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Geometric inequalities in real Banach spaces with applications

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ABSTRACT. In this paper, new geometric inequalities are established in real Banach spaces. As an application, a new iterative algorithm is proposed for approximating a solution of a split equality fixed point problem (SEFPP) for a quasi-ϕ-nonexpansive semigroup. It is proved that the sequence generated by the algorithm converges *strongly* to a solution of the SEFPP in p-uniformly convex and uniformly smooth real Banach spaces, $p > 1$. Furthermore, the theorem proved is applied to approximate a solution of a variational inequality problem. All the theorems proved are applicable, in particular, in L_p, l_p and the Sobolev spaces, $W_p^m(\Omega)$, for p such that $2 < p < \infty$.

1. INTRODUCTION

In nonlinear operator theory, the study of iterative algorithms for approximating solutions of nonlinear equations in real Banach spaces has become an area of intensive research efforts and a flourishing research area for several authors. However, most of the algorithms proposed and studied are largely confined to real Hilbert spaces. This is understandable because, as is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the nicest geometric properties, most of which characterize inner product spaces and make problems posed in real Hilbert spaces more manageable than those posed in more general Banach spaces. However, as has rightly been observed by M. Hazewinkel, Series Editor, Kluwe Publishers, *"... many, and probably most, mathematical objects and models do not live naturally in Hilbert spaces"*. It is obvious that to extend results established in Hilbert spaces to more general Banach spaces, analogues of geometric identities that characterize inner product spaces have to be developed in more general Banach spaces. Early results in this direction can be found in Bynum [5], Chidume [12], [13], [14], Reich [22],. Most of these analogues now in use have been developed between the mid 1980s and early 1990s (see e.g., Alber [1], Xu [23], Xu and Roach[24], and the references contained in them).
Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) = \{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|,\ \|u^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality map on E*. If $p = 2$, J_2 is called the *normalized duality map* and is denoted by J. In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2}Jx, \quad \text{and} \quad \langle x, J_px \rangle = \|x\|^p,\ \forall x \in E.$$

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It is well-known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive. Furthermore, if E is uniformly smooth, then J is uniformly continuous on bounded sets.

Definition 1.1. *Let E be a real normed space with dimension $E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by*

$$\delta_E(\epsilon) := \left\{1 - \left|\left|\frac{u+v}{2}\right|\right| : ||u|| = ||v|| = 1;\ \epsilon = ||u-v||\right\}.$$

Let $p > 1$ be a real number and $\delta_E : (0,2] \to [0,1]$ be the modulus of convexity of E. Then, a normed space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$.

It is well known that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $1 < p < \infty$, are all p-uniformly convex and that the following estimates hold:

$$\delta_{l_p}(\epsilon) = \delta_{L_p}(\epsilon) = \delta_{W_m^p(\Omega)}(\epsilon) = \begin{cases} \frac{p-1}{8}\epsilon^2 + o(\epsilon^2) > \frac{p-1}{8}\epsilon^2,\ 1 < p < 2; \\ 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}} > \frac{1}{p}\left(\frac{\epsilon}{2}\right)^p\ p \geq 2. \end{cases}$$

(see, e.g., [20], see also [11], page 44).

In developing analogues of Hilbert space identities in more general Banach spaces, the *generalized* duality map, $J_2 = J$, which has become a most important tool in nonlinear operator theory, plays a central role. Most of the geometric inequalities which have been used involve the normalized duality map. Also, from the estimates above, we see that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $1 < p \leq 2$, are all 2-*uniformly convex*. Consequently, several of the extensions of Hilbert space results to more general Banach spaces have been extensions to 2-*uniformly convex spaces*. These spaces **do not include**, for example, the important spaces L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, **for** $2 < p < \infty$.

It is our purpose in Section 3 to establish new geometric inequalities in real Banach spaces which will be useful tools for extending results in Hilbert spaces to, in particular, L_p, l_p and the Sobolev spaces, $W_p^m(\Omega)$, for p such that $2 < p < \infty$. In Section 4, as an application, a new iterative algorithm is proposed for approximating a solution of a *split equality fixed point problem (SEFPP)* for *quasi-ϕ-nonexpansive semigroups*. Using some of the new geometric inequalities, it is proved that the sequence generated by the algorithm converges *strongly* to a solution of the SEFPP in p-uniformly convex and uniformly smooth real Banach spaces, $p > 1$. Furthermore, in Section 5, the theorem proved in Section 4 is applied to approximate a solution of a variational inequality problem. All the theorems proved in this paper are applicable, in particular, in L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for p such that $2 < p < \infty$.

2. Some Known Geometric Inequalities in Real Banach Spaces

For p-uniformly convex smooth normed spaces, the following lemma is well known.

Lemma 2.1. *(Xu, [23]) Let E be a p-uniformly convex and smooth normed real space. Then, there exist constants $d_p > 0$ and $c_p > 0$ such that for every $x, y \in E$, there exists $J_p(x)$ such that the following inequalities hold:*

$$(a)\ ||x+y||^p \geq ||x||^p + p\langle y, J_p(x)\rangle + d_p||y||^p,$$

$$(b)\ ||\lambda x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p - c_p W_p(\lambda)||x-y||^p,$$

for all $\lambda \in [0,1]$ and $W_p(\lambda) := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$, and,

$$(c)\ \langle x-y, J_p(x) - J_p(y)\rangle \geq c_2||x-y||^p.$$

Alber [1] introduced the concept of *generalized projection*.

Definition 2.2. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed and convex subset of E. The mapping $\Pi_C : E \to C$ defined by $x^* = \Pi_C x \in C$ such that*

$$\psi(x^*, x) = \inf_{y \in C} \psi(y, x)$$

is called the generalized projection of E onto C.

Lemma 2.2. *(Alber, [1]) Let C be a nonempty closed and convex subset of a smooth and strictly convex Banach space E. Then,*

$$\psi(x, \Pi_C y) + \psi(\Pi_C y, y) \leq \psi(x, y), \ \forall\, x \in C,\ y \in E.$$

The following important and well known lemma will be used in the sequel.

Lemma 2.3. *(Kamimura and Takahashi, [19]) Let E be a uniformly convex and uniformly smooth real Banach space and $\{x_n\},\ \{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim\limits_{n\to\infty} \psi(x_n, y_n) = 0$, then, $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

3. New Geometric Inequalities in p-uniformly Convex and Smooth Spaces

Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^*. For $p \geq 2$, we define the following new functionals: $\psi_p : E \times E \to \mathbb{R}^+$ by

$$\psi_p(x, y) : \ = \ \|x\|^p - p\langle x, J_p y\rangle + (p-1)\|y\|^p, \ \forall x, y \in E. \tag{3.1}$$

Define $V_p : E \times E^* \to \mathbb{R}$ by

$$V_p(x, x^*) : \ = \ \|x\|^p - p\langle x, x^*\rangle + (p-1)\|x^*\|^{\frac{p}{p-1}}, \ \forall x \in E,\ x^* \in E^*. \tag{3.2}$$

Observe that,

$$\psi_p(x, J_p^{-1}u^*) \ = \ V_p(x, u^*), \ \forall x \in E,\ u^* \in E^*. \tag{3.3}$$

Remarks.

- ψ_p is the Bregman distance for the strictly convex functional $f(x) = ||x||^p,\ p > 1$. Hence,
  $\psi_p(x, y) \geq 0,\ \forall\, x, y \in E$.
- Clearly, $\psi_p(x, x) = 0,\ \forall\, x \in E$.
- If $p = 2$, we shall denote $\psi_2(x, y)$ simply as $\psi(x, y)$, and $V_2(x, x^*)$ as $V(x, x^*)$ so that

$$\psi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \ \forall x, y \in E.$$
$$V(x, x^*) := \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E,\ x^* \in E^*.$$

These equations were first defined by Alber, [1].

We now prove the following *new lemmas* that yield *new geometric inequalities* which are of independent interest. Some of them will be used in the sequel. We first state the following mathematical analysis inequalities which will be used in the proofs of the lemmas.

Proposition 3.1. (a) *For $a,\ b \in \mathbb{R}^+,\ p \geq 2$, the following inequality holds.*

$$a^{\frac{p}{p-1}} + \frac{p}{p-1} a^{\frac{1}{p-1}} b \leq (a+b)^{\frac{p}{p-1}}.$$

(b) *Let $a,\ b \in \mathbb{R}^+,\ p \geq 2$. Then,*

$$\Big(a^p + b^p\Big)^{\frac{1}{p-1}} \ \leq \ a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}. \tag{3.4}$$

Proof. (a). Define $f_b : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_a(x) := (a+x)^{\frac{p}{p-1}} - \frac{p}{p-1} a^{\frac{1}{p-1}} x - a^{\frac{p}{p-1}}, \ \forall\, x \in \mathbb{R}^+.$$

Then, $f_a(0) = 0$, and

$$f_a^{'}(x) = \frac{p}{p-1}\Big[(a+x)^{\frac{1}{p-1}} - a^{\frac{1}{p-1}}\Big] > 0,$$

since $x > 0$. So, f_a is increasing which implies that $f_a(x) > f_a(0)$. Take $x = b$ and the result follows.

(b) Define $f_a : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_a(x) := a^{\frac{p}{p-1}} + x^{\frac{p}{p-1}} - \Big(a^p + x^p\Big)^{\frac{1}{p-1}}.$$

Clearly, $f_a(0) = 0$. Furthermore,

$$\begin{aligned} f_a^{'}(x) &= \frac{p}{p-1}\Big[x^{\frac{1}{p-1}} - \frac{x^{(p-1)}}{(a^p + x^p)^{\frac{p-2}{p-1}}}\Big] > 0 \\ \text{since } (a^p + x^p)^{\frac{p-2}{p-1}} &> x^{\frac{p(p-2)}{p-1}} \text{ and } a > 0. \end{aligned}$$

This implies that f_a is increasing. Since $f_a(0) = 0$, it follows that for all $x > 0$, $f_a(x) > f_a(0)$ $(= 0)$. This completes the proof by taking $x = b$. $\square$

Remark 3.1. *Proposition 3.1 (a) and (b) are used below in the proofs of Lemma 3.5 and Lemma 3.6, respectively.*

Proposition 3.2. *For $a,\ b\ \in \mathbb{R}^+$ $p \geq 2$, the fdollowing inequality holds.*

$$a^{\frac{p}{p-1}} + \frac{p}{p-1} a^{\frac{1}{p-1}} b \leq (a+b)^{\frac{p}{p-1}}.$$

Proof. Define $f_a : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_a(x) := (a+x)^{\frac{p}{p-1}} - \frac{p}{p-1} a^{\frac{1}{p-1}} x - a^{\frac{p}{p-1}}, \ \forall\, x \in \mathbb{R}^+.$$

Then, $f_a(0) = 0$, and

$$f_a^{'}(x) = \frac{p}{p-1}\Big[(a+x)^{\frac{1}{p-1}} - a^{\frac{1}{p-1}}\Big] > 0,$$

since $x > o$. So, f_a is increasing which implies that $f_a(x) > f_a(0)$. Take $x = b$ and the vresult follows. $\square$

Proposition 3.3. *Let $a,\ b \in \mathbb{R}^+,\ p \geq 2$. Then,*

$$\Big(a^p + b^p\Big)^{\frac{1}{p-1}} \leq a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}.$$

Proof. Define $f_a : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_a(x) := a^{\frac{p}{p-1}} + x^{\frac{p}{p-1}} - \Big(a^p + x^p\Big)^{\frac{1}{p-1}}.$$

Clearly, $f_a(0) = 0$. Furthermore,

$$\begin{aligned} f_a^{'}(x) &= \frac{p}{p-1}\Big[x^{\frac{1}{p-1}} - \frac{x^{(p-1)}}{(a^p+x^p)^{\frac{p-2}{p-1}}}\Big] > 0 \\ \text{since } (a^p+x^p)^{\frac{p-2}{p-1}} &> x^{\frac{p(p-2)}{p-1}} \text{ and } a > 0. \end{aligned}$$

This implies that f_a is increasing. Since $f_a(0) = 0$, it follows that for all $x \geq 0$, $f_a(x) \geq f_a(0)$ $(= 0)$. This completes theb proof by setting $x = b$. □

Lemma 3.4. *Let E be a p-uniformly convex and uniformly smooth real Banach space. For $p \geq 2$, wen have:*

$$J_p^{-1}u^* = ||u^*||^{\frac{-(p-2)}{p-1}} J^{-1}u^*.$$

Proof. First, using the definition f J_p, we compute as follows:

$$\begin{aligned} ||u^*|| &= ||J_pJ_p^{-1}u^*|| = ||J_p^{-1}u^*||^{p-2}J(J_p^{-1}u*)|| \\ &= ||J_p^{-1}u^*||^{p-2}||.||J_p^{-1}u^*|| = ||J_p^{-1}u^*||^{p-1} \\ \Rightarrow ||J_p^{-1}u^*|| &= ||u^*||^{\frac{1}{p-1}}. \end{aligned}$$

Now, again using the definition of J_p, we obtain:

$$\begin{aligned} u^* &= J_p(J_p^{-1}u^*) = ||J_p^{-1}u^*||^{p-2}J(J_p^{-1}u^*) \\ \Rightarrow J^{-1}u^* &= ||J_p^{-1}u^*||^{p-2}J^{-1}J(J_p^{-1}u^*) \\ \Rightarrow J^{-1}u^* &= ||J_p^{-1}u^*||^{(p-2)}(J_p^{-1}u^*) \\ \Rightarrow J^{-1}u^* &= ||u^*||^{\frac{(p-2)}{p-1}}(J_p^{-1}u^*) \\ \text{so that, } J_p^{-1}u^* &= ||u^*||^{\frac{-(p-2)}{p-1}}J^{-1}u^*. \end{aligned}$$

□

Corollary 3.1. *Let E be a p-uniformly convex and uniformly smooth real Banach space. For $p \geq 2$, wen have:*

$$||J_p^{-1}u^*|| = ||u^*||^{\frac{1}{p-1}}.$$

Proof. This follows from Lemma 3.4 □

Lemma 3.5. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p \geq 2$,*

$$V_p(u,u^*) + p\langle J_p^{-1}u^* - u, v^*\rangle \leq V_p(u, u^*+v^*),\ \forall u \in E,\ u^*, v^* \in E^*. \tag{3.5}$$

Proof. We compute as follows:

$$\begin{aligned} V_p(u,u^*) + p\langle J_p^{-1}u^* - u, v^*\rangle &= \|u\|^p - p\langle u,u^*\rangle + (p-1)\|u^*\|^{\frac{p}{p-1}} + p\langle J_p^{-1}u^*, v^*\rangle - p\langle u, v^*\rangle \\ &\leq \|u\|^p - p\langle u, u^*+v^*\rangle + (p-1)\Big(\|u^*\|^{\frac{p}{p-1}} + \frac{p}{(p-1)}\|u^*\|^{\frac{1}{p-1}}\|v^*\|\Big) \\ &\leq \|u\|^p - p\langle u, u^*+v^*\rangle + (p-1)\|u^*+v^*\|^{\frac{p}{p-1}} \\ &= V_p(u, u^*+v^*), \end{aligned}$$

establishing the lemma. □

Lemma 3.6. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p > 1$,*

$$\psi_p\big(x, J_p^{-1}(\lambda J_pu + (1-\lambda)J_pv)\big) \leq \lambda\psi_p(x,u) + (1-\lambda)\psi_p(x,v),\ \forall x,u,v \in E. \tag{3.6}$$

Proof. Using definition of ψ_p and Lemma 2.1(b), we compute as follows:

$$\begin{aligned}
\psi_p\big(x, J_p^{-1}(\lambda J_p u + (1-\lambda)J_p v)\big) &= V_p\Big(x, \lambda J_p u + (1-\lambda)J_p v\Big) \\
&= ||x||^p - p\langle x, \lambda J_p u + (1-\lambda)J_p v\rangle \\
&\quad +(p-1)||\lambda J_p u + (1-\lambda)J_p v||^{\frac{p}{p-1}} \\
&\leq ||x||^p - p\lambda\langle x, J_p u\rangle - p(1-\lambda)\langle x, J_p v\rangle \\
&\quad +(p-1)\Big(\lambda||J_p u||^p + (1-\lambda)||J_p v||^p\Big)^{\frac{1}{p-1}} \\
&\leq \lambda||x||^p - p\lambda\langle x, J_p u\rangle + (p-1)\lambda||J_p u||^{\frac{p}{p-1}} \\
&\quad +(1-\lambda)||x||^p - p(1-\lambda)\langle x, J_p v\rangle \\
&\quad +(p-1)(1-\lambda)||J_p v||^{\frac{p}{p-1}} \\
&= \lambda\Big[||x||^p - p\langle x, J_p u\rangle + (p-1)||J_p u||^{\frac{p}{p-1}}\Big] \\
&\quad +(1-\lambda)\Big[||x||^p - p\langle x, J_p v\rangle + (p-1)||J_p v||^{\frac{p}{p-1}}\Big] \\
&= \lambda V_p(x, J_p u) + (1-\lambda)V_p(x, J_p v) \\
&= \lambda\psi_p(x, J_p^{-1}J_p u) + (1-\lambda)\psi_p(x, J_p^{-1}J_p v) \\
&= \lambda\psi_p(x, u) + (1-\lambda)\psi_p(x, v),
\end{aligned}$$

establishing the lemma. □

Lemma 3.7. *Let E be a p-uniformly convex and smooth real Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. Then, $\psi_p(u_n, v_n) \to 0$ implies $\|u_n - v_n\| \to 0$, as $n \to \infty$.*

Proof. Using Lemma 2.1(b), we proceed as follows: replacing y by $(x-y)$ and x by y in the following inequality:

$$||x+y||^p \geq ||x||^p + p\langle y, J_p(x)\rangle + d_p||y||^p,$$

we obtain: $\forall x, y \in E$,

$$\begin{aligned}
d_p\|x-y\|^p &\leq \|x\|^p - \|y\|^p - p\langle x-y, J_p y\rangle \\
&= \Big(\|x\|^p - p\langle x, J_p y\rangle + (p-1)||y||^p\Big) \\
&= \psi_p(x, y).
\end{aligned}$$

This inequality now yields that $d_p\|x-y\|^p \leq \psi_p(x, y)$, from which the lemma follows. □

Lemma 3.8. *Let E be a p-uniformly convex and smooth real Banach space with dual space E^*. For $p > 1$, let $J_p : E \to E^*$ be the generalized duality map. Then,*

$$\|J_p^{-1}u - J_p^{-1}v\| \leq \kappa_p\|u-v\|^{\frac{1}{p-1}}, \ \forall\, u, v \in E^*, \tag{3.7}$$

where $\kappa_p = \left(\frac{1}{c_2}\right)^{\frac{1}{p-1}}$, for some constant $c_2 > 0$.

Proof. We first recall that since E is reflexive, J_p is surjective. For E, the following inequality holds (Lemma 2.1 (c)):

$$\langle x-y, J_p x - J_p y\rangle \geq c_2\|x-y\|^p, \ \forall\, x, y \in E, \tag{3.8}$$

for some constant $c_2 > 0$. This implies that

$$||J_p x - J_p y|| \quad\geq\quad c_2||x-y||^{p-1}, \ \forall\, x, y \in E. \tag{3.9}$$

This inequality implies that J_p is injective. Hence, $J_p^{-1}: E^* \to E$ exists. For $u,\ v \in E^*$, let $J_p^{-1}u = x,\ J_p^{-1}v = y$. Substituting in inequality (3.9), we obtain inequality (3.7), establishing the Lemma. □

Remark 3.2. *Lemmas 3.5 and 3.6 were established for $p = 2$ by Alber [1]. Since for $p > 1$, p-uniformly convex spaces are reflexive and strictly convex, we can use, in this paper, $\psi(x, y)$ and $V(x, x^*)$ instead of $\psi_p(x, y)$ and $V_p(x, x^*)$, respectively, whenever it is convenient to do so.*

Lemma 3.9. *Let C be a nonempty closed and convex subset of a p-uniformly convex and smooth real Banach space, E. Let $x \in E$ be arbitrary and let $P_C(x) = x^*$, where $P_C : E \to C$ denotes the metric projection of E onto C. Then, for arbitrary $u \in C$, the following inequality holds:*

$$\langle u - x^*,\ J_p(x - x^*)\rangle \le 0. \tag{3.10}$$

Proof. Since E is p-uniformly convex and smooth, there exists a constant $d_p > 0$ such that the following inequality holds, (Lemma 2.1(a)):

$$||x + y||^p \ge ||x||^p + p\langle y, J_p(x)\rangle + d_p||y||^p,\ \forall\, x, y \in E. \tag{3.11}$$

From this inequality, it follows that

$$||x + y||^p \le ||x||^p + p\langle y, J_p(x + y)\rangle - d_p||y||^p,\ \forall\, x, y \in E. \tag{3.12}$$

By the convexity of C, we have that $x_\lambda := x^* - \lambda(x^* - u) \in C$, $\lambda \in (0, 1)$. Hence, using inequality (3.12),

$$\begin{aligned} ||x - x^*||^p &\le ||x - x_\lambda||^p = ||(x - x^*) + \lambda(x^* - u)||^p \\ &\le ||x - x^*||^p + p\lambda\langle x^* - u, J_p\big(x - x^* + \lambda(x^* - u)\big)\rangle - d_p\lambda^p||x^* - u||^p, \end{aligned}$$

so that

$$p\langle x^* - u, J_p(x - x^* + \lambda(x^* - u))\rangle \ge d_p\lambda^{p-1}||x^* - u||^p.$$

Letting $\lambda \to 0^+$, we obtain

$$\langle u - x^*,\ J_p(x - x^*)\rangle \le 0,$$

establishing the lemma. □

3.1. **Analytical representations of generalized duality maps in L_p, l_p, and W_m^p, spaces, $1 < p < \infty$.** Using the following analytic representations of the *normalized* duality map, J, in L_p, l_p, and W_m^p, $1 < p < \infty$ (see e.g., Lindenstrauss and Tzafriri [20])

$$\begin{aligned} Jz &= y \in l_q,\ y = \{|z_1|^{p-2}z_1, |z_2|^{p-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ J^{-1}z &= y \in l_p,\ y = \{|z_1|^{q-2}z_1, |z_2|^{q-2}z_2, ...\},\ z = \{z_1, z_2, ...\}, \\ Jz &= \|z\|_{L_p}^{2-p}|z(s)|^{p-2}z(s) \in L_q(G),\ s \in G, \\ J^{-1}z &= |z(s)|^{q-2}z(s) \in L_p(G),\ s \in G, \text{ and} \\ Jz &= \sum_{|\alpha| \le m} (-1)^{|\alpha|}D^\alpha(|D^\alpha z(s)|^{p-2}D^\alpha z(s)) \in W_{-m}^q(G), m > 0, s \in G, \end{aligned}$$

and the relation:

$$J_p(x) = ||x||^{(p-2)}J(x),\ J_p^{-1}u^* = ||u^*||^{(p-2)}J^{-1}u^*,$$

we obtain the analytic representations of the *generalized* duality map, J_p, in these spaces.

Let C be a nonempty closed and convex subset of a real Banach space E, with dual space, E^*.

For any map $T : C \to E$, we shall denote the set of fixed points of T by $F(T) := \{x \in C : Tx = x\}$.

Now,

1. Let E_1, E_2 and E_3 be real normed spaces;
2. Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be nonlinear maps with $F(T) \neq \emptyset$, $F(S) \neq \emptyset$;
3. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear maps.

The *split equality fixed point problem* (SEFPP) is to

$$\text{find } \ x \in F(T), \ \ y \in F(S) \ \text{ such that } \ Ax = By.$$

We shall denote the set of solutions of $SEFPP$ by $\mathcal{F} := \{(x, y) \in F(T) \times F(S) : Ax = By\}$.

The $SEFPP$ studied by Moudafi [21], in the setting of real Hilbert spaces has recently attracted the attention and interest of numerous researchers due to its various applications, for example, in game theory, in intensity-modulated radiation therapy preparation, in decomposition methods for partial differential equations, in fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstruction (see, e.g., Censor and Segal [9], Attouch *et al.* [3], Byrne [6, 7], and the references therein).

The $SEFPP$ is a generalization of the *split common fixed point problem* $(SCFPP)$, which is applicable in several important real-life problems (see, e.g. Censor and Segal [9] and the references therein).

In 2014, Zhao [28] proposed and studied the following algorithm for approximating a solution of $SEFPP$ in real Hilbert spaces:

$$\begin{cases} x_0 \in H_1, \ y_o \in H_2, \\ u_n = x_n - \gamma_n U^*(Ux_n - Vy_n), \quad x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Tu_n), \\ v_n = y_n - \gamma_n V^*(Ux_n - Vy_n), \quad y_{n+1} = \beta_n y_n + (1 - \beta_n)Sv_n), \ n \geq 0, \end{cases} \tag{3.13}$$

where T, S are *quasi-nonexpansive maps* with $F(T) \neq \emptyset$, $F(S) \neq \emptyset$, and $U, \ V$ are bounded linear maps from H_1 and H_2 to H_3, respectively, and H_3 is an arbitrary Hilbert space, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{\gamma_n\}$ is a sequence of positive numbers satisfying appropriate conditions. Assuming $(I - T)$ and $(I - S)$ are demi-closed at *zero*, he proved that the sequence generated by algorithm (3.13) converges *weakly* to a solution of $SEFPP$.

Chidume *et al.* [16] in 2015 studied the convergence of the sequence generated by following algorithm in real Hilbert spaces:

$$\begin{cases} x_1 \in H_1, \ y_1 \in H_2, \\ x_{n+1} = (1 - \alpha)(x_n - \gamma U^*(Ux_n - Vy_n) + \alpha T(x_n - \gamma U^*(Ux_n - Vy_n)), \\ y_{n+1} = (1 - \alpha)(y_n - \gamma V^*(Ux_n - Vy_n)) + \alpha S(y_n - \gamma V^*(Ux_n - Vy_n)), \ n \geq 1, \end{cases} \tag{3.14}$$

where T, S are *demi-contractive maps*, and $U, \ V$ are bounded linear maps from H_1 and H_2 to H_3, respectively, and H_3 is an arbitrary Hilbert space, $\alpha, \ \gamma$ are positive constants satisfying appropriate conditions. Assuming $I - T$ and $I - S$ are both demi-closed at zero and semi-compact, they proved that the sequence generated by algorithm (3.14) converges *strongly* to a solution of $SEFPP$.

Recently, other iterative algorithms for approximating a solution of the $SEFPP$ in Hilbert spaces have been proposed and studied by several authors (see, e.g. Zhao *et al.* [27], Giang *et al.* [18], Chang *et al.* [10], Zhao and He [26], and the references therein).

In 2018, Zhaoli *et al.*, [25] studied the split feasibility problem and fixed point problem in a 2-*uniformly convex and* 2-*uniformly smooth* real Banach space. They considered the following algorithm:

(3.15)

$$\begin{cases} x_1 \in E_1,\ C_1 = E_1, \\ e_n = J_1^{-1}(J_{E_1}x_n - \gamma A^* J_2(P_Q - I)Ax_n), \quad y_n = J_1^{-1}[(1-\beta_n)J_1 e_n + \beta_n J_1 S e_n], \\ C_{n+1} = \{v \in C_n : \psi(v, y_n) \le \psi(v, x_n);\ \psi(v, z_{n,t}) \le \psi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ n \ge 1, \end{cases}$$

where S is a closed quasi-ψ-nonexpansive map, P_Q is the metric projection of E_2 onto Q, $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}, $\{\beta_n\} \subset [\delta, 1)$, $\delta > 0$ and γ is a positive constant satisfying $0 < \gamma < 1/||A||^2 k^2$, $k > 0$ is best smoothness constant of the underlying space. They proved strong convergence of the sequence generated by algorithm (3.15).

Remark 3.3. *A real Banach space that is* 2*-uniformly convex and* 2*-uniformly smooth is necessarily a real Hilbert space. Consequently, the results of Zhaoli et al., [25] are still in real Hilbert spaces.*

In 2019, Chidume *et al.* [15] studied the following *Krasnoselkii-type* algorithm for the SEFPP for quasi-ϕ-nonexpansive semigroups:

$$(3.16) \qquad \begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2,\ e_n \in J_{E_3}(Ax_n - By_n) \\ u_n = J_{E_1}^{-1}(J_{E_1}x_n - \gamma A^* e_n), \quad z_{n,t} = J_{E_1}^{-1}(\beta J_{E_1}x_n + (1-\beta)J_{E_1}T(t)u_n), \\ v_n = J_{E_2}^{-1}(J_{E_2}y_n + \gamma B^* e_n), \quad w_{n,t} = J_{E_2}^{-1}(\beta J_{E_2}y_n + (1-\beta)J_{E_2}S(t)v_n), \\ C_{n+1} = \{v \in C_n : \sup\limits_{t \ge 0} \psi(v, z_{n,t}) \le \psi(v, x_n) + \psi(z, y_n),\ \forall\, z \in \mathfrak{S}\}, \\ Q_{n+1} = \{z \in Q_n : \sup\limits_{t \ge 0} \psi(z, w_{n,t}) \le \psi(v, x_n) + \psi(z, y_n),\ \forall\, v \in \mathfrak{T}\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1,\ n \ge 1, \end{cases}$$

where E_1 and E_2 are 2-*uniformly convex and uniformly smooth real Banach spaces*, E_3 is a real Banach space, $\mathcal{T} := \{T(t) : t \ge 0\}$ and $\mathcal{S} := \{S(t) : t \ge 0\}$ are *closed quasi-*ψ*-nonexpansive semigroups*, A and B are bounded linear maps, $\mathfrak{T} := \bigcap\limits_{t \ge 0} F(T(t))$; $\mathfrak{S} := \bigcap\limits_{t \ge 0} F(S(t))$, $\beta \in (0, 1)$ and γ is some positive constants satisfying appropriate mild conditions.
They proved that the sequence generated by algorithm (3.16) converges *strongly* to some point in the solution set of SEFPP. Furthermore, they applied this result to approximate a solution of a *split equality variational inequality problem in a* 2*-uniformly convex and uniformly smooth real Banach space*.

Remark 3.4. *While* 2*-uniformly convex and uniformly smooth real Banach spaces are more general than real Hilbert spaces, (they include* $L_p,\ l_p,\ W_p^m(\Omega)$ *spaces, for* $1 < p \le 2$*); they exclude some very important real Banach spaces. In particular, they exclude* $L_p,\ l_p,\ W_p^m(\Omega)$ *spaces, for* $2 < p < \infty$.

4. Application to split equality fixed point problem for quasi-ϕ-nonexpansive semigroups

As an application of our main results in Section 3, it is our purpose in this section to introduce a new iterative algorithm for approximating a solution of the $SEFPP$ in p-*uniformly convex and uniformly smooth real Banach spaces* for quasi-ϕ-nonexpansive semigroups. These spaces include, in particular, $L_p,\ l_p$, and the Sobolev spaces, $W_p^m(\Omega)$, **for** p **such that** $2 < p < \infty$.

We begin with the following definitions.

Definition 4.3. *Let C be a nonempty closed and convex subset of a real Banach space E and $T : C \to C$ be a map.*

(1) *T is said to be quasi-ψ-nonexpansive if $F(T) \neq \emptyset,\ \psi(p, Tx) \leq \psi(p, x),\ \forall\, p \in F(T),\ x \in C$.*

(2) *T is demi-closed at zero if for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x^*$ and $||x_n - Tx_n|| \to 0$, then, $x^* = Tx^*$.*

Definition 4.4. *A one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ of maps from C into itself is called a strongly continuous semigroup of Lipschitzian maps on E if it satisfies the following conditions:*
(i) $T(0)x = x$, for all $x \in E$;
(ii) $T(s + t) = T(s)T(t)$, for all $s, t \geq 0$;
(iii) for each $x \in E$, the map $t \mapsto T(t)x$ is continuous;
(iv) for each $t > 0$, there exists a bounded measurable function $T(t) : C \to C$ such that

$$||T(t)x - T(t)y|| \leq L(t)||x - y||, \ \text{for all} \ x, y \in C.$$

(v) A strongly continuous semigroup of Lipschitzian maps $\mathcal{T}$ is called strongly continuous non-expansive if $L(t) = 1$, for each $t \geq 0$.

(vi) A one parameter family $\mathcal{T}$ is called quasi-ψ-nonexpansive semigroup, if conditions (i) - (iii) hold; and $\mathfrak{T} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and $\psi(p, T(t)x) \leq \psi(p, x),\ \forall\, p \in \mathfrak{T},\ x \in C$.

In Theorem 4.1 below, the setting is as follows:

1. E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces, E_3 is an arbitrary smooth real Banach space;
2. $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive mappings;
3. $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear mappings with adjoints $A^*,\ B^*$, respectively;
4. $J_{p_{(1)}}, J_{p_{(2)}}, J_{p_{(3)}}$ are the *generalized duality maps* on E_1, E_2, E_3, respectively, $J^{-1}_{p_{(1)}}, J^{-1}_{p_{(2)}}, J^{-1}_{p_{(3)}}$ are the *generalized duality maps* on E_1^*, E_2^*, E_3^*, respectively, and $\alpha \in (0, 1)$.
5. $\mathfrak{T} := \bigcap_{t\geq 0} F(T(t))$; and $\mathfrak{S} := \bigcap_{t\geq 0} F(S(t))$.
6. κ_p is the constant appearing in Lemma 3.8.
7. $\mathcal{F} = \mathfrak{T} \times \mathfrak{S}$.

The Algorithm.

$$(4.17) \quad \begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ C_1 = E_1,\ Q_1 = E_2,\ e_n = J_{p(3)}(Ax_n - By_n); \\ u_n = J^{-1}_{p(1)}(J_{p(1)}x_n - \gamma A^* e_n), \quad z_{n,t} = J^{-1}_{p(1)}(\alpha J_{p(1)}x_n + (1-\alpha)J_{p(1)}T(t)u_n); \\ v_n = J^{-1}_{p(2)}(J_{p(2)}y_n + \gamma B^* e_n), \quad w_{n,t} = J^{-1}_{p(2)}(\alpha J_{p(2)}y_n + (1-\alpha)J_{p(2)}S(t)v_n); \\ C_{n+1} = \{v \in C_n : \sup\limits_{t\geq 0} \psi_p(v, z_{n,t}) \leq \psi_p(v, x_n) + \psi_p(z, y_n),\ \forall\, z \in \mathfrak{S}\}; \\ Q_{n+1} = \{s \in Q_n : \sup\limits_{t\geq 0} \psi_p(s, w_{n,t}) \leq \psi_p(z, x_n) + \psi_p(s, y_n),\ \forall\, z \in \mathfrak{T}\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad y_{n+1} = \Pi_{Q_{n+1}} y_1,\ n \geq 1. \end{cases}$$

Theorem 4.1. *Let $\mathcal{T} := \{T(t) : t \geq 0\} : E_1 \to E_1$ and $\mathcal{S} := \{S(t) : t \geq 0\} : E_2 \to E_2$ be closed quasi-ψ-nonexpansive semigroups such that $\mathfrak{T} \neq \emptyset$ and $\mathfrak{S} \neq \emptyset$. Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated iteratively by algorithm (4.17). Assume $\mathcal{F} := \{(x, y) \in \mathfrak{T} \times \mathfrak{S} : Ax = By\} \neq \emptyset$, $\beta \in (0, 1)$ and γ is such that $0 < \gamma < \left[\frac{1}{\kappa_p\left(||A||^{\frac{p}{p+1}} + ||B||^{\frac{p}{p+1}}\right)}\right]^{(p-1)}$. Then, $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \mathcal{F}$.*

Proof. The proof is divided into four steps.

Step 1. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are well-defined. First, we prove that C_n and Q_n are closed and convex. Clearly, $C_1 = E_1$ and $Q_1 = E_2$ are closed and convex. Assume that C_n and Q_n are closed and convex for some $n \geq 1$. Then, from the definition of C_{n+1}, we have that:

$$\begin{aligned} C_{n+1} &= \{v \in C_n : \sup_{t\geq 0} \psi_p\,(v, z_{n,t}) \leq \psi_p(v, x_n) + \psi_p(z, y_n),\ \forall\, z \in \mathfrak{S}\} \\ &= \bigcap_{t\geq 0}\{v \in C_n : \psi_p(v, z_{n,t}) \leq \psi_p(v, x_n) + \psi_p(z, y_n),\ \forall\, z \in \mathfrak{S}\} \\ &= \bigcap_{t\geq 0}\{v \in C_n : p\langle v, J_{p(1)}x_n - J_{p(1)}z_{n,t}\rangle \leq ||J_{p(1)}x_n||^p - ||J_{p(1)}z_{n,t}||^p + \psi_p(z, y_n),\ \forall\, z \in \mathfrak{S}\}. \end{aligned}$$

Thus, C_{n+1} is closed and convex. Similarly, Q_{n+1} is closed and convex. Hence, C_n and Q_n are closed and convex. Therefore, $\{x_n\}$ and $\{y_n\}$ are well defined.

Next, we prove that $\mathcal{F} \subset C_n \times Q_n,\ \forall\ n \geq 1$. Clearly, $\mathcal{F} \subset C_1 \times Q_1$. Assume that $\mathcal{F} \subset C_n \times Q_n$, for some $n \geq 1$. Let $(p^*, q^*) \in \mathcal{F}$. Using Lemma 3.6, we obtain:

$$\begin{aligned} \psi_p(p^*, z_{n,t}) &= \psi_p(p^*, J_{p(1)}^{-1}(\alpha J_{p(1)}x_n + (1-\alpha)J_{p(1)}T(t)u_n)) \\ &\leq \alpha\psi_p(p^*, x_n) + (1-\alpha)\psi_p(p^*, u_n). \end{aligned} \tag{4.18}$$

Now,

$$\begin{aligned} \psi_p(p^*, u_n) &= \psi_p\Big(p^*, J_{p(1)}^{-1}\Big(J_{p(1)}x_n - \gamma A^* e_n\Big)\Big) = V(p^*, J_{p(1)}x_n - \gamma A^* e_n) \\ &= ||p^*||^p - p\langle p^*, J_{p(1)}x_n\rangle + p\gamma\langle p^*, A^* e_n\rangle + (p-1)||J_{p(1)}x_n - \gamma A^* e_n||^{\frac{p}{p-1}}. \end{aligned} \tag{4.19}$$

Replacing x by $x - y$ in the inequality $||x + y||^p \geq ||x||^p + p\langle y, J_p x\rangle + d_p||y||^p$, we obtain that

$$||x - y||^p \leq ||x||^p - p\langle y, J_p(x - y)\rangle.$$

Using this inequality in (4.19), we obtain:

$$\begin{aligned} \psi_p(p^*, u_n) &\leq ||p^*||^p - p\langle p^*, J_{p(1)}x_n\rangle + p\gamma\langle p^*, A^* e_n\rangle \\ &+ (p-1)\Big[||J_{p(1)}x_n||^p - p\gamma\langle u_n, A^* e_n\rangle\Big]^{\frac{1}{p-1}} \\ &\leq ||p^*||^p - p\langle p^*, J_{p(1)}x_n\rangle + p\gamma\langle Ap^*, e_n\rangle \\ &+ (p-1)\Big[||J_p x_n||^{\frac{p}{p-1}} - p(p-1)\gamma\langle Au_n, e_n\rangle\Big] \\ &= V_p(p^*, J_p x_n) + \gamma p\Big[\langle Ap^*, e_n\rangle - (p-1)\langle Au_n, e_n\rangle\Big] \\ &= \psi_p(p^*, x_n) + \gamma p\Big[\langle Ap^*, e_n\rangle - (p-1)\langle Au_n, e_n\rangle\Big] \end{aligned}$$

Hence, it follows from inequality (4.18) that:

$$\psi_p(p^*, z_{n,t}) \leq \psi_p(p^*, x_n) + \gamma(1-\alpha)p\langle Ap^*, e_n\rangle - \gamma(1-\alpha)p(p-1)\langle Au_n, e_n\rangle. \tag{4.20}$$

Similarly, using w_n, y_n, and B, we get that:

$$\psi_p(q^*, w_{n,t}) \leq \psi_p(q^*, y_n) - \gamma(1-\alpha)p\langle Bq^*, e_n\rangle + \gamma(1-\alpha)p(p-1)\langle Bv_n, e_n\rangle. \tag{4.21}$$

From inequalities (4.20) and (4.21) and using the fact that $Ap^* = Bq^*$, we get that:

$$\psi_p(p^*, z_{n,t}) + \psi_p(q^*, w_{n,t}) \leq \psi_p(p^*, x_n) + \psi_p(q^*, y_n) - \gamma(1-\alpha)p(p-1)\langle Au_n - Bv_n, e_n\rangle. \tag{4.22}$$

Now, $e_n = J_{p(3)}(Ax_n - By_n)$. Set $\sigma := \gamma(1-\alpha)p(p-1)$. So, by Lemma 3.8, we have that:

$$
\begin{aligned}
& -\sigma\langle Au_n - Bv_n, e_n\rangle \\
= \quad & -\sigma||Ax_n - By_n||^p - \sigma\langle Au_n - Bv_n, e_n\rangle + \sigma\langle Ax_n - By_n, e_n\rangle \\
= \quad & -\sigma||Ax_n - By_n||^p + \sigma\langle A(x_n - u_n), e_n\rangle + \sigma\langle B(v_n - y_n), e_n\rangle \\
\le \quad & -\sigma||Ax_n - By_n||^p + \sigma\Big[||A||.||x_n - J^{-1}_{p(1)}(J_{p(1)}x_n - \gamma A^*e_n|| \\
& + ||B||.||y_n - J^{-1}_{p(2)}(J_{p(2)}y_n + \gamma B^*e_n)\Big]||e_n|| \\
\text{(4.23)} \quad \le \quad & -\sigma||Ax_n - By_n||^p + \sigma\gamma^{\frac{1}{p-1}}k_p\Big[||A||.||A^*e_n||^{\frac{1}{p-1}} + ||B||.||B^*e_|{|}^{\frac{1}{p-1}}\Big].||e_n||.
\end{aligned}
$$

But,

$$||A||.||A^*e_n||^{\frac{1}{p-1}}||e_n|| \le ||A||^{\frac{p}{p-1}}.||Ax_n - By_n||^p,$$

and,

$$||B||.||B^*e_n||^{\frac{1}{p-1}}||e_n|| \le ||A||^{\frac{p}{p-1}}.||Ax_n - By_n||^p.$$

Substituting these inequalities in inequality (4.23), we obtain:

$$-\sigma\langle Au_n - Bv_n, e_n\rangle \le -\sigma||Ax_n - By_n||^p + \sigma\gamma^{\frac{1}{p-1}}k_p\Big[||A||^{\frac{p}{p-1}} + ||B||^{\frac{p}{p-1}}\Big].||Ax_n - By_n||^p.$$

Substituting this inequality in inequality (4.22), we obtain:

$$
\begin{aligned}
& \psi_p(p^*, z_{n,t}) + \psi_p(q^*, w_{n,t}) \\
\text{(4.24)} \le \quad & \psi_p(p^*, x_n) + \psi_p(q^*, y_n) - \sigma\Big[1 - \gamma^{\frac{1}{p-1}}k_p\Big(||A||^{\frac{p}{p-1}} + ||B||^{\frac{p}{p-1}}\Big)\Big].||Ax_n - By_n||^p \\
\le \quad & \psi_p(p^*, x_n) + \psi_p(q^*, y_n),
\end{aligned}
$$

establishing that $\mathcal{F} \subset C_n \times Q_n, \ \forall\, n \ge 1$.

Step 2. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are convergent.

From the definition of $\{x_n\}$ and Lemma 2.2, we have that $\psi(x_n, x_1) \le \psi(p^*, x_1) - \psi\big(p^*, x_n) \le \psi(p^*, x_1)$, $\forall\ (p^*, q^*) \in \mathcal{F} \subset C_n \times Q_n$. This implies that $\{\psi(x_n, x_1)\}$ is bounded. Hence, $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}x_1$, we have that $\psi(x_n, x_1) \le \psi(x_{n+1}, x_1)$ and this implies that $\{\psi(x_n, x_1)\}$ is nondecreasing. Hence, $\lim\limits_{n\to\infty}\psi(x_n, x_1)$ exists. Furthermore, for $m \ge n$, we have that:

$$
\begin{aligned}
\psi(x_m, x_n) = \psi(\Pi_{C_m}x_1, \Pi_{C_n}x_1) \quad & \le \quad \psi(\Pi_{C_m}x_1, x_1) - \psi(\Pi_{C_n}x_1, x_1) \\
& = \quad \psi(x_m, x_1) - \psi(x_n, x_1) \to 0 \ (as\ n \to \infty).
\end{aligned}
$$

It follows from Lemma 2.3 that $||x_n - x_m|| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is Cauchy. Thus, there exists $x^* \in E_1$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Following a similar argument, there exists $y^* \in E_2$ such that $\lim\limits_{n\to\infty} y_n = y^*$.

Step 3. We prove that $(x^*, y^*) \in \mathcal{F}$ and $Ax^* = By^*$.

For $m \ge n$, $(x_m, y_m) \in C_m \times Q_m$. We have that $\sup\limits_{t\ge0}\psi_p(x_m, z_{n,t}) \le \psi_p(x_m, x_n) + \psi_p(y_m, y_n) \to 0$ as $m, n \to \infty$, $\sup\limits_{t\ge0}\psi_p(y_m, w_{n,t}) \le \psi_p(x_m, x_n) + \psi_p(y_m, y_n) \to 0$ as $m, n \to \infty$. Hence, for each $t \ge 0$, and by Lemma 3.7, we have that $||x_m - z_{n,t}|| \to 0$ as $m, n \to \infty$ and $||y_m - w_{n,t}|| \to 0$ as $m, n \to \infty$. Therefore, for each $t \ge 0$, $z_{n,t} \to x^*$ as $n \to \infty$ and $w_{n,t} \to y^*$ as $n \to \infty$.

Set $\eta =: \sigma\Big[1-\gamma^{\frac{1}{p-1}}k_p\Big(||A||^{\frac{p}{p-1}}+||B||^{\frac{p}{p-1}}\Big)\Big]$. Then, it follows, from inequality (4.24) that:

$$\begin{aligned}\lim_{n\to\infty}||Ax_n-By_n||^p &\le \frac{1}{\eta}\lim_{n\to\infty}\Big(\psi_p(p^*,x_n)-\psi_p(p^*,z_{n,t})+\psi_p(q^*,y_n)-\psi_p(q^*,w_{n,t})\Big)\\ &= \frac{1}{\eta}\Big(\psi_p(p^*,x^*)-\psi_p(p^*,x^*)+\psi_p(q^*,y^*)-\psi_p(q^*,y^*)\Big)=0. \quad (4.25)\end{aligned}$$

Hence, we have: $Ax^*=Bx^*$.

Step 4. Finally, we show that $(x^*,y^*)\in\mathfrak{T}\times\mathfrak{S}$. From Lemma 3.8, we have that:

$$||u_n-x^*||\le\kappa_p\Big(||J_{p(1)}x_n-J_{p(1)}x^*||+\gamma||A||.||Ax_n-By_n||^{p-1}\Big)^{\frac{1}{p-1}}\to 0 \text{ as } n\to\infty,$$

which implies that $\lim\limits_{n\to\infty}u_n=x^*$. Furthermore,

$$\begin{aligned}||J_{p(1)}z_{n,t}-J_{p(1)}x^*|| &= ||\alpha J_{p(1)}x_n+(1-\alpha)J_{p(1)}T(t)u_n-J_{p(1)}x^*||\\ &\ge (1-\alpha)||J_{p(1)}T(t)u_n-J_{p(1)}x^*||-\alpha||J_{p(1)}x^*-J_{p(1)}x_n||.\end{aligned}$$

This implies that for each $t\ge 0$, $\lim\limits_{n\to\infty}||J_{p(1)}T(t)u_n-J_{p(1}x^*||=0$. Since $J^{-1}_{p(1)}$ is norm-to-norm uniformly continuous on bounded sets, it follows that for each $t\ge 0$, $T(t)u_n\to x^*$ as $n\to\infty$. Since, $\mathcal{T}:=\{T(t):t\ge 0\}$ is closed and $\lim\limits_{n\to\infty}u_n=x^*$, we have that for each $t\ge 0$, $T(t)x^*=x^*$, which implies that $x^*\in F(T(t))$, for each $t\ge 0$. Hence, $x^*\in\mathfrak{T}:=\bigcap\limits_{t\ge 0}F(T(t))$. Following the same argument, we also have that $y^*\in\mathfrak{S}:=\bigcap\limits_{t\ge 0}F(S(t))$. These results now imply that $(x^*,y^*)\in\mathcal{F}$. The proof is complete. □

Remark 4.5. *Theorem 4.1 is applicable in L_p, l_p and the Sobolev spaces since these spaces are p-uniformly convex and uniformly smooth for $p\in(2,\ \infty)$. As far as we know, there is no theorem in the literature for iteratively approximating a solution of the problem discussed in Theorem 4.1 in these real Banach spaces.*

5. Application to split equality variational inequality problem in Banach spaces

In this section, we suppose that C and Q are nonempty closed and convex subsets of E_1 and E_2, respectively, E_3 is an arbitrary smooth real Banach space. A *variational inequality problem* (VIP) in a real Banach space is the problem of finding a point $u^*\in C$ such that for some $j(v-u^*)\in J(v-u^*)$,

$$\langle\mathcal{A}u^*,j(v-u^*)\rangle\ge 0,\ \forall\, v\in C, \quad (5.26)$$

where $\mathcal{A}:C\to E_1$ is a map. We denote the set of solutions of VIP by $VI(\mathcal{A},C)$.

A map $\mathcal{A}:C\to E_1$ is called *accretive* if for each $u,v\in C$, there exists $j(u-v)\in J(u-v)$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,j(u-v)\rangle\ge 0. \quad (5.27)$$

Let $\mathcal{A}_1:C\to E_1$ and $\mathcal{A}_2:Q\to E_2$ be two accretive maps, where C and Q are nonempty closed and convex subsets of E_1 and E_2, respectively.
The *split equality variational inequality problem* is a problem of finding $u^*\in C,\ v^*\in Q$ such that

$$\langle\mathcal{A}_1u^*,j_1(u-u^*)\rangle\ge 0,\ \forall\, u\in C, \text{ and } \langle\mathcal{A}_2v^*,j_2(v-v^*)\rangle\ge 0,\ \forall\, v\in Q,\ Au^*=Bv^*,$$

where $A : C \to E_3$ and $B : Q \to E_3$ are bounded linear maps. We shall denote the set of solutions of split equality variational inequality problem by:

$$\mathcal{G} := \{(u^*, v^*) \in VI(\mathcal{A}_1, C) \times VI(\mathcal{A}_1, Q) : Au^* = Bv^*\}.$$

For $r > 0$, $u \in E_1$ and $v \in E_2$, define maps $T_r : E_1 \to C$ and $S_r : E_2 \to Q$ as follows:

$$T_r(u) := \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, j_1(z - u)\rangle \geq 0, \ \forall\, y \in C\},$$

and

$$S_r(v) := \{z \in Q : g(z, w) + \frac{1}{r}\langle w - z, j_2(z - v)\rangle \geq 0, \ \forall\, w \in Q\}$$

The split equality variational inequality problem with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$ is equivalent to the following split equality fixed point problem:

$$\text{find } u^* \in F(T_r), \ v^* \in F(S_r) \quad \text{such that} \quad Au^* = Bv^*,$$

where A and B are bounded linear maps. (see e.g., [15] for details).

Now, the setting for our next theorem is as follows:

1. C and Q are nonempty closed and convex subsets of p-uniformly convex and uniformly smooth real Banach spaces E_1 and E_2, respectively; E_3 is an arbitrary smooth real Banach space.
2. $\mathcal{A}_i : E_i \to E_i, \ i = 1, 2$ are continuous accretive maps;
3. $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints, A^* and B^*, respectively.
4. T_r and S_r are as defined above.

We now have the following theorem.

Theorem 5.2. *Let $\{(x_n, y_n)\}$ be a sequence in $E_1 \times E_2$ generated iteratively by algorithm (4.17). Assume $\mathcal{G} := \{(x, y) \in VI(\mathcal{A}_1, C) \times VI(\mathcal{A}_2, Q) : Ax = By\} \neq \emptyset$, $\alpha \in (0, 1)$ and γ is such that, $0 < \gamma < \left[\frac{1}{\kappa_p ||A||^{\frac{p}{p+1}} + ||B||^{\frac{p}{p+1}}}\right]^{p-1}$, then, $\{(x_n, y_n)\}$ converges strongly to some point $(x^*, y^*) \in \mathcal{G}$.*

Proof. Set $T_r = T(t)$ and $S_r = S(t)$. Then, T_r and S_r are quasi-ψ-nonexpansive semigroups. Hence, the conclusion follows directly from the proof of Theorem 4.1. □

Remark 5.6. *In the proofs of our theorems in this paper, the condition on γ involves the norms, $||A||$ and $||B||$ of A and B, respectively. This is not a drawback on implementing the algorithm because one does not have to know the values of these norms to use the algorithms. For computational purposes, these norms can be replaced with two constants associated with the maps A and B, which are easy to compute, as follows. To assert that A is a bounded linear map, one has to show that*

$$||Ax|| \leq K_1||x||, \ \forall x \in E,$$

for some constant $K_1 > 0$. This constant $K_1 > 0$ which is an upper bound for $||A||$ is generally easy to obtain (since it is not unique) for any bounded linear map. In fact, one has to almost necessarily know it before one can assert that A is a bounded linear map. Similarly, to assert that B is a bounded linear map, one has to show that

$$||Bx|| \leq K_2|x||, \ \forall x \in E,$$

and some constant $K_2L > 0$. Again, this constant $K_2 > 0$ is an upper bound for $||B||$ and is generally easy to obtain for any bounded linear map. From the proof of Theorem 4.1, it is clear *that, for computational purposes, the norms $||A||$ and $||B||$ of A and B, respectively, can be replaced with K_1 and K_2, respectively, so that the condition*

$$0 < \gamma < \left[\frac{1}{\kappa_p\big(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\big)}\right]^{p-1},$$

can be replaced with the condition

$$0 < \gamma < \left[\frac{1}{\kappa_p\big(K_1^{\frac{p}{p-1}} + K_2^{\frac{p}{p-1}}\big)}\right]^{p-1},$$

where $K_1 > 0$ and $K_2 > 0$ are easily determined.

Conclusion. In this paper, *new important geometric inequalities are established*. These inequalities are of independent interest in the sense that they can be used in several problems in nonlinear operator theory. As an application, an iterative algorithm for approximating a solution of split equality fixed point problems for a quasi-ϕ-nonexpansive semigroup was proposed and studied. A *strong convergence* theorem of the sequence generated by the algorithm (4.17) was established without imposing any compactness-type condition on either the space or the operators. Furthermore, the theorem proved was applied to approximate a solution of a split equality variational inequality problem in a p-uniformly convex and uniformly smooth real Banach space, $p > 2$.

The results of this paper complement related recent important results in the literature that had been proved only in the setting of 2-*uniformly convex and uniformly smooth real Banach spaces* (which, in particular, *do not include* the important L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, for $2 < p < \infty$). On the other hand, the theorems proved in this paper are applicable in L_p, l_p and the Sobolev spaces, $W_p^m(\Omega$, for p such that $2 < p < \infty$..

References

[1] Alber Y., *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlininear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.
Vol. 21, (2001), No. 1, pp. 17-39.

[2] Alber Y. and Ryazantseva I., *Nonlinear Ill Posed problems of Monotone Type*, Springer, London, UK, 2006.

[3] Attouch, H.; Bolte, J.; Redont, P.; Soubeyran, A. Alternating proximal algorithms for weakly coupled convex minimization problems. Applications to dynamical games and PDE's. *J. Convex Anal.* **15** (2008), no. 3, 485–506.

[4] Berinde V., *Iterative approximation of fixed points Lecture Notes in Mathematics 1912*, Springer Berlin, 2007.

[5] Bynum, W. L. Weak parallelogram laws for Banach spaces. *Canad. Math. Bull.* **19** (1976), no. 3, 269–275.

[6] Byrne C. Iterative oblique projection onto convex subsets and the split feasibility problem. *Inverse Problems* **18** (2002),441–453.

[7] Byrne C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Problems* **20** (2004), no. 1, 103–120.

[8] Censor Y., Bortfeld T. and Martin B., *et al. A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys Med Biol. 2006 , 51: 2353-2365.

[9] Censor, Y.; Segal, A. The split common fixed point problem for directed operators. *J. Convex Anal.* **16** (2009), no. 2, 587–600.

[10] Chang S.S., Wang L. and Zhao Y., *Strongly convergent iterative methods for split equality variational inclusion problems in Banach spaces*, Acta Math Scientia. 2016, 36B(6): 1641-1650.

[11] Chidume C.E., *Geometric Properties of Banach Spaces and Nonlinear Iterations, vol. 1965 of Lectures Notes in Mathematics*, Springer, London, UK, 2009.

[12] Chidume C.E., *Iterative approximation of fixed points of Lipschitz strictly pseudocontractive mappings,* Proc. Amer. Math. Soc **98**(1987), no. 4, 283-288.

[13] Chidume C.E., *The iterative solution of the equation $f \in x + Tx$ for a monotone operator T in L^p spaces,* J. Math. Anal. Appl. Vol. 116 (1986), no. 2, 531-537.

[14] Chidume C.E., *Iterative construction of fixed points for multi-valued operators of the monotone type,* Applicable Anal. **23** (1986), no. 3, 209-218.

[15] Charles E. Chidume, Monday O. Nnakwe and A. U. Bello; *An Iterative Algorithm for Solving Split Equality Fixed Point Problems for a Class of Nonexpansive-type Semigroups*, Advances in Nonlinear Variational Inequalities, Volume 20(2017), Number 1, 1-10.

[16] Chidume C. E., Ndambomve P. and Bello A. U., *The split equality fixed point problem for demi-contractive mappings,* J. Nonlinear Anal. Optim. 2015, 6(1):61-69.

[17] Cioranescu I., *Geometry of Banach Spaces, Duality Mappings and Nonlinear problems*, vol. 62, Kluwer Academic Publisher 1990.

[18] Giang D.M., Strodiot J.J. and Nguyen V.H., *Strong convergence of an iterative method for solving the multiple-set split equality fixed point problem in a real Hilbert space*, RACSAM. 2017, doi:10.1007/s13398-016-0338-7.

[19] Kamimura S. and Takahashi W., *Strong convergence of a proximal-type algorithm in a Banach space*, SIAMJ. Optim., vol. 13 (2002), no. 3, pp. 938-945.

[20] Lindenstrauss , J. and Tzafriri L., Classical Banach spaces, II: Function Spaces, Ergebnisse Math. Grenzgebiete Bd. **97**, Springer-Verlag, Berlin, 1979.

[21] Moudafi A., *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, JNonlinear Convex Anal.2014, 15(4) : 809-818.

[22] Reich, S., Constructive techniques for accretive and monotone operators, in "Applied Nonlinear Analysis", Academic Press, New York, (1979), 335-345.

[23] Xu H. K., *Inequalities in Banach spaces with applications,* Nonlinear Analysis: Theory, Methods and Applications, Vol. 16, (1991), No 12, pp 1127-1138.

[24] Xu, Z. B. and Roach, G. F., Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. **157** (1991), 189-210.

[25] Zhaoli Ma, Lin Wang and Shih-sen Chang, *On the split feasibility problem and fixed point problem of quasi-ϕ-nonexpansive mapping in Banach spaces*, 2018, Numer Algor https://doi.org/10.1007/s11075-018-0523-1.

[26] Zhao J. and He S. *Solving the general split common fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operator norms*, Filomat. 2017,31(3): 559-573.doi:10.2298/FIL1703559Z.

[27] Zhao J., Zong H. and Liu G., *et al.*, *Solving variational inequality and split equality common fixed-point problem without prior knowledge of operator norms*, J Nonlinear Sci Appl. 2016,9:5428-5440.

[28] Zhao J., *Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms*, Optim. 2014. doi:10.1080/02331934.2014. 883515.

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