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In memoriam Prof. Charles E. Chidume (1947-2021)

Most continuous and increasing functions have two different fixed points

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ABSTRACT. We study the space of all continuous and increasing self-mappings of the real interval $[0, 1]$ equipped with the topology of uniform convergence. In particular, we show that most such functions have at least two different fixed points.

1. INTRODUCTION

For nearly sixty years now, there has been considerable research activity regarding the fixed point theory of certain nonlinear mappings. See, for example, [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 23, 24] and references cited therein. This activity stems from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point problems and variational inequalities, which find important applications in engineering, medical and the natural sciences [3, 6, 21, 22, 23, 24].

In the present paper we consider the space of all continuous and increasing self-mappings of the real interval $[0, 1]$.

It is well known that a continuous and monotone self-mapping of an interval in a Banach space X ordered by a closed and convex cone X_+ has a fixed point if the norm is an increasing function on X_+ and the following property holds:

(a) Every increasing and bounded (in the sense of the order) sequence converges.

See, for example, Theorem 3.1 on page 42 of [14].

If property (a) is not assumed, then the situation becomes more difficult and less understood. Nevertheless, some existence results were obtained for certain subspaces of the space of increasing operators using the Baire category approach (see [18] and references mentioned therein). For instance, a generic existence result was obtained for the subspace of increasing mappings A which satisfy the condition

$$A(\alpha x) \geq \alpha A(x)$$

for each $\alpha \in [0, 1]$ and each x belonging to the domain of A. For this subspace, we established the existence of a set $\mathcal{F}$, which is a countable intersection of open and everywhere dense subsets of the space, such that each mapping in $\mathcal{F}$ has a unique fixed point and all its iterates converge uniformly to this fixed point.

In the present paper we answer the question if such a result is true for the whole space of continuous and increasing mappings. It turns out that such a result is not true even in

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the one-dimensional case. As a matter of fact, we show below (see Theorem 2.1) that most continuous and increasing functions $f : [0,1] \to [0,1]$ have at least two different fixed points.

2. Results

Denote by $\mathcal{A}$ the set of all continuous and increasing functions $f : [0,1] \to [0,1]$, that is, all continuous self-mappings of $[0,1]$ such that

$$f(x) \le f(y) \text{ for all } x, y \in [0,1] \text{ satisfying } x \le y.$$

For each $f, g \in \mathcal{A}$, set

$$d(f,g) := \sup\{|f(x) - g(x)| : \ x \in [0,1]\}.$$

It is clear that $(\mathcal{A}, d)$ is a complete metric space. We begin with a few results regarding this space. The proof of our main result (Theorem 2.5 below) is relegated to the next section.

Proposition 2.1. *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that for each $f \in \mathcal{F}$,*

$$\{0,1\} \cap f([0,1]) = \emptyset.$$

Proof. Let $f \in \mathcal{A}$ and take $\gamma \in (0,1)$. Define

$$f_\gamma(x) := (1-\gamma)f(x) + 2^{-1}\gamma, \ x \in [0,1].$$

It is clear that $f_\gamma \in \mathcal{A}$ and

$$f_\gamma([0,1]) \subset [2^{-1}\gamma, 1 - 2^{-1}\gamma].$$

There exists an open neighborhood $\mathcal{U}(f,\gamma)$ of f_γ in $\mathcal{A}$ such that for each $g \in \mathcal{U}(f,\gamma)$,

$$g([0,1]) \subset [4^{-1}\gamma, 1 - 4^{-1}\gamma].$$

Define

$$\mathcal{F} := \cup\{\mathcal{U}(f,\gamma) : \ f \in \mathcal{A}, \ \gamma \in (0,1)\}.$$

It is clear that $\mathcal{F}$ is an open and everywhere dense set in $\mathcal{A}$ and that for each $f \in \mathcal{F}$, we have

$$\{0,1\} \cap f([0,1]) = \emptyset.$$

This completes the proof of Proposition 2.1. □

Proposition 2.2. *There exists a set $\mathcal{F} \subset \mathcal{A}$, which is countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that each $f \in \mathcal{F}$ is a strictly increasing function.*

Proof. Given $f \in \mathcal{A}$ and $\gamma \in (0,1)$, define

$$f_\gamma(x) := (1-\gamma)f(x) + \gamma x, \ x \in [0,1].$$

Clearly, $f_\gamma \in \mathcal{A}$.

Let $i \ge 1$ be an integer. Then for each $x_1, x_2 \in [0,1]$ satisfying $x_2 - x_1 \ge i^{-1}$, we have

$$f_\gamma(x_2) - f_\gamma(x_1) \ge \gamma(x_2 - x_1) \ge \gamma i^{-1}.$$

This implies that there exists an open neighborhood $\mathcal{U}(f,\gamma,i)$ of f_γ in $\mathcal{A}$ such that for each $g \in \mathcal{U}(f,\gamma,i)$ and each $x_1, x_2 \in [0,1]$ satisfying $x_2 - x_1 \ge i^{-1}$, we have

$$g(x_2) - g(x_1) \ge \gamma(2i)^{-1}.$$

Define

$$\mathcal{F} := \cap_{q=1}^{\infty} \cup \{\mathcal{U}(f,\gamma,i) : \ f \in \mathcal{A}, \ \gamma \in (0,1), \ i \ge q \text{ is an integer}\}.$$

It is not difficult to see that $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$ and that each $f \in \mathcal{F}$ is a strictly increasing function. Proposition 2.2 is proved. □

Example 2.1. *Let the function f be defined by*

$$f(x) = 0,\ x \in [0, 1/3],$$
$$f(x) = 1,\ x \in [2/3, 1]$$

and

$$f(x) = 3x - 1,\ x \in (1/3, 2/3).$$

Clearly, $f \in \mathcal{A}$. It is not difficult to see that for each $h \in \mathcal{A}$ satisfying $\|h - g\| < 1/4$, we have

$$h([0, 1/3]) \subset (f([0, 1/3]) + [-1/4, 1, 4]) \cap [0, 1] \subset [0, 1/4]$$

and

$$h([2/3, 1]) \subset (f([2/3, 1]) + [-1/4, 1, 4]) \cap [0, 1] \subset [3/4, 1].$$

These equations imply that there exist two points

$$\xi_1 \in [0, 1/3],\ \xi_2 \in [2/3, 1]$$

such that $h(\xi_i) = \xi_i$, $i = 1, 2$.

Example 2.2. *Let $a \in (0, 1)$, $p \geq 1$ and let the function f be defined by $f(x) = ax^p$ for all $x \in [0, 1]$. Then zero is the unique fixed point of f.*

Theorem 2.1. *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that each function $f \in \mathcal{F}$ has at least two different fixed points.*

3. Proof of Theorem 2.1

Given $f \in \mathcal{A}$ and $\gamma \in (0, 1)$, define

$$f_\gamma(x) := (1 - \gamma)f(x) + 4^{-1}\gamma + 2^{-1}\gamma x,\ x \in [0, 1]. \tag{3.1}$$

Clearly, $f_\gamma \in \mathcal{A}$, the function f_γ is strictly increasing,

$$f_\gamma([0, 1]) \subset [4^{-1}\gamma, 1 - 4^{-1}\gamma] \tag{3.2}$$

and

$$\|f_\gamma - f\| \leq \gamma. \tag{3.3}$$

There exists a point $x_* \in [0, 1]$ such that

$$f_\gamma(x_*) = x_*. \tag{3.4}$$

In view of (3.2) and (3.4),

$$x_* \in [4^{-1}\gamma, 1 - 4^{-1}\gamma]. \tag{3.5}$$

Let $i \geq 1$ be an integer. Since the function f_γ is continuous and strictly increasing, there exists

$$\Delta_0 \in (0, 8^{-1}\gamma) \tag{3.6}$$

such that

$$f_\gamma(x_*) < f_\gamma(x_* + \Delta_0) \leq f_\gamma(x_*) + 1/i \tag{3.7}$$

and

$$f_\gamma(x_*) - 1/i \leq f_\gamma(x_* - \Delta_0) < f_\gamma(x_*). \tag{3.8}$$

By (3.5), (3.7) and (3.8), there exist positive numbers Δ_1 and Δ_2 such that

$$\Delta_1 + \Delta_2 < \Delta_0 \tag{3.9}$$

and

$$\Delta_1 < \min\{f_\gamma(x_* + \Delta_0) - x_*,\ x_* - f_\gamma(x_* - \Delta_0)\}. \tag{3.10}$$

By (3.10), there exists a number $\lambda > 1$ such that

$$(2\lambda - 1)\Delta_1 < \Delta_1 + \Delta_2 \tag{3.11}$$

and

$$x_* + \lambda\Delta_1 < f(x_* + \Delta_0),\ x_* - \lambda\Delta_1 > f(x_* - \Delta_0). \tag{3.12}$$

Now define a function $g : [0,1] \to R^1$ as follows. For each point $x \in [0, x_* - \Delta_0] \cup [x_* + \Delta_0, 1]$, set

$$g(x) := f_\gamma(x). \tag{3.13}$$

Next, define

$$g(x) := x_* + \lambda(x - x_*),\ x \in [x_* - \Delta_1, x_* + \Delta_1], \tag{3.14}$$

$$g(x) := x_* + \lambda\Delta_1,\ x \in (x_* + \Delta_1, x_* + \Delta_1 + \Delta_2] \tag{3.15}$$

and

$$g(x) := x_* - \lambda\Delta_1,\ x \in [x_* - \Delta_1 - \Delta_2, x_* - \Delta_1). \tag{3.16}$$

Equations (3.10), (3.15) and (3.16) imply that

$$g(x_* + \Delta_1 + \Delta_2) = x_* + \lambda\Delta_1 < f_\gamma(x_* + \Delta_0) \tag{3.17}$$

and

$$g(x_* - \Delta_1 - \Delta_2) = x_* - \lambda\Delta_1 > f_\gamma(x_* - \Delta_0). \tag{3.18}$$

For each $x \in [x_* + \Delta_1 + \Delta_2, x_* + \Delta_0]$, set

$$g(x) := x_* + \lambda\Delta_1 + (\Delta_0 - \Delta_1 - \Delta_2)^{-1}(f_\gamma(x_* + \Delta_0) - x_* - \lambda\Delta_1)(x - x_* - \Delta_1 - \Delta_2) \tag{3.19}$$

and for each $x \in [x_* - \Delta_0, x_* - \Delta_1 - \Delta_2]$, set

$$g(x) := x_* - \lambda\Delta_1 + (\Delta_0 - \Delta_1 - \Delta_2)^{-1}(-f_\gamma(x_* - \Delta_0) + x_* - \lambda\Delta_1)(x - x_* - \Delta_1 - \Delta_2). \tag{3.20}$$

Put

$$f_{\gamma,i} := g.$$

It follows from (3.13)–(3.16), (3.19) and (3.20) that the function g is continuous and increasing and that $g \in \mathcal{A}$. In view of (3.7), (3.8) and (3.13), we have

$$\|g - f_\gamma\| \le 4i^{-1}. \tag{3.21}$$

By (3.15) and (3.16),

$$g([x_* + \Delta_1, x_* + \Delta_1 + \Delta_2]) = \{x_* + \lambda\Delta_1\} \tag{3.22}$$

and

$$g([x_* - \Delta_1 - \Delta_2, x_* - \Delta_1]) = \{x_* - \lambda\Delta_1\}. \tag{3.23}$$

Denote by $\mathcal{U}(f, \gamma, i)$ the open neighborhood of $g = f_{\gamma,i}$ in $\mathcal{A}$ such that for each $h \in \mathcal{U}(f, \gamma, i)$,

$$d(h, f_{\gamma,i}) < (\lambda - 1)\Delta_1 \tag{3.24}$$

(see (3.11)). Consider

$$h \in \mathcal{U}(f, \gamma, i). \tag{3.25}$$

It follows from (3.11) and (3.22)–(3.25) that

$$\begin{aligned} h([x_* + \Delta_1, x_* + \Delta_1 + \Delta_2]) &\subset g([x_* + \Delta_1, x_* + \Delta_1 + \Delta_2]) + [-(\lambda - 1)\Delta_1,\ (\lambda - 1)\Delta_1] \\ &\subset [x_* + \Delta_1, x_* + (2\lambda - 1)\Delta_1] \subset [x_* + \Delta_1, x_* + \Delta_1 + \Delta_2] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} h([x_* - \Delta_1 - \Delta_2, x_* - \Delta_1]) &\subset g([x_* - \Delta_1 - \Delta_2, x_* - \Delta_1]) + [-(\lambda - 1)\Delta_1,\ (\lambda - 1)\Delta_1] \\ &\subset [x_* - (2\lambda - 1)\Delta_1, x_* - \Delta_1] \subset [x_* - \Delta_1 - \Delta_2, x_* - \Delta_1]. \end{aligned} \tag{3.27}$$

In view of (3.26) and (3.27), there exist points

$$\xi_1 \in [x_* + \Delta_1, x_* + \Delta_1 + \Delta_2]$$

and

$$\xi_2 \in [x_* - \Delta_1 - \Delta_2, x_* - \Delta_1]$$

such that

$$h(\xi_i) = \xi_i,\ i = 1, 2.$$

Thus any function in $\mathcal{U}(f, \gamma, i)$ has at least two different fixed points. Define

$$\mathcal{F} := \cup\{\mathcal{U}(f, \gamma, i) :\ f \in \mathcal{A},\ \gamma \in (0, 1),\ i \geq 1 \text{ is an integer}\}.$$

It is clear that $\mathcal{F}$ is an open set. Since the set $\{f_\gamma :\ f \in \mathcal{A},\ \gamma \in (0, 1)\}$ is everywhere dense, inequality (3.21) implies that $\mathcal{F}$ is also an everywhere dense set in $\mathcal{A}$. By construction, each function in $\mathcal{F}$ has at least two different fixed points. This completes the proof of Theorem 2.1.

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