alpha(\omega)n}\|x\|, \tag{4.21}$$

for all $(n,\omega,x)\in\mathbb{Z}_+\times\Omega\times X$. Further, using (4.21), (4.18), the exponential growth property and Lemma 4.1 we have that

$$\begin{aligned}\|\phi(m,\omega)x\|_{\theta^m\omega}&=\sup_{k\geq 0} e^{-\beta(\theta^m\omega)k}\|\phi(k,\theta^m\omega)\phi(m,\omega)x\|\\&=\sup_{k\geq 0} e^{-\beta(\omega)k}\|\phi(k+m,\omega)x\|\leq\sup_{k\geq 0} e^{-\beta(\omega)k}N(\omega)e^{-\alpha(\omega)(k+m)}\|x\|\\&=N(\omega)\|x\|e^{-\alpha(\omega)m}\sup_{k\geq 0} e^{-(\beta(\omega)+\alpha(\omega))k}\\&\leq N(\omega)\|x\|e^{-\alpha(\omega)m}\leq N(\omega)\|x\|_\omega e^{-\alpha(\omega)m}.\end{aligned}$$

Now, let $\xi : \Omega \to (0, +\infty)$ be a $\theta-$invariant random variable, which, without loss of generality satisfy $0 < \xi < \alpha$. Then

$$\sum_{m=0}^{\infty} e^{\xi(\omega)m} \|\phi(m,\omega)x\|_{\theta^m\omega} \le \sum_{m=0}^{\infty} e^{\xi(\omega)m} N(\omega)\|x\|_\omega e^{-\alpha(\omega)m}$$
$$= N(\omega)\|x\|_\omega \frac{e^{\alpha(\omega)}}{e^{\alpha(\omega)} - e^{\xi(\omega)}}$$

which for $D(\omega) = \frac{e^{\alpha(\omega)}N(\omega)}{e^{\alpha(\omega)}-e^{\xi(\omega)}}$ satisfy (4.20).

Sufficiency. Let $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. By direct calculations one obtains that

$$e^{\xi(\omega)m}\|\phi(m,\omega)x\| \le e^{\xi(\omega)m}\|\phi(m,\omega)x\|_{\theta^m\omega}$$
$$\le D(\omega)\|x\|_\omega \le D(\omega)M(\omega)\|x\|,$$

and so

$$\|\phi(m,\omega)x\| \le K(\omega)e^{-\xi(\omega)m}\|x\|.$$

Thus the proof ends. □

5. Applications to the inverse and adjoint systems

To any RDTS (θ, ϕ) we can associate in a natural way two random one-sided discrete-time systems (θ, ϕ^{-1}) and (θ, ϕ^*), i.e. the inverse and the adjoint one. Further, we will review some properties for these systems. Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$. Then $\phi(n, \omega)^{-1} : X \to X$ satisfy

$$\phi(n,\omega)\phi(n,\omega)^{-1}x = \phi(n,\omega)^{-1}\phi(n,\omega)x = x,$$

for all $x \in X$. Respectively, $\phi(n,\omega)^* : X^* \to X^*$ satisfy

$$\phi(n,\omega)^* y^*(x) = y^*(\phi(n,\omega)x),$$

for all $y^* \in X^*$ and $x \in X$, where X^* is the topological dual space of X.

Proposition 5.2. *The RDTS (θ, ϕ) is NES if and only if there exists a $\theta-$invariant random variable $\alpha : \Omega \to (0, +\infty)$ and a function $N : \Omega \to [1, +\infty)$ such that*

$$\|\phi(n,\omega)\| \le N(\omega)e^{-\alpha(\omega)n}, \tag{5.22}$$

for all $(n, \omega) \in \mathbb{Z}_+ \times \Omega$.

Proof. For the necessity part, passing to supremum with $\|x\| = 1$ in (2.1) we obtain that (5.22) is true, for all $(n, \omega) \in \mathbb{Z}_+ \times X$. For the sufficient part, let $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. Then

$$\|\phi(n,\omega)x\| \le \|\phi(n,\omega)\| \cdot \|x\| \le N(\omega)e^{-\alpha(\omega)n}\|x\|,$$

which confirms that the RDTS (θ, ϕ) is NES. □

Proposition 5.3. *The RDTS (θ, ϕ) is NES if and only if the RDTS (θ, ϕ^*) is NES.*

Proof. Necessity. Let $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$ and $y^* \in X^*$. Let α and N as in Definition 2.1 such that (2.1) is valid. Then

$$\|\phi(n,\omega)^* y^*(x)\| = \|y^*(\phi(n,\omega)x)\| \le \|y^*\| \cdot \|\phi(n,\omega)x\|$$
$$\le N(\omega)e^{-\alpha(\omega)n}\|y^*\| \cdot \|x\|.$$

Passing to supremum with $\|x\| = 1$ we have that

$$\|\phi(n,\omega)^* y^*\| \le N(\omega)e^{-\alpha(\omega)n}\|y^*\|.$$

This allows us to conclude that the RDTS (θ, ϕ^*) is NES.
Sufficiency. One sees that $\|\phi(n,\omega)^*\| = \|\phi(n,\omega)\|$. Finally, using Proposition 5.2 we get

$$\|\phi(n,\omega)\| = \|\phi(n,\omega)^*\| \leq N(\omega)e^{-\alpha(\omega)n}.$$

Hence, RDTS (θ, ϕ) is NES. □

Theorem 5.8. *The following hold:*

(a) *If RDTS (θ, ϕ) is NES then RDTS (θ, ϕ^{-1}) is NEI.*
(b) *If RDTS (θ, ϕ) is NEI then RDTS (θ, ϕ^{-1}) is NES.*
(c) *The RDTS (θ, ϕ) is NES (respectively NEI) if and only if RDTS (θ, ϕ^{-1}) is NEI (respectively NES).*

Proof. Let $(n,\omega) \in \mathbb{Z}_+ \times \Omega$. Let $y \in X$. Then there is a unique $x \in X$ such that $\phi(n,\omega)x = y$ and so $x = \phi(n,\omega)^{-1}y$.
(a). Let α and N as in Definition 2.1 such that (2.1) is valid. By direct computation one obtains

$$\|y\| = \|\phi(n,\omega)x\| \leq N(\omega)e^{-\alpha(\omega)n}\|x\| = N(\omega)e^{-\alpha(\omega)n}\|\phi(n,\omega)^{-1}y\|,$$

respectively,

$$e^{\alpha(\omega)n}\|y\| \leq N(\omega)\|\phi(n,\omega)^{-1}y\|.$$

(b). Let α and N as in Definition 3.3 such that (3.11) is valid. Then

$$\|y\| = \|\phi(n,\omega)x\| \geq N(\omega)^{-1}e^{\alpha(\omega)n}\|x\| = N(\omega)^{-1}e^{\alpha(\omega)n}\|\phi(n,\omega)^{-1}y\|.$$

Hence

$$\|\phi(n,\omega)^{-1}y\| \leq N(\omega)e^{-\alpha(\omega)n}\|y\|.$$

(c). Necessity follows using (a) and (b), respectively. For the sufficiency part we use the same arguments for the RDTS (θ, ϕ^{-1}) using the fact that $(\theta, \phi) = (\theta, (\phi^{-1})^{-1})$. □

Theorem 5.9. *The RDTS (θ, ϕ) is NEI if and only if RDTS (θ, ϕ^*) is NEI.*

Proof. We have that RDTS (θ, ϕ) is NEI if and only if (θ, ϕ^{-1}) is NES (via Theorem 5.8). This is equivalent with $(\theta, (\phi^{-1})^*) = (\theta, (\phi^*)^{-1})$ is NES, respectively $(\theta, ((\phi^*)^{-1})^{-1}) = (\theta, \phi^*)$ is NEI, which completes the proof. □

6. Conclusions

Nonuniform exponential stability and nonuniform exponential instability in the context of random semi-dynamical systems have been introduced. Based on this, we have derived a set of necessary and sufficient conditions that assure the existence of Lyapunov functions. Thus, various results have been extended from the deterministic case of linear cocycles over semiflows to the stochastic discrete-time one-sided dynamical systems in the nonuniform framework.

References

[1] Aitken, V. C.; Schwartz, H. M. On the exponential stability of discrete-time systems with applications in observer design. *IEEE Trans. Automat. Control* **39** (1994), no. 9, 1959–1962.
[2] Arnold, L. Random dynamical systems. *Springer Monographs in Mathematics*. Springer-Verlag, Berlin, (1998).
[3] Nguyen, D. C. Topological dynamics of random dynamical systems. *Oxford Mathematical Monographs*. The Clarendon Press, Oxford University Press, New York, (1997).
[4] Datko, R. Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.* **3** (1972), 428–445.
[5] Doan, T. S. Lyapunov exponents for random dynamical systems, *PhD Technischen Universitat Dresden*, (2009).

[6] Dragičević, D.; Sasu, A. L.; Sasu, B. Input-output criteria for stability and expansiveness of dynamical systems, *Appl. Math. Comput.* **414** (2022), 1–22.
[7] Dragičević, D. On the exponential stability and hyperbolicity of linear cocycles, *Linear Multilinear Algebra* **69**, (2021), no. 2, 259–277.
[8] Kreysszig, E. Introductory functional analysis with applications, *Wiley Classics Library. John Wiley & Sons, Inc., New York*, (1978).
[9] Lupa, N.; Popescu, L. H. A complete characterization of exponential stability for discrete dynamics, *J. Difference Equ. Appl.* **23** (2017), no. 12, 2072–2092.
[10] Megan, M.; Sasu, A. L.; Sasu, B. On uniform exponential stability of linear skew-product semiflows in Banach spaces, *Bull. Belg. Math. Soc. Simon Stevin* **9** (2002), no. 1, 143–154.
[11] Megan, M.; Sasu, A. L.; Sasu, B. Perron conditions for uniform exponential expansiveness of linear skew-product flows, *Monatsh. Math.* **138** (2003), no. 2, 145-157.
[12] Pötzsche, C. Geometric Theory of Discrete Nonautonomous Dynamical Systems, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, (2010).
[13] Popa, I.-L.; Ceauşu, T.; Megan, M. Nonuniform power instability and Lyapunov sequences, *Appl. Math. Comput.* **247** (2014), 969–975.
[14] Popa, I.-L.; Ceauşu, T.; Megan, M. On exponential stability for linear discrete-time systems in Banach spaces, *Comput. Math. Appl.* **63** (2012), no. 11, 1497–1503.
[15] Popa, I.-L. Lyapunov functions for random semi-dynamical systems in terms of tempered exponential splitting, *Math. Methods Appl. Sci.* **44** (2021), no. 15. 11923–11932.
[16] Przyluski, K. M.; Rolewicz, S. On stability of linear time-varying infinite-dimensional discrete-time systems, *Systems Control Lett.* **4** (1984), no. 5, 307–315.
[17] Rolewicz, S. On uniform N-equistability, *J. Math. Anal. Appl.* **115** (1986), no. 2, 434–441.
[18] Sasu, A. L. Integral characterizations for stability of linear skew-product semiflows, *Math. Inequal. Appl.* **7** (2004), no. 4, 535–542.
[19] Sasu, A. L.; Megan, M.; Sasu, B. On Romewicz-Zabczyk techniques in the stability theory of dynamical systems, *Fixed Point Theory* **13** (2012), 205–236.
[20] Zabczyk, Z. Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control Optim.* **12** (1974), 721–735.
[21] Zhou, L.; Lu, K.; Zhang, W. Roughness of tempered exponential dichotomies for infinite-dimensional random difference equations, *J. Differential Equations* **254** (2013), no. 9, 4024–4046.

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