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In memoriam Prof. Charles E. Chidume (1947-2021)

Modified inertial Mann's algorithm and inertial hybrid algorithm for k-strict pseudo-contractive mappings

SUPARAT BAIYA[1], KASAMSUK UNGCHITTRAKOOL[1,2,*]

ABSTRACT. In this work, we introduce and study the modified inertial Mann's algorithm and inertial hybrid algorithm for approximating some fixed points of a k-strict pseudo-contractive mapping in Hilbert spaces. Weak convergence to a solution of fixed-point problems for a k-strict pseudo-contractive mapping is obtained by using the modified inertial Mann's algorithm. In order to obtain strong convergence, we introduce an inertial hybrid algorithm by using the inertial extrapolation method mixed with the convex combination of three iterated vectors and forcing for strong convergence by the hybrid projection method for a k-strict pseudo-contractive mapping in Hilbert spaces. The strong convergence theorem of the proposed method is proved under mild assumptions on the scalars. For illustrating the performance of the proposed algorithms, we provide some new nonlinear k-strict pseudo-contractive mappings which are not nonexpansive to create some numerical experiments to show the advantage of the two new inertial algorithms for a k-strict pseudo-contractive mapping.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Throughout the paper, we let I be the identity mapping. A mapping $T : C \to C$ is said to be a k-strict pseudo-contractive mapping if there exists a constant $k \in (-\infty, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for all $x, y \in C$; see [8–11, 13, 34] for more details. If $k = -1$, then T is said to be firmly nonexpansive. If $k = 0$, then T is said to be nonexpansive. The set of all fixed points of T is denoted by $Fix(T) = \{x \in C : Tx = x\}$. On the other hand, a mapping $U : C \to H$ is called α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha\|Ux - Uy\|^2$$

for all $x, y \in C$. The set of all zeros of U is denoted by $U^{-1}(0) = \{x \in C : Ux = 0\}$. It is clear that the class of firmly nonexpansive mappings and the class of nonexpansive mappings are strictly included in the class of k-strict pseudo-contractive mappings; see [33, 34] for more details. k-strict pseudo-contractive mappings were first proposed by Browder and Petryshyn [13] in 1967. It has been found in practice that k-strict pseudo-contractive mappings play an important role and have more practical applications than firmly nonexpansive mappings and nonexpansive mappings do in solving inverse problems (see Scherzer [30]). Indeed, if U is α-inverse strongly monotone operator, then $T := I - U$ is a $(1 - 2\alpha)$-strict pseudo-contractive mapping, and so we can translate a problem of zeros for U in the form of fixed point problem for T, and vice versa (see e.g. [33, 34]).

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Corresponding author: *Kasamsuk Ungchittrakool; kasamsuku@nu.ac.th

The fixed point problem is to find a point

$$x \in C \quad \text{such that} \quad Tx = x. \tag{1.2}$$

There are several methods for solving (1.2). One of the most popular methods is Mann's algorithm [20] which was introduced in 1953. The form of Mann's algorithm is as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$ which satisfies some appropriate assumptions. Reich [29] proved the fundamental results of convergence, that is, if sequence $\{\alpha_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = +\infty$ then the sequence $\{x_n\}$ generated by Mann's algorithm (1.3) converges weakly to a fixed point of a nonexpansive mapping T.

Later, Marino and Xu [21] developed the result of Reich [29] to the class of k-strict pseudo-contractive mappings in the framework of real Hilbert space, that is, if the control sequcnce $\{\alpha_n\}$ is chosen so that $k < \alpha_n < 1$ for all n and $\sum_{n=1}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$ then the sequence $\{x_n\}$ converges weakly to a fixed point of a k-strict pseudo-contractive mapping. Some important iterative methods for fixed point problems of nonexpansive mappings and k-strict pseudo-contractive mappings have been collected in the literature (see [8–11, 16, 20, 24, 27]).

Strong convergence is often much more desirable than weak convergence in many problems that arise in infinite-dimensional spaces (see [6, 17, 19, 21, 25, 26, 34, 35] and references therein). In 2003, Nakajo and Takahashi [24] introduced a hybrid algorithm for Mann's iteration as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, z - x_n \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \subset [0, 1)$ and P_K denotes the metric projection from H onto a closed convex subset K of H.

In 2007, Marino and Xu [21] introduced a hybrid algorithm for a k-strict pseudo-contractive mapping as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.5}$$

Polyak [28] introduced an inertial-type algorithm. He presents a two-step iterative method in which the next iterate is defined by using the previous two iterates. It is an acceleration process of incorporating an inertial term in an algorithm to speed up or accelerates the rate of convergence of the sequence generated by the algorithm. Consequently, many researchers have adopted inertial-type algorithms to speed up the convergence process, see for example [1–5, 12, 15] and the references therein.

In 2008, Mainge [18] introduced the following inertial Mann's algorithm:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = w_n + \lambda_n[T(w_n) - w_n)], \end{cases}$$

for each $n \ge 1$ and showed that the iterative sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(A1) $\theta_n \in [0, \theta)$ for any $\theta \in [0, 1)$,

(A2) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$,
(A3) $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < 1$.
To satisfy the condition (A2) of the sequence $\{x_n\}$, one needs to calculate $\{\theta_n\}$ at each step (see [23]).

Later, Bot and Csetnek [12] have revised the above conditions to simple to prove the theorems as the following:
(B1) $\theta_n \in [0, \theta)$ for all $\theta \in [0, 1), \theta_1 = 0$ and $\{\theta_n\}$ is nondecreasing,
(B2) $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$ and $0 < \lambda \leq \lambda_n \leq \frac{\delta-\theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$ for each $n \geq 1$, where $\lambda, \sigma, \delta > 0$.

By using the concept of the inertial method, Shehu et al. [31] introduced an algorithm by the technique of Halpern method and error terms for solving a fixed point of a nonexpansive mapping which was defined as follows:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \beta_n w_n + \gamma_n T w_n + e_n, \end{cases}$$

for each $n \geq 1$, where $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1), \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1]$ and $\{e_n\}$ is a sequence in H.

In 2018, Dong et al. [15] introduced a modified inertial Mann's algorithm by combining the accelerated Mann's algorithm and the inertial extrapolation. They proved the weak convergence of the proposed algorithm for a nonexpansive mapping.

Algorithm (Modified inertial Mann's algorithm).

Let $T : H \to H$ be a self mapping such that $Fix(T) \neq \emptyset$. Choose $\mu \in (0, 1], \lambda > 0$ and $x_0, x_1 \in H$ arbitrarily and set $d_0 = (Tx_0 - x_0)/\lambda$. Compute d_{n+1} and x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ d_{n+1} = \dfrac{1}{\lambda}(Tw_n - w_n) + \beta_n d_n, \\ y_n = w_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\gamma_n w_n + (1 - \mu\gamma_n) y_n, \end{cases} \tag{1.6}$$

for each $n \geq 1$, where $\{\theta_n\} \subset [0, \theta]$ is nondecreasing with $\theta_1 = 0$ and $0 \leq \theta < 1, \{\gamma_n\}$ satisfies
(D1) $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$ and $0 < 1 - \mu\gamma \leq 1 - \mu\gamma_n \leq \frac{\delta-\theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$, where $\gamma, \sigma, \delta > 0$ and $\{\beta_n\}$ satisfies
(D2) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Moreover, there are more some additional assumptions as follows:

Assumption 1.1. *The sequence $\{w_n\}$ defined in (1.6) satisfies*
(D3) $\{Tw_n - w_n\}$ is bounded;
(D4) $\{Tw_n - y\}$ is bounded for any $y \in Fix(T)$.

Remark 1.1. *If the considered mapping T is a k-strict pseudo-contractive mapping, then it is not hard to verify that Assumption 1.1 is equivalent to $\{w_n\}$ is bounded.*

Furthermore, they introduced an inertial CQ-algorithm for a nonexpansive mapping by combining the CQ-algorithm (1.4) and the inertial extrapolation, and analyzed its strong convergence. Set $x_0, x_1 \in H$ arbitrarily. Define a sequence $\{x_n\}$ as following:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1 - \beta_n)w_n + \beta_n T w_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|w_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

for each $n \geq 0$ where $\{\theta_n\} \subset [\theta_1, \theta_2], \theta_1 \in (-\infty, 0], \theta_2 \in [0, \infty), \{\beta_n\} \subset [\beta, 1], \beta \in (0, 1]$.

Motivated by the research works as mentioned in the direction as above, it is the driving force for us to develop the modified inertial Mann's algorithm based on Dong et al. [15] in (1.6) and prove the weak convergence for a k-strict pseudo-contractive mapping by using the conditions in Assumption 1.1. Moreover, the inertial extrapolation method combined with the convex combination of three iterated vectors and forcing for strong convergence by the hybrid projection method is provided to solve a fixed point problem for a k-strict pseudo-contractive mapping in Hilbert spaces. The performance of these two newly created algorithms demonstrates some numerical advantages which will be illustrated in the last section.

2. Preliminaries

We will use the following notation:
(1) $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.
(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Now, we present some fact and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities which will be used in the various places in the proofs of the results of this paper:*
(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, for all $x, y \in H$.
(2) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$ for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$ and for all $x, y, z \in H$.

Lemma 2.2 ([7])**.** *Let $\{\Psi_n\}, \{\delta_n\}$ and $\{\theta_n\}$ be the sequence in $[0, +\infty)$ such that $\Psi_{n+1} \leq \Psi_n + \theta_n(\Psi_n - \Psi_{n-1}) + \delta_n$ for all $n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \geq 1$. Then the following hold:*
(1) $\sum_{n\geq 1}[\Psi_n - \Psi_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;
(2) there exists $\Psi^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \Psi_n = \Psi^*$.

Lemma 2.3 ([7])**.** *Let C be a nonempty set of a real Hilbert space H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*
(1) for any $x \in C, \lim_{n\to\infty} \|x_n - x\|$ exists;
(2) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.4 ([21])**.** *Let $T : C \to C$ be a self-mapping.*
(1) If T is a k-strict pseudo-contraction, then T satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\| \quad \text{for all } x, y \in C.$$

(2) If T is a k-strict pseudo-contraction, then the mapping $I - T$ is demiclosed at zero. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - T)x_n \to 0$ strongly, then $(I - T)x^* = 0$.

Lemma 2.5 ([32])**.** *Let C be a closed convex subset of a real Hilbert space H and let P_C be the (metric or nearest point) projection from H onto C (i.e., for $x \in H, P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0 \quad \text{for all } y \in C.$$

Lemma 2.6 ([22]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is a sequence such that $\omega_w(x_n) \subset C$ and satisfies the condition:*

$$\|x_n - u\| \le \|u - q\| \quad \text{for all } n.$$

Then $x_n \to q$ as $n \to \infty$.

3. A modified inertial Mann's algorithm for k-strict pseudo-contractive mappings

In this section, we study and prove the weak convergence of a modified inertial Mann's algorithm in (1.6) for a k-strict pseudo-contractive mapping under the Assumption 1.1 which was introduced by Dong et al. [15].

Theorem 3.1. *Let $T : H \to H$ is a k-strict pseudo-contractive mapping for some $0 \le k < 1$ and $k < \mu\gamma_n$ with $Fix(T) \neq 0$. Let $\{d_n\}$ and $\{x_n\}$ be the sequences generated by Algorithm (1.6) and let Assumption 1.1 hold. Then the following hold:*
1. $\{d_n\}$ is bounded;
2. $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$;
3. the sequence $\{x_n\}$ converges weakly to a point of $Fix(T)$.

Proof. 1. The proof follows from Theorem 3.1 in [15].
2. Note that $w_n = x_n + \theta_n(x_n - x_{n-1})$ for each $n \ge 1$. From (1.6), we get

$$\begin{aligned} x_{n+1} &= \mu\gamma_n w_n + (1 - \mu\gamma_n)(Tw_n + \lambda\beta_n d_n) \\ &= w_n + (1 - \mu\gamma_n)(Tw_n - w_n + \lambda\beta_n d_n). \end{aligned} \tag{3.1}$$

Let arbitrarily $y \in Fix(T)$. Using Lemma 2.1 and T is a k-strict pseudo-contractive mapping, we get

$$\begin{aligned}
&\|x_{n+1} - y\|^2 \\
&= \mu\gamma_n\|w_n - y\|^2 + (1 - \mu\gamma_n)\|Tw_n - y + \lambda\beta_n d_n\|^2 - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \mu\gamma_n\|w_n - y\|^2 + (1 - \mu\gamma_n)(\|Tw_n - y\|^2 + 2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) \\
&\quad - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&\le \mu\gamma_n\|w_n - y\|^2 + (1 - \mu\gamma_n)(\|w_n - y\|^2 + k\|(I - T)w_n - (I - T)y\|^2) \\
&\quad + (1 - \mu\gamma_n)(2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - y\|^2 + (1 - \mu\gamma_n)k\|Tw_n - w_n\|^2 + (1 - \mu\gamma_n)(2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) \\
&\quad - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - y\|^2 + (1 - \mu\gamma_n)k\|(Tw_n - w_n + \lambda\beta_n d_n) - \lambda\beta_n d_n\|^2 \\
&\quad + (1 - \mu\gamma_n)(2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - y\|^2 + (1 - \mu\gamma_n)k(\|Tw_n - w_n + \lambda\beta_n d_n\|^2 - 2\langle Tw_n - w_n + \lambda\beta_n d_n, \lambda\beta_n d_n\rangle \\
&\quad + \lambda^2\beta_n^2\|d_n\|^2) + (1 - \mu\gamma_n)(2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) \\
&\quad - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - y\|^2 + (1 - \mu\gamma_n)k(\|Tw_n - w_n + \lambda\beta_n d_n\|^2 - 2\lambda\beta_n \langle Tw_n - w_n, d_n\rangle - \lambda^2\beta_n^2\|d_n\|^2) \\
&\quad + (1 - \mu\gamma_n)(2\lambda\beta_n \langle Tw_n - y, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2) - \mu\gamma_n(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - y\|^2 - (\mu\gamma_n - k)(1 - \mu\gamma_n)\|Tw_n - w_n + \lambda\beta_n d_n\|^2
\end{aligned}$$

$$+(1-\mu\gamma_n)\beta_n\left(2\lambda\langle Tw_n-y,d_n\rangle-2\lambda k\langle Tw_n-w_n,d_n\rangle+(1-k)\lambda^2\beta_n\|d_n\|^2\right)$$

$$=\|w_n-y\|^2-(\mu\gamma_n-k)(1-\mu\gamma_n)\|Tw_n-w_n+\lambda\beta_n d_n\|^2+\beta_n\varphi_n \tag{3.2}$$

where

$$\varphi_n=(1-\mu\gamma_n)\Big[2\lambda\langle Tw_n-y,d_n\rangle-2\lambda k\langle Tw_n-w_n,d_n\rangle+(1-k)\lambda^2\beta_n\|d_n\|^2\Big].$$

From (D1), (D3), (D4) and $\{d_n\}$ is bounded, it follows that $\{\varphi_n\}$ is bounded. Then there exists $M_1>0$ such that $\varphi_n\le M_1$ for all $n\ge 1$. By Lemma 2.1, we get

$$\begin{aligned}\|w_n-y\|^2&=\|(1+\theta_n)(x_n-y)-\theta_n(x_{n-1}-y)\|^2\\&=(1+\theta_n)\|x_n-y\|^2-\theta_n\|x_{n-1}-y\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2,\end{aligned}$$

which with (3.2) implies

$$\begin{aligned}&\|x_{n+1}-y\|^2-(1+\theta_n)\|x_n-y\|^2+\theta_n\|x_{n-1}-y\|^2\\&\le-(\mu\gamma_n-k)(1-\mu\gamma_n)\|Tw_n-w_n+\lambda\beta_n d_n\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2+\beta_n\varphi_n.\end{aligned}\tag{3.3}$$

From (1.6) and (3.1), we get

$$\begin{aligned}\|Tw_n-w_n+\lambda\beta_n d_n\|^2&=\left\|\frac{x_{n+1}-w_n}{1-\mu\gamma_n}\right\|^2=\left\|\frac{x_{n+1}-x_n-\theta_n(x_n-x_{n-1})}{1-\mu\gamma_n}\right\|^2\\&=\frac{\|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2-2\theta_n\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle}{(1-\mu\gamma_n)^2}\\&\ge\frac{1}{(1-\mu\gamma_n)^2}[\|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\\&\quad+\theta_n(-\rho_n\|x_{n+1}-x_n\|^2-\frac{1}{\rho_n}\|x_n-x_{n-1}\|^2)],\end{aligned}$$

where we note $\rho_n=1/(\theta_n+\delta(1-\mu\gamma_n))$. Thus

$$\begin{aligned}&-(\mu\gamma_n-k)(1-\mu\gamma_n)\|Tw_n-w_n+\lambda\beta_n d_n\|^2\\&\le-\frac{\mu\gamma_n}{(1-\mu\gamma_n)}\Big[\|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\\&\quad+\theta_n\big(-\rho_n\|x_{n+1}-x_n\|^2-\frac{1}{\rho_n}\|x_n-x_{n-1}\|^2\big)\Big].\end{aligned}\tag{3.4}$$

Replace (3.4) in the inequality (3.3), we get

$$\begin{aligned}&\|x_{n+1}-y\|^2-(1+\theta_n)\|x_n-y\|^2+\theta_n\|x_{n-1}-y\|^2\\&\le-\frac{\mu\gamma_n}{(1-\mu\gamma_n)}\Big[\|x_{n+1}-x_n\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\\&\quad+\theta_n\big(-\rho_n\|x_{n+1}-x_n\|^2-\frac{1}{\rho_n}\|x_n-x_{n-1}\|^2\big)\Big]\\&\quad+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2+\beta_n\varphi_n\\&\le\frac{\mu\gamma_n(\theta_n\rho_n-1)}{(1-\mu\gamma_n)}\|x_{n+1}-x_n\|^2+\Theta_n\|x_n-x_{n-1}\|^2+\beta_n\varphi_n,\end{aligned}\tag{3.5}$$

where

$$\Theta_n=\theta_n(1+\theta_n)+\theta_n\mu\gamma_n\frac{1-\rho_n\theta_n}{\rho_n(1-\mu\gamma_n)}>0\tag{3.6}$$

since $\rho_n\theta_n < 1$ and $(1-\mu\gamma_n) \in (0,1)$. We choose $\delta = (1-\rho_n\theta_n)/\rho_n(1-\mu\gamma_n)$ and from (3.6), it follows

$$\Theta_n = \theta_n(1+\theta_n) + \theta_n\mu\gamma_n\delta \le \theta(1+\theta) + \theta\delta. \tag{3.7}$$

We now apply some techniques from [1] adapted to our setting. Define the sequence $\{\phi_n\}$ and $\{\Psi_n\}$ by

$$\phi_n = \|x_n - y\|^2 \text{ and } \Psi_n = \phi_n - \theta_n\phi_{n-1} + \Theta_n\|x_n - x_{n-1}\|^2 + \beta_n\varphi_n,$$

for all $n \ge 1$. Using the monotonicity of $\{\theta_n\}$ and the fact that $\phi_n \ge 0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\Psi_{n+1} - \Psi_n \le \phi_{n+1} - (1+\theta_n)\phi_n + \theta_n\phi_{n-1} + \Theta_{n+1}\|x_{n+1} - x_n\|^2 \\ - \Theta_n\|x_n - x_{n-1}\|^2 + \beta_{n+1}\varphi_{n+1} - \beta_n\varphi_n.\end{aligned}$$

By (3.5), we obtain that

$$\begin{aligned}\Psi_{n+1} - \Psi_n &\le \frac{\mu\gamma_n(\theta_n\rho_n - 1)}{1-\mu\gamma_n}\|x_{n+1} - x_n\|^2 + \Theta_{n+1}\|x_{n+1} - x_n\|^2 + \beta_{n+1}\varphi_{n+1} \\ &= \Big(\frac{\mu\gamma_n(\theta_n\rho_n - 1)}{1-\mu\gamma_n} + \Theta_{n+1}\Big)\|x_{n+1} - x_n\|^2 + \beta_{n+1}\varphi_{n+1}. \end{aligned} \tag{3.8}$$

We now claim that

$$\frac{\mu\gamma_n(\theta_n\rho_n - 1)}{1-\mu\gamma_n} + \Theta_{n+1} \le -\sigma. \tag{3.9}$$

Indeed, by the choice of ρ_n, we have

$$\begin{aligned}\frac{\mu\gamma_n(\theta_n\rho_n - 1)}{1-\mu\gamma_n} + \Theta_{n+1} \le -\sigma &\iff (1-\mu\gamma_n)(\Theta_{n+1} + \sigma) + \mu\gamma_n(\theta_n\rho_n - 1) \le 0 \\ &\iff (1-\mu\gamma_n)(\Theta_{n+1} + \sigma) - \frac{\delta(1-\mu\gamma_n)\mu\gamma_n}{\theta_n + \delta(1-\mu\gamma_n)} \le 0 \\ &\iff (\theta_n + \delta(1-\mu\gamma_n))(\Theta_{n+1} + \sigma) + \delta(1-\mu\gamma_n) \le \delta.\end{aligned}$$

Employing (3.7), we have

$$\begin{aligned}&(\theta_n + \delta(1-\mu\gamma_n))(\Theta_{n+1} + \sigma) + \delta(1-\mu\gamma_n) \\ &\le (\theta + \delta(1-\mu\gamma_n))(\theta(1+\theta) + \theta\delta + \sigma) + \delta(1-\mu\gamma_n) \le \delta,\end{aligned}$$

where the last inequality follows by using the upper bound for sequence $\{1-\mu\gamma_n\}$ in (D1). Hence the claim in (3.9) is true. It follows from (3.8) and (3.9), we have

$$\Psi_{n+1} - \Psi_n \le -\sigma\|x_{n+1} - x_n\|^2 + \beta_{n+1}\varphi_{n+1} \le \beta_{n+1}M_2, \tag{3.10}$$

which implies

$$\Psi_n - \Psi_1 \le M_3,$$

where $M_3 = M_2\sum_{n=2}^{\infty}\beta_n$. The boundedness for $\{\theta_n\}$ delivers

$$-\theta\phi_{n-1} \le \phi_n - \theta\phi_{n-1} \le \Psi_n \le \Psi_1 + M_3.$$

Thus we obtain

$$\phi_n \le \theta^n\phi_0 + (\Psi_1 + M_3)\sum_{n=1}^{k-1}\theta^n \le \theta^n\phi_0 + \frac{1}{1-\theta}(\Psi_1 + M_3),$$

where we notice that $\Psi_1 = \varphi_1 \geq 0$ (due to the relation $\theta_1 = 0$). Using (3.10) and the boundedness of $\{\Psi_n\}$, we obtain

$$\sigma \sum_{k=1}^{n} \|x_{k+1} - x_k\|^2 \leq \Psi_1 - \Psi_{n+1} + M_3 \leq \Psi_1 + \theta\phi_n + M_3$$
$$\leq \Psi_1 + \theta^n \phi_0 + \frac{1}{1-\theta}(\Psi_1 + M_3) + M_3,$$

which shows $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$.
3. We show this by using Lemma 2.3. For arbitrary $y \in Fix(T)$, by (3.5), (3.7) and Lemma 2.2, we get the result that $\lim_{n\to\infty} \|x_n - y\|$ exists (we take into consideration also that, in (3.5), $\theta_n \rho_n < 1$). On the other hand, we let x be a sequential weak cluster point of $\{x_n\}$, that is, there exists a subsequence $\{x_{n_k}\}$ which converge weakly to x. By part (2), the definition of w_n and $\{\theta_n\} \subset [0, 1)$, we get $w_{n_k} \rightharpoonup x$ as $k \to \infty$. Furthermore, from (3.1), we get

$$\|Tw_n - w_n\| = \left\| \frac{x_{n+1} - w_n}{1 - \mu\gamma_n} - \lambda\beta_n d_n \right\| = \left\| \frac{x_{n+1} - x_n - \theta_n(x_n - x_{n-1})}{1 - \mu\gamma_n} - \lambda\beta_n d_n \right\|$$
$$\leq \frac{\|x_{n+1} - x_n\| + \theta_n\|x_n - x_{n-1}\|}{1 - \mu\gamma_n} + \lambda\beta_n\|d_n\|.$$

By (D2), (1.3) and (1.5), we obtain $\|Tw_{n_k} - w_{n_k}\| \to 0$ as $k \to \infty$. Applying Lemma 2.4 for the sequence $\{w_{n_k}\}$, we conclude that $x \in Fix(T)$. This completes the proof. □

Since a nonexpansive mapping is a 0-strict pseudo-contractive mapping, we have the following consequence of Theorem 3.1 which improves on the main result of Dong et al. [15].

Corollary 3.2 ([15, Theorem 3.1]). *Let $T : H \to H$ is a nonexpansive mapping with $Fix(T) \neq 0$. Let $\{d_n\}$ and $\{x_n\}$ be the sequences generated by Algorithm (1.6) and let Assumption 1.1 holds. Then the following hold:*
1. $\{d_n\}$ is bounded;
2. $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$;
3. the sequence $\{x_n\}$ converges weakly to a point of $Fix(T)$.

4. An inertial hybrid algorithm for k-strict pseudo-contractive mappings

In this section, we introduce an inertial hybrid algorithm by using the inertial extrapolation method combined with the convex combination of three iterated vectors and forcing for strong convergence by the hybrid projection method for a k-strict pseudo-contractive mapping in Hilbert spaces. The strong convergence theorem is proved under mild assumptions on the scalars.

Theorem 4.1. *Let $T : H \to H$ be a k-strict pseudo-contractive mapping for some $0 \leq k < 1$ with $Fix(T) \neq \emptyset$ and let $\{\theta_n\} \subset [\theta_1, \theta_2], \theta_1 \in (-\infty, 0], \theta_2 \in [0, \infty)$. Set $x_0, x_1 \in H$ arbitrarily. Define a sequence $\{x_n\}$ by the following algorithm:*
(4.1)
$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_n + \beta_n w_n + \gamma_n T w_n, \\ C_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 + 2(1-\alpha_n)\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + \Phi_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\Phi_n = \theta_n^2(1-\alpha_n-\alpha_n\beta_n)\|x_n-x_{n-1}\|^2+\gamma_n(k-\beta_n)\|w_n-Tw_n\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2$. *Assume that the control sequence* $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ *and* $\{\gamma_n\}\subset(0,1]$ *such that* $\alpha_n+\beta_n+\gamma_n=1$. *Then the iterative sequence* $\{x_n\}$ *converges in norm to* $P_{Fix(T)}(x_0)$.

Proof. We first to show that $Fix(T)\subset C_n$ for all $n\geq 0$. Using Lemma 2.1 (2) and (1.1), we get that for all $z\in Fix(T)$,

$$\begin{aligned}
\|y_n-z\|^2 &= \|\alpha_n(x_n-z)+\beta_n(w_n-z)+\gamma_n(Tw_n-z)\|^2\\
&= \alpha_n\|x_n-z\|^2+\beta_n\|w_n-z\|^2+\gamma_n\|Tw_n-z\|^2-\alpha_n\beta_n\|x_n-w_n\|^2\\
&\quad -\alpha_n\gamma_n\|x_n-Tw_n\|^2-\beta_n\gamma_n\|w_n-Tw_n\|^2\\
&\leq \alpha_n\|x_n-z\|^2+\beta_n\|w_n-z\|^2+\gamma_n(\|w_n-z\|^2+k\|w_n-Tw_n\|^2)\\
&\quad -\alpha_n\beta_n\|x_n-w_n\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2-\beta_n\gamma_n\|w_n-Tw_n\|^2\\
&= \alpha_n\|x_n-z\|^2+(1-\alpha_n)\|w_n-z\|^2+\gamma_n(k-\beta_n)\|w_n-Tw_n\|^2 \qquad (4.2)\\
&\quad -\alpha_n\beta_n\|x_n-w_n\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2.
\end{aligned}$$

By the definition of w_n in (4.1), we have

$$\|w_n-z\|^2=\|(x_n-z)+\theta_n(x_n-x_{n-1})\|^2=\|x_n-z\|^2+2\theta_n\langle x_n-z,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2 \tag{4.3}$$

and

$$\|x_n-w_n\|^2=\|x_n-(x_n+\theta_n(x_n-x_{n-1}))\|^2=\theta_n^2\|x_n-x_{n-1}\|^2. \tag{4.4}$$

Substitute (4.3) and (4.4) into (4.2) to get

$$\begin{aligned}
&\|y_n-z\|^2\\
&\leq \alpha_n\|x_n-z\|^2+(1-\alpha_n)\big(\|x_n-z\|^2+2\theta_n\langle x_n-z,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\big)\\
&\quad +\gamma_n(k-\beta_n)\|w_n-Tw_n\|^2-\alpha_n\beta_n\theta_n^2\|x_n-x_{n-1}\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2\\
&= \|x_n-z\|^2+2(1-\alpha_n)\theta_n\langle x_n-z,x_n-x_{n-1}\rangle+\Phi_n,
\end{aligned}$$

where $\Phi_n=\theta_n^2(1-\alpha_n-\alpha_n\beta_n)\|x_n-x_{n-1}\|^2+\gamma_n(k-\beta_n)\|w_n-Tw_n\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2$. This implies that $z\in C_n$ and hence $Fix(T)\subset C_n$ for all $n\geq 0$. Since

$$\begin{aligned}
C_n &= \left\{z\in H:\|y_n-z\|^2\leq\|x_n-z\|^2+2(1-\alpha_n)\theta_n\langle x_n-z,x_n-x_{n-1}\rangle+\Phi_n\right\}\\
&= \left\{z\in H:\|y_n\|^2-\|x_n\|^2\leq\langle 2(y_n-x_n-(1-\alpha_n)\theta_n(x_n-x_{n-1})),z\rangle+a\right\},
\end{aligned}$$

where $a=2(1-\alpha_n)\theta_n\langle x_n-x_{n-1},x_n\rangle+\Phi_n$. So, C_n is closed and convex. Therefore $\{x_n\}$ is well defined.

We next to show that $Fix(T)\subset Q_n$ for all $n\geq 0$ by induction. For $n=0$, we have $Fix(T)\subset H=Q_0$. Assume that $Fix(T)\subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n\cap Q_n$, by Lemma 2.5 we get

$$\langle x_{n+1}-z,x_{n+1}-x_0\rangle\leq 0\quad\text{for all } z\in C_n\cap Q_n.$$

From $Q_{n+1}=\{z\in H:\langle x_{n+1}-z,x_{n+1}-x_0\rangle\leq 0\}$, it follows $C_n\cap Q_n\subset Q_{n+1}$. Since $Fix(T)\subset C_n$ and the assumption that $Fix(T)\subset Q_n$, we have $Fix(T)\subset C_n\cap Q_n$. So, we have $Fix(T)\subset Q_{n+1}$ implies that $Fix(T)\subset Q_n$ for all $n\geq 0$.

Since $Fix(T)$ is a nonempty closed convex subset of H, there exists a unique element $q\in Fix(T)$ such that $q=P_{Fix(T)}x_0$. From the definition of Q_n actually implies $x_n=P_{Q_n}(x_0)$. This together with that fact that $Fix(T)\subset Q_n$ further implies $\|x_n-x_0\|\leq\|p-x_0\|$ for all $p\in Fix(T)$. Due to $q=P_{Fix(T)}\in Fix(T)$, we get

$$\|x_n-x_0\|\leq\|q-x_0\|, \tag{4.5}$$

which implies that $\{x_n\}$ bounded.

The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. From Lemma 2.1 (1) implies that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2\\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle\\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we obtain that

$$\begin{aligned}\sum_{n=1}^{N} \|x_{n+1} - x_n\|^2 &\leq \sum_{n=1}^{N}(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2) = \|x_{N+1} - x_0\|^2 - \|x_1 - x_0\|^2\\ &\leq \|q - x_0\|^2 - \|x_1 - x_0\|^2.\end{aligned}$$

So, it follows that the series $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent and thus

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.7}$$

From (4.4), we get

$$\|w_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

Since

$$\begin{aligned}\|w_n - x_{n+1}\|^2 &= \|(x_n - x_{n+1}) + \theta_n(x_n - x_{n-1})\|^2\\ &= \|x_n - x_{n+1}\|^2 + 2\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{4.9}$$

From the fact $x_{n+1} \in C_n$, we get

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 2(1 - \alpha_n)\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \Phi_n, \tag{4.10}$$

where $\Phi_n = \theta_n^2(1-\alpha_n-\alpha_n\beta_n)\|x_n-x_{n-1}\|^2+\gamma_n(k-\beta_n)\|w_n-Tw_n\|^2-\alpha_n\gamma_n\|x_n-Tw_n\|^2$. Since $y_n = \alpha_n x_n + \beta_n w_n + \gamma_n T w_n$ and using Lemma 2.1 (2), this implies that

$$\begin{aligned}\|y_n - x_{n+1}\|^2 &= \|\alpha_n(x_n - x_{n+1}) + \beta_n(w_n - x_{n+1}) + \gamma_n(Tw_n - x_{n+1})\|^2\\ &= \alpha_n\|x_n - x_{n+1}\|^2 + \beta_n\|w_n - x_{n+1}\|^2 + \gamma_n\|Tw_n - x_{n+1}\|^2\\ &\quad - \alpha_n\beta_n\|x_n - w_n\|^2 - \alpha_n\gamma_n\|x_n - Tw_n\|^2 - \beta_n\gamma_n\|w_n - Tw_n\|^2.\end{aligned} \tag{4.11}$$

Substituting (4.10) with (4.11), we get that

$$\begin{aligned}\gamma_n\|Tw_n - x_{n+1}\|^2 &\leq (1 - \alpha_n)\|x_n - x_{n+1}\|^2 + 2(1 - \alpha_n)\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle\\ &\quad + \theta_n^2(1 - \alpha_n - \alpha_n\beta_n)\|x_n - x_{n-1}\|^2 + k\gamma_n\|w_n - Tw_n\|^2\\ &\quad - \beta_n\|w_n - x_{n+1}\|^2 + \alpha_n\beta_n\|x_n - w_n\|^2,\end{aligned} \tag{4.12}$$

Consider,

$$\begin{aligned}\|Tw_n - x_{n+1}\|^2 &= \|(Tw_n - w_n) - (x_{n+1} - w_n)\|^2\\ &= \|Tw_n - w_n\|^2 - 2\langle Tw_n - w_n, x_{n+1} - w_n\rangle + \|x_{n+1} - w_n\|^2.\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) with $0 \leq k < 1$ and $\{\gamma_n\} \subset (0, 1]$ for all n, we obtain that

$$\begin{aligned}\|Tw_n - w_n\|^2 \leq \frac{1}{\gamma_n(1 - k)}\Big(&(1 - \alpha_n)\|x_n - x_{n+1}\|^2 + 2(1 - \alpha_n)\theta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle\\ &+ \theta_n^2(1 - \alpha_n - \alpha_n\beta_n)\|x_n - x_{n-1}\|^2 - (\beta_n + \gamma_n)\|w_n - x_{n+1}\|^2\\ &+ \alpha_n\beta_n\|x_n - w_n\|^2 + 2\gamma_n\langle Tw_n - w_n, x_{n+1} - w_n\rangle\Big).\end{aligned}$$

By (4.7), (4.8) and (4.9), it can be concluded that

$$\|Tw_n - w_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.14}$$

By subtracting and adding, and using the triangle inequality, we get that

$$\|Tx_n - x_n\| \leq \|Tx_n - Tw_n\| + \|Tw_n - w_n\| + \|w_n - x_n\|.$$

By Lemma 2.4, we have

$$\|Tx_n - x_n\| \leq \frac{1+k}{1-k}\|x_n - w_n\| + \|Tw_n - w_n\| + \|w_n - x_n\|. \tag{4.15}$$

From (4.8), (4.14) and let $n \to \infty$ in (4.15). Thus we have

$$\|Tx_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.16}$$

From (4.16) and Lemma 2.4 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is $\omega_w(x_n) \subset Fix(T)$. This fact, the inequality (4.5) and Lemma 2.6 ensure strong convergence of $\{x_n\}$ to $P_{Fix(T)}x_0$. This completes the proof. □

Corollary 4.2 ([21, Theorem 4.1]). *Let $T : H \to H$ be a k-strict pseudo-contractive mapping for some $0 \leq k < 1$ with $Fix(T) \neq \emptyset$. Assume that the control sequence $\{\alpha_n\} \subset [0, 1)$. Set $x_0 \in H$ chosen arbitrarily. Define a sequence $\{x_n\}$ by the following:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0, \ n \geq 1. \end{cases}$$

Then the iterative sequence $\{x_n\}$ converges strongly to $P_{Fix(T)}x_0$.

Proof. If $\theta_n = 0$ and $\beta_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 4.1 then $w_n = x_n$ and $\gamma_n = (1-\alpha_n)$, respectively. Therefore, we obtain the result of Marino and Xu [21, Theorem 4.1]. □

5. Numerical Experiments

In this section, two examples of k-strict pseudo-contractive mappings are introduced and studied. In order to illustrate the development of the theory that plays an important role in numerical results in this research, we have compared the numerical results between our algorithms for k-strict pseudo-contractive mappings and the previous existing results.

Firstly, some examples for k-strict pseudo-contractive mappings are provided on real Euclidean space $\mathbb{R}$ and Euclidean space $\mathbb{R}^2$, respectively.

Example 5.1. Let $C = H = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = -\left(\tan^{-1}(x) + \frac{\sin(x) + \cos(x)}{4}\right)$$

for all $x \in \mathbb{R}$. Then, T is a k-strict pseudo-contractive mapping with $k = \frac{5}{9}$ which is not nonexpansive.

Example 5.2. Let $C = H = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2\tan^{-1}(x) \\ 2\cot^{-1}(y) \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Then, T is a k-strict pseudo-contractive mapping with $k = \frac{3}{4}$ which is not nonexpansive.

Because Example 5.1 and example 5.2 are very basic and are not difficult to prove and calculate, the details of proof and calculations are therefore not provided.

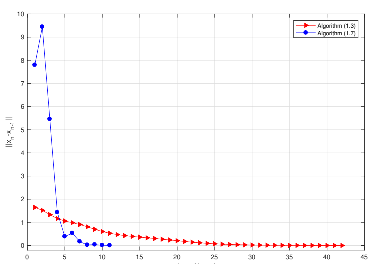
We next to show that a numerical example for supporting Theorem 3.1 which compare Mann's algorithm, that is, Algorithm (1.3) and a modified inertial Mann's algorithm, that is, Algorithm (1.6) with a k-strict pseudo-contractive mapping T defined in Example 5.1 and Example 5.2. First of all, in the Table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively. For Example 5.1, we set $\alpha_n = 0.9$ in Algorithm (1.3) and set different initial values $x_0 \neq x_1, \lambda = 0.9, \mu = 0.8, \theta_1 = 0, \theta_n = 1/(n+1)^2, \gamma_n = 0.9, \beta_n = 1/(n+1)$ in Algorithm (1.6). For Example 5.2, we set $\alpha_n = 0.9$ in Algorithm (1.3) and set different initial values $x_0 = x_1, \lambda = 0.5, \mu = 0.9, \theta_1 = 0, \theta_n = 1/(n+1), \gamma_n = 0.8, \beta_n = 1/(n+1)^2$ in Algorithm (1.6).

TABLE 1. Comparison of Algorithm (1.3) and Algorithm (1.6) for Example 5.1.

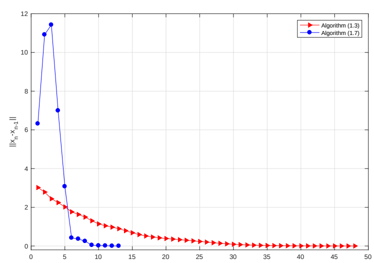
inertial		Algorithm (1.3)		Algorithm (1.6)	
x_0	x_1	Iter.	Sec.	Iter.	Sec.
15	23	43	0.004780	12	0.007699
29	-26	49	0.002477	14	0.006077
-17	22	43	0.002214	13	0.004657
-47	-35	52	0.002403	15	0.004109

TABLE 2. Comparison of Algorithm (1.3) and Algorithm (1.6) for Example 5.2.

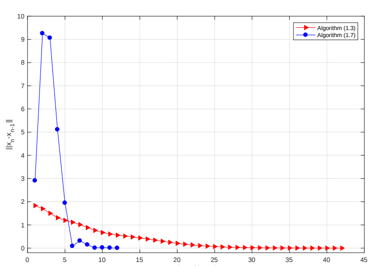
	Algorithm (1.3)		Algorithm (1.6)	
$x_0 = x_1$	Iter.	Sec.	Iter.	Sec.
(12,3)	35	0.003009	13	0.006721
(20,-30)	57	0.003302	16	0.007122
(-15,15)	50	0.003240	14	0.008172



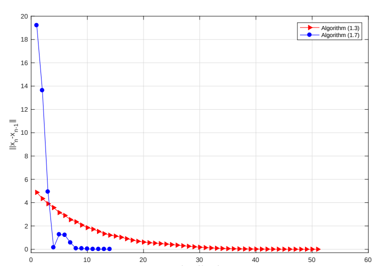
(A) Case: $x_0 = 15$ and $x_1 = 23$.



(B) Case: $x_0 = 29$ and $x_1 = -26$.

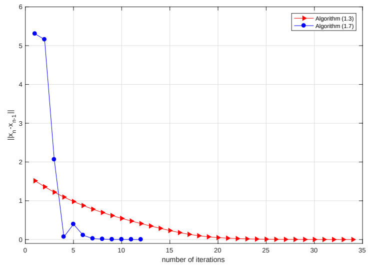


(C) Case: $x_0 = -17$ and $x_1 = 22$.

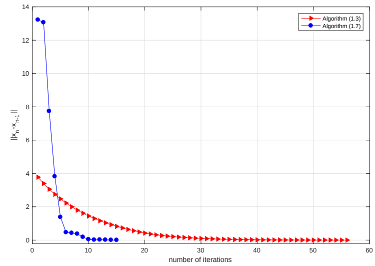


(D) Case: $x_0 = -47$ and $x_1 = -35$.

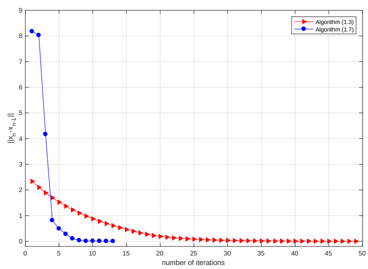
FIGURE 1. The results computed by Algorithm (1.3) and Algorithm (1.6) for Example 5.1.



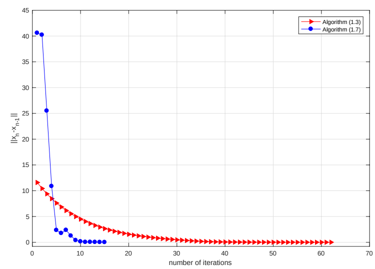
(A) Case: $x_0 = x_1 = (12, 3)$.



(B) Case: $x_0 = x_1 = (20, -30)$.



(C) Case: $x_0 = x_1 = (-15, 15)$.



(D) Case: $x_0 = x_1 = (-99, -55)$.

FIGURE 2. The results computed by Algorithm (1.3) and Algorithm (1.6) for Example 5.2.

We next design a new algorithm which equivalent to Algorithm (4.1), by using the concept of Dong and Lu [14], we obtain the specific expression of $P_{C_n \cap Q_n} x_0$ of Algorithm (4.1) as following:

$$\begin{cases} x_0, x_1 \in H \text{ chosen arbitrarily}, \\ w_n = x_n + \theta_n (x_n - x_{n-1}), \\ y_n = \alpha_n x_n + \beta_n w_n + \gamma_n T w_n, \\ u_n = w_n - y_n - \alpha_n \theta_n (x_n - x_{n-1}), \\ v_n = \dfrac{1}{2}\Big(\|x_n\|^2 + 2(1-\alpha_n)\theta_n \langle x_n, x_n - x_{n-1}\rangle + {\theta_n}^2 (1 - \alpha_n - \alpha_n \beta_n)\|x_n - x_{n-1}\|^2 \\ \qquad\qquad + \gamma_n (k - \beta_n)\|w_n - T w_n\|^2 - \alpha_n \gamma_n \|x_n - T w_n\|^2 - \|y_n\|^2 \Big), \\ C_n = \{z \in H : \langle u_n, z\rangle \le v_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = p_n, \ \text{if } p_n \in Q_n, \\ x_{n+1} = q_n, \ \text{if } p_n \notin Q_n. \end{cases}$$

where

$$p_n = x_0 - \frac{\langle u_n, x_0\rangle - v_n}{\|u_n\|^2} u_n, \qquad z_n = x_n - \frac{\langle u_n, x_n\rangle - v_n}{\|u_n\|^2} u_n,$$

$$q_n = \left(1 - \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, z_n - p_n\rangle}\right) p_n + \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, z_n - p_n\rangle} z_n,$$

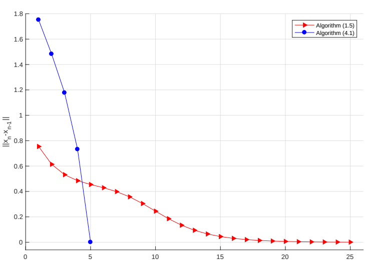
Now, we use this iteration algorithm to create a numerical example for supporting Theorem 4.1 and provide a comparison among Algorithm (1.5) and Algorithm (4.1) with a k-strict pseudo-contractive mapping T defined in Example 5.1 and Example 5.2. For Example 5.1, we set $\alpha_n = 0.6$ in Algorithm (1.5) and set different initial values $x_0 \neq x_1, \theta_n = 0.4, \alpha_n = 0.3, \beta_n = 0.3$ and $\gamma_n = 0.4$ in Algorithm (4.1). For Example 5.2, we set different initial values $x_0 = x_1$ and set the parameters correspond to the Example 5.2. Denote $E(x) = \|x_n - x_{n-1}\|$. We take $E(x) < \varepsilon$ as the stopping criterion and $\varepsilon = 10^{-4}$.

TABLE 3. Comparison of Algorithm (1.5) and Algorithm (4.1) for Example 5.1.

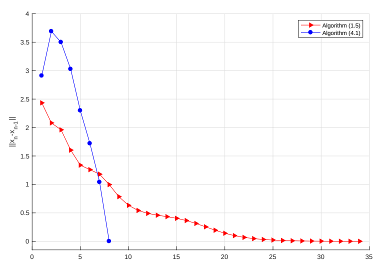
inertial		Algorithm (1.5)		Algorithm (4.1)	
x_0	x_1	Iter.	Sec.	Iter.	Sec.
9	7	17	0.022053	5	0.011346
19	18	20	0.020251	6	0.007112
-9	-7	16	0.015129	5	0.012575
-45	-41	23	0.020495	6	0.006705

TABLE 4. Comparison of Algorithms (1.5) and Algorithm (4.1) for Example 5.2.

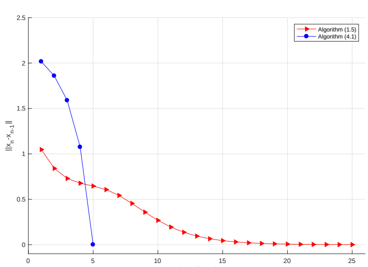
	Algorithm (1.5)		Algorithm (4.1)	
$x_0 = x_1$	Iter.	Sec.	Iter.	Sec.
(5,9)	712	0.054894	15	0.009313
(11,-15)	1566	0.101804	21	0.009859
(-18,35)	1778	0.116099	16	0.009950
(-9,-8)	699	0.060291	19	0.012281



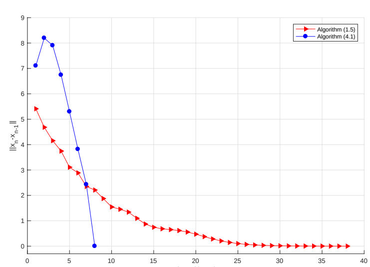
(A) Case: $x_0 = 9, x_1 = 7$.



(B) Case: $x_0 = 19, x_1 = 18$.

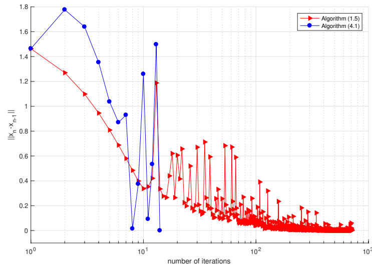


(C) Case: $x_0 = -9, x_1 = -7$.

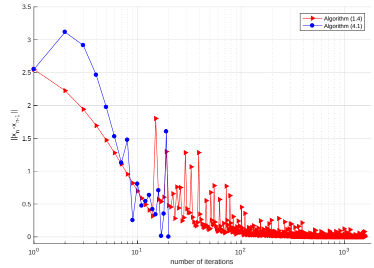


(D) Case: $x_0 = -45, x_1 = -41$.

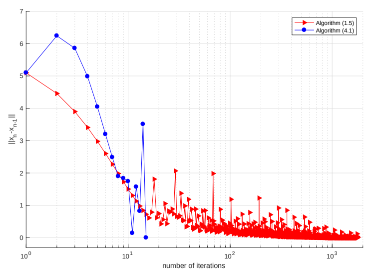
FIGURE 3. The results computed by Algorithm (1.5) and Algorithm (4.1) for Example 5.1.



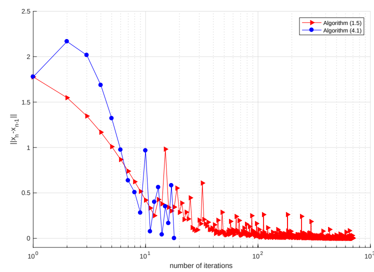
(A) Case: $x_0 = x_1 = (5, 9)$.



(B) Case: $x_0 = x_1 = (11, -15)$.



(C) Case: $x_0 = x_1 = (-18, 35)$.



(D) Case: $x_0 = x_1 = (-9, -8)$.

FIGURE 4. The results computed by Algorithm (1.5) and Algorithm (4.1) for Example 5.2.

6. CONCLUSIONS

In this paper, we proposed weak and strong convergence theorems for fixed points of k-strict pseudo-contractive mappings by using a modified inertial Mann's algorithm (1.6) and an inertial hybrid algorithm (4.1), respectively.

We also provided some new examples as appeared in Example 5.1 and Example 5.2 for k-strict pseudo-contractive mappings on the real space $\mathbb{R}$ and the Euclidean space $\mathbb{R}^2$, respectively.

The results obtained from numerical experiments using these samples showed that the modified inertial Mann's algorithm (1.6) and the inertial hybrid algorithm (4.1) represent more numerical advantages than the Mann's algorithm (1.3) and the hybrid algorithm (1.5) as shown in Table 1, Table 2, Table 3, Table 4, Figure 1, Figure 2 Figure 3 and Figure 4, respectively.

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[1]DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND
Email address: s.baiya20@hotmail.com
Email address: kasamsuku@nu.ac.th

[2]RESEARCH CENTER FOR ACADEMIC EXCELLENCE IN MATHEMATICS,
FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND
Email address: kasamsuku@nu.ac.th