A parallel method for common variational inclusion and common fixed point problems with applications

THANASAK MOUKTONGLANG$^{1,2}$, KANYUTA POochenapan$^{1,2}$, AND RAWeROTE SUPARATULATORN$^2$

ABSTRACT. In this paper, we construct a new parallel method to solve common variational inclusion and common fixed point problems in a real Hilbert space. We obtain a weak convergence theorem by using this method. Besides, numerical results on the signal recovery problem consisting of various blurred filters present that our proposed method outperforms the two previous methods.

1. Introduction

Throughout this article, let $\mathcal{H}$ be a real Hilbert space equipped with their own inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and define $\mathcal{K} = \{1,2,\ldots,K\}$, where $K$ is positive integer. The problem of identifying a point $\overline{x} \in \mathcal{H}$ such that

$$0 \in (F + G)\overline{x}$$

is called the variational inclusion problem, where $F : \mathcal{H} \to \mathcal{H}$ is a single valued mapping and $G : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued mapping. The solution set of the problem (1.1) is represented as $(F + G)^{-1}(0)$. The problem (1.1) can be interpreted as a model of numerous issues in different research fields, such as machine learning [8, 21], signal processing [7, 26] and image recovery [17, 20]. Many splitting algorithms have been introduced and improved to find a solution to the variational inclusion problem (1.1), one of the most famous splitting algorithms is the forward-backward splitting algorithm, see in [12, 18] for more details. In 2015, O’Donoghue and Candès [16] showed the forward-backward splitting algorithm, which is reduced to the proximal gradient algorithm for convex optimization problems. It is well known that the problem (1.1) is equivalent to the following fixed point equation $x = J^G_\gamma (I-\gamma F)x$, where $J^G_\gamma$ is the resolvent operator of $G$ defined by $J^G_\gamma = (I + \gamma G)^{-1}$ such that $\gamma > 0$. Before that in 1964, the inertial extrapolation technique was proposed by Polyak [19] to speed up the convergence of iterative algorithms which is called the heavy ball method. Moudafi and Oliny [15] in 2003 introduced the inertial proximal algorithm to solve the problem (1.1), which was developed from the forward-backward splitting algorithm with the inertial extrapolation technique. Some very recent results on the modified forward-backward splitting method have also been in [1, 5, 6, 14].

Many real-world problems necessitate finding a solution that satisfies several constraints. These constraints can be reformulated using a nonlinear functional model. We are motivated to study common variational inclusion and common fixed point problems in this paper because the problem can be utilized to solve real-world problems such as...
signal recovery and image recovery problems with various blurred filters, see [23, 25]. We also show that our problem can be applied for solving signal recovery with various blurred filters as shown in Section 4. This problem consists of finding a point $\bar{x} \in \mathcal{H}$ such that

\begin{equation}
0 \in (F_i + G_i)\bar{x} \quad \text{and} \quad \bar{x} = S_i\bar{x},
\end{equation}

where $F_i, S_i$ are single valued mappings on $\mathcal{H}$ and $G_i : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued mapping for all $i \in \mathcal{K}$. For finding a common fixed point of a finite family of $G$-nonexpansive mappings $\{S_i\}_{i \in \mathcal{K}}$, Suparatulatorn et al. [23] introduced Algorithm 1 in $\mathcal{H}$ with directed graphs. This algorithm is defined as follows:

**Algorithm 1**: Parallel monotone hybrid algorithm

**Initialization**: Select an arbitrary element $v_1 \in C_1 \subseteq \mathcal{H}$ and set $k := 1$.

**Iterative Steps**: Construct $\{v_k\}$ by using the following steps:

**Step 1.** For any $i \in \mathcal{K}$, set $u_k^i = \rho_k^i v_k + (1 - \rho_k^i) S_i v_k$, where $\{\rho_k^i\} \subset [0, 1]$, and compute

$$
\bar{u}_k = \text{argmax} \left\{ \|u_k^i - v_k\| : i \in \mathcal{K} \right\}.
$$

**Step 2.** Compute

$$
v_{k+1} = P_{C_{k+1}} v_1,
$$

where $C_{k+1} = \{ c \in C_k : \|c - \bar{u}_k\| \leq \|c - v_k\| \}$. Replace $k$ by $k + 1$ and then repeat **Step 1.**

Suparatulatorn et al. [24] recently proposed Algorithm 2 to solve a common variational inclusion problem under Lipschitz continuity and monotonicity of $F_i$, and maximal monotonicity of $G_i$, for all $i \in \mathcal{K}$. This algorithm is defined as follows:

**Algorithm 2**: Parallel inertial Tseng type algorithm

**Initialization**: Given $\lambda_i \in (0, 1)$ and $\gamma_i^j > 0$ for all $i \in \mathcal{K}$. Select arbitrary elements $v_0, v_1 \in \mathcal{H}$ and set $k := 1$.

**Iterative Steps**: Construct $\{v_k\}$ by using the following steps:

**Step 1.** Set $r_k = v_k + \xi_k (v_k - v_{k-1})$, where $\{\xi_k\} \subset [0, \infty)$, and compute, for all $i \in \mathcal{K}$,

$$
s_k^i = J_{\frac{\gamma_i^j}{\gamma_i^j}} (I - \gamma_i^j F_i) r_k.
$$

If $r_k = s_k^i$ for all $i \in \mathcal{K}$, then stop and $r_k \in \bigcap_{i \in \mathcal{K}} (F_i + G_i)^{-1} (0)$. Otherwise

**Step 2.** Compute, for all $i \in \mathcal{K}$,

$$
t_k^i = s_k^i - \gamma_i^j (F_i s_k^i - F_i r_k) \quad \text{and} \quad \bar{t}_k = \text{argmax} \left\{ \|t_k^i - r_k\| : i \in \mathcal{K} \right\}.
$$

**Step 3.** Compute

$$
v_{k+1} = a_k \varphi(v_k) + (1 - a_k - b_k) v_k + b_k \bar{t}_k,
$$

where $\{a_k\}, \{b_k\} \subset (0, 1), \varphi$ is a contractive on $\mathcal{H}$, and update, for all $i \in \mathcal{K}$,

$$
\gamma_{k+1}^i = \begin{cases} 
\min \left\{ \lambda_i \frac{\|r_k - s_k^i\|}{\|F_i r_k - F_i s_k^i\|} : \gamma_k^i \right\} & \text{if } F_i r_k \neq F_i s_k^i; \\
\gamma_k^i & \text{otherwise.}
\end{cases}
$$

Replace $k$ by $k + 1$ and then repeat **Step 1.**

Furthermore, Suparatulatorn and Chaichana [25] studied an image recovery problem in which several blurred filters are considered and the mathematical model used there is
the common variational inclusion problem. Several interesting outcomes for the problem (1.2) and related problems were published, see [9, 10, 22, 27].

Inspired by the previous works, we develop a novel parallel algorithm based on the inertial Mann iteration process to prove a weak convergence result for solving the problem (1.2) under some control conditions in \( \mathcal{H} \). Additionally, we compare our algorithm with Algorithm 1 and Algorithm 2 in order to solve the signal recovery problem involving multiple blurring filters.

2. Preliminaries

In this section, we collect some necessary definitions and lemmas for proving our main result. We denote \( \rightarrow \) and \( \to \) as weak and strong convergence, respectively. Denote the set of the fixed point of the mapping \( S \) by \( \text{Fix}(S) \). For each \( x, y \in \mathcal{H} \), we have the following facts:

\[
\|x + y\|^2 = \|x\|^2 + 2 \langle x, y \rangle + \|y\|^2 \tag{2.3}
\]

and

\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle. \tag{2.4}
\]

Definition 2.1. A self-mapping \( S: \mathcal{H} \to \mathcal{H} \) is said to be

(i) \( L \)-Lipschitz continuous if there is \( L > 0 \) such that for all \( x, y \in \mathcal{H} \),

\[ \|Sx - Sy\| \leq L \|x - y\|, \]

(ii) nonexpansive if \( S \) is \( L \)-Lipschitz continuous when \( L = 1 \),

(iii) \( \mu \)-demicontractive if \( \text{Fix}(S) \neq \emptyset \) and there is \( \mu \in [0, 1) \) such that for all \( x \in \mathcal{H} \) and all \( p \in \text{Fix}(S) \),

\[ \|Sx - p\|^2 \leq \|x - p\|^2 + \mu \|x - Sx\|^2, \]

this is equivalent to

\[ \langle x - p, Sx - x \rangle \leq \frac{\mu - 1}{2} \|x - Sx\|^2 \]

for all \( x \in \mathcal{H} \) and all \( p \in \text{Fix}(S) \).

Definition 2.2. Let \( G: \mathcal{H} \to 2^\mathcal{H} \) be a multivalued mapping. Then \( G \) is said to be

(i) monotone if for all \( (x, u), (y, v) \in \text{graph}(G) \) (the graph of mapping \( G \)),

\[ \langle u - v, x - y \rangle \geq 0, \]

(ii) maximal monotone if for every \( (x, u) \in \mathcal{H} \times \mathcal{H} \), \( \langle u - v, x - y \rangle \geq 0 \) for all \( (y, v) \in \text{graph}(G) \) if and only if \( (x, u) \in \text{graph}(G) \).

Definition 2.3. [28] Assume that \( S: \mathcal{H} \to \mathcal{H} \) is a mapping with \( \text{Fix}(S) \neq \emptyset \). Then, \( I - S \) is said to be demiclosed at zero if for any \( \{v_k\} \subset \mathcal{H} \), the following implication holds:

\[ v_k \rightharpoonup v \text{ and } (I - S)v_k \to 0 \implies v \in \text{Fix}(S). \]

Lemma 2.1. [4] If \( G: \mathcal{H} \to 2^\mathcal{H} \) is a maximal monotone mapping and \( F: \mathcal{H} \to \mathcal{H} \) is a Lipschitz continuous and monotone mapping, then the mapping \( F + G \) is maximal monotone.

Lemma 2.2. [2] Let \( \{a_k\} \) and \( \{b_k\} \) be nonnegative sequences of real numbers satisfying \( \sum_{k=1}^{\infty} b_k < \infty \) and \( a_{k+1} \leq a_k + b_k \). Then, \( \{a_k\} \) is a convergent sequence.
Lemma 2.3. [3, Opial] Let $\Phi$ be a nonempty set of $\mathcal{H}$ and $\{v_k\}$ be a sequence in $\mathcal{H}$. Suppose the following assertions hold.

(i) For every $v \in \Phi$, the sequence $\{\|v_k - v\|\}$ converges.

(ii) Every weak sequential cluster point of $\{v_k\}$ belongs to $\Phi$. Then $\{v_k\}$ weakly converges to a point in $\Phi$.

3. MAIN RESULTS

In this section, we propose a new method for solving the problem (1.2). For the convergence analysis of the proposed method, we assume the following assumptions, for all $i \in \mathcal{K}$.

Assumption 1. $\mathcal{H}$ is a real Hilbert space, $F_i : \mathcal{H} \to \mathcal{H}$ is $L_i$-Lipschitz continuous and monotone mapping and $G_i : \mathcal{H} \to 2^\mathcal{H}$ is maximal monotone operator.

Assumption 2. $\{p^i_k\} \subset [0, \infty)$, $\{q^i_k\} \subset [1, \infty)$ such that $\sum_{k=1}^{\infty} p^i_k < \infty$ and $\lim_{k \to \infty} q^i_k = 1$.

Assumption 3. $S_i : \mathcal{H} \to \mathcal{H}$ is $\mu_i$-demicontractive mapping such that $I - S_i$ is demiclosed at zero.

Assumption 4. $\{\xi^i_k\} \subset [0, \xi)$, $\{\alpha^i_k\} \subset (\mu_i, \bar{\alpha}_i) \subset (0, 1)$, for some $\xi, \bar{\alpha}_i > 0$.

Assumption 5. $\Phi := \bigcap_{i \in \mathcal{K}} (F_i + G_i)^{-1}(0) \cap \bigcap_{i \in \mathcal{K}} Fix(S_i)$ is nonempty.

Algorithm 3

Initialization: Given $\lambda_i \in (0, 1)$ and $\gamma^i_1 > 0$ for all $i \in \mathcal{K}$. Select arbitrary elements $v_0, v_1 \in \mathcal{H}$ and set $k := 1$.

Iterative Steps: Construct $\{v_k\}$ by using the following steps:

Step 1. Set $r_k = v_k + \xi_k(v_k - v_{k-1})$ and compute, for all $i \in \mathcal{K}$,

$$s^i_k = J^{G_i}_{\gamma^i_k}(I - \gamma^i_k F_i)r_k.$$ 

Step 2. Compute, for all $i \in \mathcal{K}$,

$$t^i_k = s^i_k - \gamma^i_k(F_i s^i_k - F_i r_k) \quad \text{and} \quad u^i_k = \alpha^i_k t^i_k + (1 - \alpha^i_k)S_i t^i_k.$$ 

Step 3. Compute

$$v_{k+1} = \arg\max \{\|u^i_k - r_k\| : i \in \mathcal{K}\}$$

and update, for all $i \in \mathcal{K}$,

$$\gamma^i_{k+1} = \left\{ \begin{array}{ll} \min \left\{ \frac{\lambda_i q^i_k \|r_k - s^i_k\|}{\|F_i r_k - F_i s^i_k\|}, \gamma^i_k + p^i_k \right\} & \text{if} \ F_i r_k \neq F_i s^i_k; \\
\gamma^i_k + p^i_k & \text{otherwise.}
\end{array} \right.$$ 

Replace $k$ by $k + 1$ and then repeat Step 1.

Lemma 3.4. Suppose that Assumptions 1-2 hold. Then the sequence $\{\gamma^i_k\}$ generated by Algorithm 3 is well defined and converges to $\gamma_i \in \left[ \min \left\{ \gamma^i_1, \frac{\lambda_i}{L_i} \right\}, \gamma^i_1 + p_i \right]$, where $p_i = \sum_{k=1}^{\infty} p^i_k$.

Proof. Since $F_i$ is an $L_i$-Lipschitz continuous mapping for all $i \in \mathcal{K}$, if $F_i s^i_k \neq F_i r_k$, then

$$\frac{\lambda_i q^i_k \|r_k - s^i_k\|}{\|F_i r_k - F_i s^i_k\|} \geq \frac{\lambda_i q^i_k \|r_k - s^i_k\|}{L_i \|r_k - s^i_k\|} = \frac{\lambda_i q^i_k}{L_i} \geq \frac{\lambda_i}{L_i}.$$
By using the same technique as in the proof of [13, Lemma 3.1], we obtain that  \( \lim_{k \to \infty} \gamma^i_k = \gamma^i \in \left[ \min \left\{ \gamma^i, \frac{\lambda^i}{L^i} \right\}, \gamma^i_1 + p_i \right] \).

**Lemma 3.5.** Let \( v \in \Phi \). Then under Assumptions 1-5, we have, for all \( i \in \mathcal{K} \),

\[
\| u_k^i - v \|^2 \leq \| r_k - v \|^2 - \left[ 1 - \left( g_k^i \right)^2 \right] \| r_k - s_k^i \|^2 - (1 - \alpha_k^i)(\alpha_k^i - \mu_i)\| S_i t_k^i - t_k \|^2
\]

and

\[
\| t_k^i - r_k \| \leq (1 + g_k^i) \| r_k - s_k^i \|,
\]

where \( g_k^i = \lambda_i g_k^i \frac{\gamma^i_k}{\gamma^i_{k+1}} \).

**Proof.** By the definitions of \( t_k^i \) and \( \gamma^i_k \), we obtain that for all \( i \in \mathcal{K} \),

\[
\| t_k^i - s_k^i \| = \gamma^i_k \| F_i s_k^i - F_i r_k \| \leq g_k^i \| s_k^i - r_k \|,
\]

which together with (2.3) indicates that for all \( i \in \mathcal{K} \),

\[
\| t_k^i - v \|^2 = \| s_k^i - v \|^2 - 2\gamma^i_k \langle s_k^i - v, F_i s_k^i - F_i r_k \rangle + \left( \gamma^i_k \right)^2 \| F_i s_k^i - F_i r_k \|^2
\]

\[
= \| r_k - v \|^2 + \| s_k^i - r_k \|^2 - 2\langle s_k^i - r_k, s_k^i - r_k \rangle + 2\langle s_k^i - r_k, s_k^i - v \rangle
\]

\[
- 2\gamma^i_k \langle s_k^i - v, F_i s_k^i - F_i r_k \rangle + \left( \gamma^i_k \right)^2 \| s_k^i - r_k \|^2
\]

\[
\leq \| r_k - v \|^2 - \left[ 1 - \left( g_k^i \right)^2 \right] \| r_k - s_k^i \|^2 - 2\langle s_k^i - v, r_k - s_k^i - \gamma^i_k (F_i r_k - F_i s_k^i) \rangle.
\]

From the definition of \( s_k^i \), we have that \( (I - \gamma^i_k F_i) r_k \in (I + \gamma^i_k G_i) s_k^i \) for all \( i \in \mathcal{K} \). This implies that there exists \( g_k^i \in G_i s_k^i \) such that \( g_k^i = \frac{1}{\gamma^i_k} \left( r_k - s_k^i - \gamma^i_k (F_i r_k - F_i s_k^i) \right) \) for all \( i \in \mathcal{K} \). Since \( F_i + G_i \) is maximal monotone, we obtain that \( \langle F_i s_k^i + g_k^i, s_k^i - v \rangle \geq 0 \) for all \( i \in \mathcal{K} \), implying that \( \langle s_k^i - v, r_k - s_k^i - \gamma^i_k (F_i r_k - F_i s_k^i) \rangle \geq 0 \) for all \( i \in \mathcal{K} \). This combined with (3.8) yields that \( \| t_k^i - v \|^2 \leq \| r_k - v \|^2 - \left[ 1 - \left( g_k^i \right)^2 \right] \| r_k - s_k^i \|^2 \) for all \( i \in \mathcal{K} \). This follows from the equivalence of demicontinuous mapping \( S_i \) and (2.3) that for all \( i \in \mathcal{K} \),

\[
\| u_k^i - v \|^2 = \| \alpha^i_k t_k^i + (1 - \alpha^i_k) S_i t_k^i - v \|^2
\]

\[
= \| t_k^i - v \|^2 + (1 - \alpha^i_k)^2 \| S_i t_k^i - t_k^i \|^2 + 2(1 - \alpha^i_k)(t_k^i - v, S_i t_k^i - t_k^i)
\]

\[
\leq \| t_k^i - v \|^2 + (1 - \alpha^i_k)^2 \| S_i t_k^i - t_k^i \|^2 + (1 - \alpha^i_k)(\mu_i - 1) \| S_i t_k^i - t_k^i \|^2
\]

\[
\leq \| r_k - v \|^2 - \left[ 1 - \left( g_k^i \right)^2 \right] \| r_k - s_k^i \|^2 - (1 - \alpha^i_k)(\alpha^i_k - \mu_i) \| S_i t_k^i - t_k^i \|^2.
\]

Further, using Cauchy-Schwarz and by (3.7), we obtain that the inequality (3.6) holds. \( \square \)

**Lemma 3.6.** Suppose that \( \lim_{k \to \infty} \| r_k - s_k^i \| = \lim_{k \to \infty} \| S_i t_k^i - t_k^i \| = 0 \) for all \( i \in \mathcal{K} \). If there exists a weakly convergent subsequence \( \{ r_{k_j} \} \) of \( \{ r_k \} \), then under Assumptions 1-5, we have that the weak limit of \( \{ r_{k_j} \} \) belongs to \( \Phi \).

**Proof.** Let \( \bar{r} \in \mathcal{H} \) such that \( r_{k_j} \rightharpoonup \bar{r} \). Since \( \lim_{k \to \infty} g_k^i = \lambda^i_i > 0 \) and by (3.6), we have \( \lim_{k \to \infty} \| t_k^i - r_k \| = 0 \). It follows that \( t_k^i \rightharpoonup \bar{r} \). This together with \( \lim_{k \to \infty} \| S_i t_k^i - t_k^i \| = 0 \), by the demiclosedness at zero of \( I - S_i, \bar{r} \in \bigcap_{i \in \mathcal{K}} Fix(S_i) \). Next, we show that \( \bar{r} \in \bigcap_{i \in \mathcal{K}} (F_i + G_i)^{-1}(0) \). Let \( (v_i, u_i) \in \text{graph}(F_i + G_i) \) for all \( i \in \mathcal{K} \), that is, \( u_i - F_i v_i \in G_i v_i \) for all \( i \in \mathcal{K} \). It implies by the definition of \( s_k^i \) that for all \( i \in \mathcal{K} \), \( \frac{1}{\gamma_{k_j}^i} \left( r_{k_j} - s_{k_j}^i - \gamma_{k_j}^i s_i v_{k_j} \right) \in G_i s_k^i \). By the maximal
monotonicity of $G_i$, we have that $\langle v_i - s_{k_i}^i, u_i - F_i v_i - \frac{1}{\gamma_{k_i}} \left( r_{k_i} - s_{k_i}^i F_i r_{k_i} \right) \rangle \geq 0$
for all $i \in \mathcal{K}$. Thus, for all $i \in \mathcal{K},$

$$\langle v_i - s_{k_i}^i, u_i \rangle \geq \left( v_i - s_{k_i}^i, F_i v_i + \frac{1}{\gamma_{k_i}} \left( r_{k_i} - s_{k_i}^i \right) F_i r_{k_i} \right)$$

$$= \langle v_i - s_{k_i}^i, F_i v_i - F_i s_{k_i}^i \rangle + \langle v_i - s_{k_i}^i, F_i s_{k_i}^i - F_i r_{k_i} \rangle$$

$$+ \frac{1}{\gamma_{k_i}} \langle v_i - s_{k_i}^i, r_{k_i} - s_{k_i}^i \rangle$$

$$\geq \langle v_i - s_{k_i}^i, F_i s_{k_i}^i - F_i r_{k_i} \rangle + \frac{1}{\gamma_{k_i}} \langle v_i - s_{k_i}^i, r_{k_i} - s_{k_i}^i \rangle.$$ 

This follows from the Lipschitz continuity of $F_i$, $\lim_{k \to \infty} \| r_k - s_k \| = 0$ and $\lim_{k \to \infty} \gamma_k = \gamma_i > 0$
that $\langle v_i - \bar{r}, u_i \rangle = \lim_{j \to \infty} \langle v_i - s_{k_j}^i, u_i \rangle \geq 0$ for all $i \in \mathcal{K}$, which, together with the maximal
monotonicity of $F_i + G_i$, we get that $0 \in (F_i + G_i) \bar{r}$ for all $i \in \mathcal{K}$, that is, $\bar{r} \in \bigcap_{i \in \mathcal{K}} (F_i + G_i)^{-1}(0)$. Therefore, $\bar{r} \in \Phi$. \hfill \qed

**Theorem 3.1.** Suppose that $\sum_{k=1}^{\infty} \xi_k \| v_k - v_{k-1} \| < \infty$, then under Assumptions 1-5, we have that
the sequence $\{ v_k \}$ generated by Algorithm 3 weakly converges to a solution of $\Phi$.

**Proof.** Let $v \in \Phi$. Since $\lim_{k \to \infty} \left[ 1 - \left( \alpha_k^i \right)^2 \right] = 1 - \lambda_i^2 > 0$, one can find $m_i \in \mathbb{N}$ such that
$1 - \left( \alpha_k^i \right)^2 > 0$ for all $i \in \mathcal{K}$ and all $k \geq k_0$, where $k_0 = \max m_i$. From Assumption 4, by
the definition of $r_k$ and using (3.5), we have $\| u_k^i - v \| \leq \| r_k \| = \| v_k + \xi_k (v_k - v_{k-1}) - v \| \leq \| v_k \| + \| v_k - v \| + \xi_k \| v_k - v_{k-1} \|$ for all $i \in \mathcal{K}$ and all $k \geq k_0$. It implies by the definition of $i$ that $\| v_{k+1} - v \| \leq \| r_k \| \leq \| v_k \| + \sum_{i \in \mathcal{K}} \xi_k \| v_k - v_{k-1} \|$ for all $k \geq k_0$. This follows that $\{ \| v_k - v \| \}$
is convergent because of using Lemma 2.2 and $\sum_{k=1}^{\infty} \xi_k \| v_k - v_{k-1} \| < \infty$. In particular, $\{ v_k \}$
is bounded and also $\{ r_k \}$. Next, applying (2.4) and (3.5), we have,

$$\| u_k^i - v \|^2 \leq \| v_k - v \|^2 + 2 \xi_k \langle v_k - v_{k-1}, r_k \rangle$$

$$\left[ 1 - \left( \alpha_k^i \right)^2 \right] \| r_k - s_k^i \|^2 \leq \| v_k - v \|^2 - \| v_{k+1} - v \|^2 + M_1 \xi_k \| v_k - v_{k-1} \|^2$$

for all $i \in \mathcal{K}$. It follows that there are $i_k \in \mathcal{K}$ and $M_1 > 0$ such that

$$\left[ 1 - \left( \alpha_k^{i_k} \right)^2 \right] \| r_k - s_k^{i_k} \|^2 \leq \| v_k - v \|^2 - \| v_{k+1} - v \|^2 + M_1 \xi_k \| v_k - v_{k-1} \|^2$$

$$- \left( 1 - \alpha_k^{i_k} \right) \| S_k t_k^{i_k} - t_k^{i_k} \|^2.$$

From Assumption 4, $\lim_{k \to \infty} \left[ 1 - \left( \alpha_k^{i_k} \right)^2 \right] > 0$ and $\sum_{k=1}^{\infty} \xi_k \| v_k - v_{k-1} \| < \infty$, and using $\lim_{k \to \infty} \| v_k - v \| \exists$, we obtain

$$\lim_{k \to \infty} \| r_k - s_k^{i_k} \| = 0$$

and so

$$\lim_{k \to \infty} \| S_k t_k^{i_k} - t_k^{i_k} \| = 0.$$
Indeed, using (3.6), (3.10) and (3.11), we deduce
\[
\|v_{k+1} - r_k\| \leq \|v_{k+1} - \ell_{k}^i\| + \|\ell_{k}^i - r_k\|
\]
\[
= \|\alpha_k \ell_{k}^i + (1 - \alpha_k)S_i \ell_{k}^i - \ell_{k}^i\| + \|\ell_{k}^i - r_k\|
\]
\[
\leq (1 - \alpha_k)^2 \|S_i \ell_{k}^i - r_k\| + (1 + \theta_k^i) \|r_k - s_k\| \to 0 \text{ as } k \to \infty.
\]
This implies by the definition of \(v_{k+1}\) that
\[
(3.12) \quad \lim_{k \to \infty} \|r_k - u_k^i\| = 0
\]
for all \(i \in K\). Again, applying (3.5), we have
\[
\left[1 - (\theta_k^i)^2\right] \|r_k - s_k\|^2 + (1 - \alpha_k^i)(\alpha_k^i - \mu_i)\|S_i \ell_{k}^i - t_k^i\|^2 \leq \|r_k - v\|^2 - \|u_k - v\|^2
\]
\[
\leq M_2 \|r_k - u_k^i\|
\]
for all \(i \in K\) and for some \(M_2 > 0\). Combining this to (3.12) with Assumption 4 and
\[
\lim_{k \to \infty} \left[1 - (\theta_k^i)^2\right] > 0,
\]
once we obtain that for all \(i \in K\),
\[
(3.13) \quad \lim_{k \to \infty} \|r_k - s_k\| = \lim_{k \to \infty} \|S_i \ell_{k}^i - t_k^i\| = 0.
\]
Finally, let \(\bar{v}\) be a weak sequential cluster point of \(\{v_k\}\), that is, it has a subsequence \(\{v_{k_j}\}\)
fulfilling \(v_{k_j} \rightharpoonup \bar{v}\) as \(j \to \infty\). Since \(\lim_{k \to \infty} \xi_k \|v_k - v_{k-1}\| = 0\), we get \(r_{k_j} \rightharpoonup \bar{v}\) as \(j \to \infty\).
Applying Lemma 3.6 to (3.13), we deduce that \(\bar{v} \in \Phi\). Using Opial’s lemma (Lemma 2.3),
we can conclude that \(\{v_k\}\) weakly converges to an element in \(\Phi\).

4. APPLICATION TO SIGNAL RECOVERY PROBLEM

The signal recovery problem consisting of various blurring filters can be expressed as:
\[
b_i = A_i x + \varepsilon_i,
\]
where \(x \in \mathbb{R}^N\) is the original signal, \(b_i \in \mathbb{R}^M\) is the observed signal with noise \(\varepsilon_i\) and \(A_i \in \mathbb{R}^{M \times N}\) (\(M < N\)) is filter matrix for all \(i \in K\). Then, we focus on the following problem:
\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_1 x - b_1\|^2 + \eta_1 \|x\|_1,
\]
\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_2 x - b_2\|^2 + \eta_2 \|x\|_1,
\]
\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_3 x - b_3\|^2 + \eta_3 \|x\|_1,
\]
\[
\vdots
\]
\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_K x - b_K\|^2 + \eta_K \|x\|_1,
\]
where \(\eta_i > 0\) for all \(i \in K\). By Proposition 3.1 (iii) of [7], this problem can be seen as
the problem (1.2) through the following settings: \(\mathcal{H} = \mathbb{R}^N\), \(F_i = \nabla h_i\), \(G_i(\cdot) = \partial \ell_i(\cdot)\) and \(S_i(\cdot) = \text{prox}_{\xi_i, \ell_i}(I - \zeta_i \nabla h_i)(\cdot)\), where \(\zeta_i > 0\), \(h_i(\cdot) = \frac{1}{2}\|A_i(\cdot) - b_i\|^2\) and \(\ell_i(\cdot) = \eta_i \|\cdot\|_1\) for all \(i \in K\). It is known that the mapping \(F_i\) is monotone and \(\|A_i\|_{2, \text{Lipschitz}}\) continuous,
and \(G_i\) is maximal monotone mapping. Besides, the mapping \(S_i\) is nonexpansive for \(\zeta_i \in \left(0, \frac{2}{\|A_i\|_2}\right)\) and hence 0-demicontractive. Numerical experiments are performed by
Matlab R2021a and run on an iMac (Apple M1 chip with 16GB of RAM). Set the original
signal $x$ is generated by the uniform distribution in $[-2, 2]$ with $m$ nonzero elements. Let $A_i$ be the Gaussian matrix generated by command $\text{randn}(M, N)$.

In the first part of the experiment, we investigate the behavior of our algorithm and then compare it with Algorithm 1 of Suantai et al. [23] and Algorithm 2 of Suparatulatorn et al. [24]. We select the signal size to be $N = 4096$ and $M = 2048$. Let the observation $b_i$ be generated by white Gaussian noise with signal-to-noise ratio $\text{SNR}=40$, $\eta_i = 1$ and $\zeta_i = \frac{1}{\|A_i\|^2}$ for all $i \in \{1, 2, 3\}$. Let $v_0, v_1$ be the vectors generated randomly. For Algorithm 1, we set $\rho_i^k = \frac{3}{4}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. For Algorithm 2, let $a_k = \frac{1}{k+1}$ and $b_k = \frac{99k}{100(k+1)}$ for all $k \in \mathbb{N}$, and define $\varphi(\cdot) = \frac{\cos(\cdot)}{10}$. For Algorithm 3, let $\alpha_i^k = \frac{1}{4}$, $p_i^k = \frac{1}{k+1}$ and $q_i^k = 1 + \frac{1}{k+1}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. Further, for Algorithm 2 and Algorithm 3, we suppose $\lambda_i = \frac{95}{100}$, $\gamma_i^1 = \frac{1}{100}$ and

$$
\xi_k = \begin{cases} 
\frac{1}{(k+1)^{1.1}} \max\left\{\|v_k - v_{k-1}\|_2, \|v_k - v_{k-1} - \frac{\|v_k - v_{k-1}\|_2^2}{2}\right\}^{\frac{1}{3}}, \\
\frac{1}{4} & \text{if } v_k \neq v_{k-1} ; \\
\frac{1}{4} & \text{otherwise}
\end{cases}
$$

for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. We use the mean-squared error to measure the restoration accuracy defined as follows: $MSE_k = \frac{1}{N} \|v_k - x\|^2 < 5 \times 10^{-5}$. The results are presented next.

<table>
<thead>
<tr>
<th>Nonzero Elements</th>
<th>$m = 64$</th>
<th>$m = 128$</th>
<th>$m = 256$</th>
<th>$m = 512$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm 1</td>
<td>Iter</td>
<td>2386</td>
<td>2461</td>
<td>2702</td>
</tr>
<tr>
<td></td>
<td>CPU Time</td>
<td>20.4741</td>
<td>21.1546</td>
<td>23.2616</td>
</tr>
<tr>
<td>Algorithm 2</td>
<td>Iter</td>
<td>749</td>
<td>950</td>
<td>1254</td>
</tr>
<tr>
<td></td>
<td>CPU Time</td>
<td>12.2045</td>
<td>15.3795</td>
<td>21.4745</td>
</tr>
<tr>
<td>Algorithm 3</td>
<td>Iter</td>
<td>215</td>
<td>230</td>
<td>232</td>
</tr>
<tr>
<td></td>
<td>CPU Time</td>
<td>5.1991</td>
<td>5.5572</td>
<td>5.8327</td>
</tr>
</tbody>
</table>

**TABLE 1.** Numerical comparison of three algorithms.

(A) The original signal and the measurements.  
(B) The reconstructed signals

**FIGURE 1.** The original signal, the measurements and the reconstructed signals by three algorithms for $m = 512$. 
Based on Table 1, Algorithm 3 requires fewer iterations and takes less time than Algorithm 1 and Algorithm 2.

The last part of the experiment considers Algorithm 3 for solving the problem (4.14) with multiple inputs $A_i$. We select the signal size to be $N = 1024$ and $M = 512$. For any $i \in \{1, 2, 3\}$, let the observation $b_i$ be generated by the white Gaussian noise $\varepsilon_i$ of the variance $\sigma_i^2$. Set $\nu_0, \nu_1, \gamma_i, \lambda_i, \eta_i, \zeta_i, \alpha_i, p_i^k, q_i^k$, and $\xi_k$ are the same as in the first part of the experiment for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. Further, we select $\sigma_i = \frac{i}{100}$ for all $i \in \{1, 2, 3\}$. The results are presented next.

<table>
<thead>
<tr>
<th>Inputting</th>
<th>$m$ Nonzero Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m = 16$ $m = 32$ $m = 64$ $m = 128$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>Iter 1085 1062 1463 2986</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.7912 0.7415 1.0678 2.1037</td>
</tr>
<tr>
<td>$A_2$</td>
<td>Iter 1049 1027 1402 2460</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.8564 0.8177 0.9194 1.6673</td>
</tr>
<tr>
<td>$A_3$</td>
<td>Iter 1088 1063 1506 2311</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.7830 0.7432 0.8780 1.5621</td>
</tr>
<tr>
<td>$A_1, A_2$</td>
<td>Iter 379 384 449 456</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.7045 0.5749 0.5747 0.7831</td>
</tr>
<tr>
<td>$A_1, A_3$</td>
<td>Iter 364 400 875 477</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.7210 0.5797 1.2161 0.7123</td>
</tr>
<tr>
<td>$A_2, A_3$</td>
<td>Iter 382 401 435 490</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.7831 0.6120 0.6199 0.6830</td>
</tr>
<tr>
<td>$A_1, A_2, A_3$</td>
<td>Iter 125 125 123 126</td>
</tr>
<tr>
<td></td>
<td>CPU Time 0.4139 0.2938 0.2731 0.2618</td>
</tr>
</tbody>
</table>

| Table 2. Numerical results of Algorithm 3. |

From Table 2, we can observe that incorporating all 3 Gaussian matrices ($A_1$, $A_2$ and $A_3$) into Algorithm 3 is more effective with respect to time and number of iterations than involving only one or two of them.
The main problem is that \( I \) is continuous and so \( \alpha \) is 3-Lipschitz continuous and monotone on \( H \). Further, for Algorithm 2 and Algorithm 3, we suppose \( \lambda_i = 3 \) for all \( i \in \mathbb{N} \) and \( \phi_i(\cdot) = \frac{1}{10} \). For Algorithm 3, let \( \alpha_k = \frac{1}{2} \), \( \rho_k = \frac{1}{(k+1)^{0.5}} \), and \( q_k = 1 + \frac{1}{k+1} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). For Algorithm 2 and Algorithm 3, we suppose \( \lambda_i = \frac{0.5}{100} \) and \( \gamma_i = \frac{7}{100} \) for all \( i \in \mathbb{N} \).

In this experiment, we compare Algorithm 3 with Algorithm 1 and Algorithm 2. Let \( v_0 = (10^5, 10^5)^t \) and \( v_1 = (10^4, 10^4)^t \). For Algorithm 1, we set \( \rho_k = \frac{3}{4} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). For Algorithm 2, select \( a_k \) and \( b_k \) as in Section 4, and define \( \phi_i(\cdot) = \frac{1}{10} \). For Algorithm 3, let \( \alpha_k = \frac{1}{2} \), \( \rho_k = \frac{1}{(k+1)^{0.5}} \), and \( q_k = 1 + \frac{1}{k+1} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). Further, for Algorithm 2 and Algorithm 3, we suppose \( \lambda_i = \frac{0.5}{100} \) and \( \gamma_i = \frac{7}{100} \) for all \( i \in \mathbb{N} \).

In this experiment, we compare Algorithm 3 with Algorithm 1 and Algorithm 2. Let \( v_0 = (10^5, 10^5)^t \) and \( v_1 = (10^4, 10^4)^t \). For Algorithm 1, we set \( \rho_k = \frac{3}{4} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). For Algorithm 2, select \( a_k \) and \( b_k \) as in Section 4, and define \( \phi_i(\cdot) = \frac{1}{10} \). For Algorithm 3, let \( \alpha_k = \frac{1}{2} \), \( \rho_k = \frac{1}{(k+1)^{0.5}} \), and \( q_k = 1 + \frac{1}{k+1} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). Further, for Algorithm 2 and Algorithm 3, we suppose \( \lambda_i = \frac{0.5}{100} \) and \( \gamma_i = \frac{7}{100} \) for all \( i \in \mathbb{N} \).

5. Numerical example

We utilize Algorithm 3 to solve the problem (1.2) with \( K = \{1, 2\} \) in the finite-dimensional Hilbert space. Suppose that \( H = \mathbb{R}^2 \) with the the Euclidean norm. For any \( i \in \{1, 2\} \), define \( F_i z = (x+y+\sin x, -x+y+\sin y)^t \) for all \( z = (x, y)^t \in H \) and set \( G_i = \partial \iota_{C_i} \), where \( \iota_{C_i} \) is the indicator function of \( C_i \) and \( C_i = [-i, i]^2 \). It is not hard to show that the mapping \( F_i \) is 3-Lipschitz continuous and monotone on \( H \), and the mapping \( G_i \) is maximal monotone for all \( i \in \{1, 2\} \). For any \( z = (x, y)^t \in H \), define \( S_1 z = -\frac{3}{2} z \) and \( S_2 z = \|A\|^{-1} A z \), where \( A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \). Then \( S_1 \) is \( \frac{1}{3} \)-demicontactive and \( S_2 \) is 0-demicontactive. Furthermore, \( S_i \) is continuous and \( I - S_i \) is demiclosed at zero for all \( i \in \{1, 2\} \). The solution of our main problem is \( x^* = 0 \).

In this experiment, we compare Algorithm 3 with Algorithm 1 and Algorithm 2. Let \( v_0 = (10^5, 10^5)^t \) and \( v_1 = (10^4, 10^4)^t \). For Algorithm 1, we set \( \rho_k = \frac{3}{4} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). For Algorithm 2, select \( a_k \) and \( b_k \) as in Section 4, and define \( \phi_i(\cdot) = \frac{1}{10} \). For Algorithm 3, let \( \alpha_k = \frac{1}{2} \), \( \rho_k = \frac{1}{(k+1)^{0.5}} \), and \( q_k = 1 + \frac{1}{k+1} \) for all \( k \in \mathbb{N} \) and all \( i \in \{1, 2\} \). Further, for Algorithm 2 and Algorithm 3, we suppose \( \lambda_i = \frac{0.5}{100} \) and \( \gamma_i = \frac{7}{100} \) for all \( i \in \mathbb{N} \).
A parallel method for common variational inclusion and common fixed point problems with applications

{1, 2}, and we set $\xi_k$ as in Section 4. We let the stopping criterion $E_k := \|v_k - x^*\| < 10^{-5}$. The numerical result is presented in Figure 5.

![Figure 5. Plots of $E_k$ over Iter.](image)

From Figure 5, we can see that the number of iterations of Algorithm 1 is 63, the number of iterations of Algorithm 2 is 58 and the number of iterations of Algorithm 3 is 19, that is, the sequence generated by Algorithm 3 improves the number of iterations.

ACKNOWLEDGMENTS

This research was partially supported by Chiang Mai University. R. Suparatulatorn was supported by Post-Doctoral Fellowship of Chiang Mai University, Thailand.

REFERENCES


---

1 Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand

Email address: thanasak.m@cmu.ac.th

Email address: kanyuta@hotmail.com

2 Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Email address: raweerote.s@gmail.com