

In memoriam Prof. Charles E. Chidume (1947-2021)

The affine Orlicz log-Minkowki inequality

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ABSTRACT. In this paper, we establish an affine Orlicz log-Minkowki inequality for the affine quermassintegrals by introducing new concepts of affine measures and Orlicz mixed affine measures, and using the newly established Orlicz affine Minkowski inequality for the affine quermassintegrals. The affine Orlicz log-Minkowski inequality in special case yields L_p -affine log-Minkowski inequality. The affine log-Minkowski inequality is also derived .

1. INTRODUCTION

In 2016, Stancu [16] established the following logarithmic Minkowski inequality.

The log-Minkowski inequality *If K and L are convex bodies in \mathbb{R}^n containing the origin in their interior, then*

$$(1.1) \quad \int_{S^{n-1}} \ln \left(\frac{h(K, u)}{h(L, u)} \right) d\bar{v}_1 \geq \frac{1}{n} \ln \left(\frac{V(K)}{V(L)} \right),$$

with equality if and only if K and L are homothetic, where dv_1 is the mixed volume measure $dv_1 = \frac{1}{n} h(K, u) dS(L, u)$, and $d\bar{v}_1 = \frac{1}{V_1(L, K)} dv_1$ is its normalization, and $V_1(L, K)$ denotes the usual mixed volume of L and K , is defined by (see [2])

$$V_1(L, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(L, u),$$

where $S(L, u)$ is the affine surface area measure of convex body L , and the support function $h(K, x)$ of K is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$.

Lutwak defined the affine quermassintegrals for a convex body K , $\Phi_0(K)$, $\Phi_1(K)$, \dots , $\Phi_n(K)$, by taking $\Phi_0(K) := V(K)$, $\Phi_n(K) := \omega_n$ and for $0 < j < n$ (also see [11]),

$$(1.2) \quad \Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\text{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n},$$

where $G_{n,j}$ denotes the Grassman manifold of j -dimensional subspaces in \mathbb{R}^n , and μ_j denotes the probability Haar measure on $G_{n,j}$, and $\text{vol}_j(K|\xi)$ denotes the j -dimensional volume of the orthogonal projection of K on j -dimensional subspace $\xi \subset \mathbb{R}^n$ and ω_j denotes the volume of j -dimensional unit ball.

Recently, the logarithmic Minkowski inequality and its dual form have attracted extensive attention and research, and the recent research can be found in the references [1], [3],

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[4], [6], [7], [8], [12], [13], [14], [17], [18], [19], [20], [21], [22] and [23]. In the paper, we generalize the usual volumes and log-Minkowski inequality (1.1) to the affine quermassintegrals and Orlicz space, respectively. The following affine Orlicz log-Minkowski inequality is established by introducing the concepts of affine and Orlicz affine measures and using the newly established Orlicz Minkowski inequality for affine quermassintegrals. The new affine Orlicz log-Minkowski inequality which in special case yields the following L_p -affine log-Minkowski inequality. The affine log-Minkowski inequality is also derived.

Theorem 1.1. (The affine Orlicz log-Minkowski inequality) *If K and L are convex bodies in \mathbb{R}^n containing the origin in their interior, $0 < j \leq n$ and $\varphi : [0, \infty) \rightarrow (0, \infty)$ is a convex and increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then*

$$(1.3) \quad \int_{G_{n,j}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{\varphi, n-j}^{-n}(L, K) \geq \ln \left(\varphi \left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right)^{1/j} \right) \right).$$

If φ is strictly convex, equality holds if and only if L and K are homothetic. Here $V_\varphi^{(j)}(L|\xi, K|\xi)$ denotes the Orlicz mixed volume of j -dimensional $L|\xi$ and $K|\xi$ in j -dimensional subspace ξ , $d\Phi_{\varphi, n-j}(L, K)$ denotes a new Orlicz affine probability measure of L and K , is defined by (see Section 3)

$$(1.4) \quad d\Phi_{\varphi, n-j}^{-n}(L, K) = \frac{1}{\Phi_{\varphi, n-j}^{-n}(L, K)} d\varphi_{\varphi, n-j}^{-n}(L, K),$$

where $d\varphi_{\varphi, n-j}(L, K)$ denotes the Orlicz affine measure, is defined by

$$(1.5) \quad d\varphi_{\varphi, n-j}^{-n}(L, K) = \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\omega_n \text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi),$$

and $\Phi_{\varphi, n-j}(L, K)$ is the Orlicz affine quermassintegral of L and K , is defined by (see [24])

$$(1.6) \quad \Phi_{\varphi, n-j}(L, K) = \omega_n \left[\int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}.$$

A special case of (1.3) is the following L_p -affine log-Minkowski inequality for the affine quermassintegrals.

The L_p -affine log-Minkowski inequality *If K and L are convex bodies in \mathbb{R}^n containing the origin in their interior, $0 < j \leq n$ and $p \geq 1$, then*

$$(1.7) \quad \int_{G_{n,j}} \ln \left(\frac{V_p^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\Phi_{p, n-j}^{-n}(L, K) \geq \frac{p}{j} \ln \left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right),$$

with equality if and only if L and K are homothetic. Here $V_p^{(j)}(L|\xi, K|\xi)$ denotes the L_p -dual mixed volume of j -dimensional convex bodies $L|\xi$ and $K|\xi$ in j -dimensional subspace ξ , $d\Phi_{p, n-j}(L, K)$ denotes a new L_p -affine probability measure of convex bodies L and K , is defined by

$$(1.8) \quad d\Phi_{p, n-j}^{-n}(L, K) = \frac{1}{\Phi_{p, n-j}^{-n}(L, K)} d\varphi_{p, n-j}^{-n}(L, K),$$

where $d\varphi_{p, n-j}(L, K)$ denotes the L_p -affine measure, is defined by

$$(1.9) \quad d\varphi_{p, n-j}^{-n}(L, K) = \frac{V_p^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\omega_n \text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi),$$

and $\Phi_{p,n-j}(L, K)$ is the L_p -affine quermassintegral of L and K , is defined by

$$(1.10) \quad \Phi_{p,n-j}(L, K) = \omega_n \left[\int_{G_{n,j}} \frac{V_p^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n},$$

which is as in (1.6).

The L_p log-Minkowski inequality has been established in [22]. Obviously, the following affine log-Minkowski inequality for the affine quermassintegrals can be derived from (1.3).

The affine log-Minkowski inequality *If K and L are convex bodies in \mathbb{R}^n containing the origin in their interior and $0 < j \leq n$, then*

$$(1.11) \quad \int_{G_{n,j}} \ln \left(\frac{V_1^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\Phi_{n-j}^{-n}(L, K) \geq \frac{1}{j} \ln \left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right),$$

with equality if and only if L and K are homothetic. Here $V_1^{(j)}(L|\xi, K|\xi)$ denotes the mixed volume of j -dimensional convex bodies $L|\xi$ and $K|\xi$ in j -dimensional subspace ξ , $d\Phi_{n-j}(L, K)$ denotes a new affine probability measure of convex bodies L and K , is defined by

$$(1.12) \quad d\Phi_{n-j}^{-n}(L, K) = \frac{1}{\Phi_{n-j}^{-n}(L, K)} d\varphi_{n-j}^{-n}(L, K),$$

where $d\varphi_{n-j}(L, K)$ denotes the affine measure, is defined by

$$(1.13) \quad d\varphi_{n-j}^{-n}(L, K) = \frac{V_1^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\omega_n \text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi),$$

and $\Phi_{n-j}(L, K)$ is the mixed affine quermassintegral of L and K , is defined by

$$(1.14) \quad \Phi_{n-j}(L, K) = \omega_n \left[\int_{G_{n,j}} \frac{V_1^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}.$$

Obviously, when $L = K$, $\Phi_{n-j}(L, K)$ becomes the well-known affine quermassintegral $\Phi_{n-j}(K)$.

It's worth mentioning that here we are establishing the affine Orlicz log-Minkowski inequality. In fact, the Orlicz log-Minkowski inequality, which is a special case of (1.3), has been established in [22] as follows:

The Orlicz log-Minkowski inequality *If K and L are convex bodies in \mathbb{R}^n that containing the origin in their interior, and $\varphi : [0, \infty) \rightarrow (0, \infty)$ is a convex and increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then*

$$\int_{S^{n-1}} \ln \left(\varphi \left(\frac{h(K, u)}{h(L, u)} \right) \right) d\bar{v}_\varphi \geq \ln \left(\varphi \left(\left(\frac{V_\varphi(K)}{V_\varphi(L)} \right)^{1/n} \right) \right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic, where dv_φ is the Orlicz mixed volume measure $dv_\varphi = \frac{1}{n} \varphi \left(\frac{h(K, u)}{h(L, u)} \right) h(L, u) dS(L, \cdot)$, and $d\bar{v}_\varphi = \frac{1}{V_\varphi(L, K)} dv_\varphi$ is its normalization, and $V_\varphi(L, K)$ denotes the usual Orlicz mixed volume of L and K , defined by (see [5])

$$V_\varphi(L, K) = \frac{1}{n} \int_{S^{n-1}} \varphi \left(\frac{h(K, u)}{h(L, u)} \right) h(L, u) dS(L, u).$$

where $S(L, \cdot)$ is the mixed surface area measure of L .

2. NOTATIONS AND PRELIMINARIES

The setting for this paper is n -dimensional Euclidean space \mathbb{R}^n . A body in \mathbb{R}^n is a compact set equal to the closure of its interior. A set K is called a convex body, if it is compact and convex subsets with non-empty interiors. Let \mathcal{K}^n denote the class of convex bodies in \mathbb{R}^n . Let \mathcal{K}_o^n denote the class of convex bodies containing the origin in their interiors in \mathbb{R}^n . We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1} . For a compact set K , we write $V(K)$ for the (n -dimensional) Lebesgue measure of K and call this the volume of K . Let d denote the Hausdorff metric on \mathcal{K}^n , i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If ξ is a subspace of \mathbb{R}^n , then it is easy to show that

$$h(K | \xi, x) = h(K, x | \xi),$$

for $x \in \mathbb{R}^n$. Let $\varphi : [0, \infty) \rightarrow (0, \infty)$ be a convex and increasing function such that $\varphi(1) = 1$ and $\varphi(0) = 0$. Let Φ denote the set of convex functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ that is increasing and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

2.1 Mixed volumes

If $K_i \in \mathcal{K}^n$ ($i = 1, 2, \dots, r$) and λ_i ($i = 1, 2, \dots, r$) are nonnegative real numbers, then the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [9])

$$(2.15) \quad V(\lambda_1 K_1 + \dots + \lambda_n K_n) = \sum_{i_1, \dots, i_n} \lambda_{i_1} \dots \lambda_{i_n} V_{i_1 \dots i_n},$$

where the sum is taken over all n -tuples (i_1, \dots, i_n) of positive integers not exceeding r . The coefficient $V_{i_1 \dots i_n}$ depends only on the bodies K_{i_1}, \dots, K_{i_n} and is uniquely determined by (2.15). It is called the mixed volume of K_{i_1}, \dots, K_{i_n} , and is written as $V(K_{i_1}, \dots, K_{i_n})$. Let $K_1 = \dots = K_{n-i} = K$ and $K_{n-i+1} = \dots = K_n = L$. Then the mixed volume $V(K_1, \dots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \dots = K_{n-i} = K$, $K_{n-i+1} = \dots = K_n = B$. The mixed volume $V_i(K, B)$ is written as $W_i(K)$ and called as quermassintegrals (or i th mixed quermassintegrals) of K . We write $W_i(K, L)$ for the mixed volume $V(K, \dots, K, \underbrace{B, \dots, B}_i, L)$ and call as mixed quermassintegrals, and

$$W_i(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u).$$

Associated with $K_1, \dots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \dots, K_{n-1}, \cdot)$ on S^{n-1} , called the mixed surface area measure of K_1, \dots, K_{n-1} , which has the property that for each $K \in \mathcal{K}^n$,

$$V(K_1, \dots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \dots, K_{n-1}, u).$$

Let $K_1 = \dots = K_{n-i-1} = K$ and $K_{n-i} = \dots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \dots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$. When $L = B$, $S(K[n-i], L[i], \cdot)$ is written as $S_i(K, \cdot)$ and called as i th mixed surface area measure.

2.2 L_p -mixed volumes

Mixed quermassintegrals are the first variation of the ordinary quermassintegrals with respect to Minkowski addition. The p -mixed quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \dots,$

$W_{p,n-1}(K, L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition, for $K, L \in \mathcal{K}^n$, and real $p \geq 1$, is defined by (see e.g. [10])

$$W_{p,i}(K, L) = \frac{p}{n-i} \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}.$$

The mixed p -quermassintegrals $W_{p,i}(K, L)$, for all $K, L \in \mathcal{K}^n$, has the following integral representation:

$$(2.16) \quad W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u),$$

where $S_{p,i}(K, \cdot)$ denotes the Boel measure on S^{n-1} . The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p},$$

where $S_i(K, \cdot)$ is a regular Boel measure on S^{n-1} . The measure $S_{n-1}(K, \cdot)$ is independent of the body K , and is just ordinary Lebesgue measure, S , on S^{n-1} . $S_i(B, \cdot)$ denotes the i -th surface area measure of the unit ball in \mathbb{R}^n . In fact, $S_i(B, \cdot) = S$ for all i . The surface area measure $S_0(K, \cdot)$ just is $S(K, \cdot)$. When $i = 0$, $S_{p,i}(K, \cdot)$ is written as $S_p(K, \cdot)$. Obviously, putting $i = 0$ in (2.16), the mixed p -quermassintegrals $W_{p,i}(K, L)$ become the well-known L_p -mixed volume $V_p(K, L)$, is defined by

$$(2.17) \quad V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u).$$

2.3 Orlicz mixed volumes

The Orlicz mixed volume was introduced by Gardner, Hug and Weil [5], as follows: for $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, the Orlicz mixed volume of L and K , is denoted by $V_\varphi(L, K)$, is defined by

$$(2.18) \quad V_\varphi(L, K) = \frac{1}{n} \int_{S^{n-1}} \varphi \left(\frac{h(K, u)}{h(L, u)} \right) h(L, u) dS(L, u).$$

If $\varphi(t) = t^p$ and $p \geq 1$, then the Orlicz mixed volume $V_\varphi(L, K)$ becomes the classical L_p -mixed volume $V_p(L, K)$. The Orlicz-Minkowski inequality is the following: for $K, L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, then

$$V_\varphi(L, K) \geq V(L) \cdot \varphi \left(\left(\frac{V(K)}{V(L)} \right)^{1/n} \right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

2.4 Orlicz mixed affine quermassintegrals

The Orlicz mixed affine quermassintegral of convex bodies K and L , is denoted by $\Phi_{\varphi,n-j}(K, L)$, is defined by (see [24])

$$(2.19) \quad \Phi_{\varphi,n-j}(K, L) := \omega_n \left[\int_{G_{n,j}} \frac{V_\varphi^{(j)}(K|\xi, L|\xi)}{\text{vol}_j(K|\xi)} \left(\frac{\text{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n},$$

where $\varphi \in \Phi$, $K, L \in \mathcal{K}_o^n$ and $0 < j \leq n$. Specifically, for $j = n$, it follows that

$$(2.20) \quad \Phi_{\varphi,0}(K, L) = \left(\frac{V(K)}{V_\varphi(K, L)} \right)^{1/n} V(K).$$

3. ORLICZ AFFINE LOG-MINKOWSKI INEQUALITY

In the section, in order to derive the Orlicz affine log-Minkowski inequality, we need to define some new affine measures. Lutwak [11] defined the affine quermassintegrals for a convex body K , $\Phi_{n-j}(K)$, for $0 \leq j \leq n$,

$$(3.21) \quad \Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\text{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}.$$

From (3.21), we introduce the following affine measure of star body K .

Definition 3.1. (The affine measure) For $K \in \mathcal{K}_o^n$ and $0 < j \leq n$, the affine measure of K , is denoted by $d\varphi_{n-j}^{-n}(K)$, is defined by

$$(3.22) \quad d\varphi_{n-j}^{-n}(K) = \left(\frac{\omega_n \text{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi).$$

From Definition 3.1, we find the following affine probability measure.

$$(3.23) \quad d\Phi_{n-j}^{-n}(K) = \frac{1}{d\Phi_{n-j}^{-n}(K)} d\varphi_{n-j}^{-n}(K).$$

For $\varphi \in \Phi$ and $0 < j \leq n$, Orlicz mixed affine quermassintegral of L and K , is denoted by $\Phi_{\varphi,n-j}(L, K)$, is defined by (see [24])

$$(3.24) \quad \Phi_{\varphi,n-j}(L, K) = \omega_n \left[\int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}.$$

From (3.24), we introduce the following Orlicz affine measure of convex bodies L and K .

Definition 3.2. (Orlicz mixed affine measure) For $L, K \in \mathcal{K}_o^n$, $0 < j \leq n$ and $\varphi \in \Phi$, the Orlicz mixed affine measure of L and K , is denoted by $d\varphi_{\varphi,n-j}^{-n}(L, K)$, is defined by

$$(3.25) \quad d\varphi_{\varphi,n-j}^{-n}(L, K) = \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\omega_n \text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi),$$

From Definition 3.2, Orlicz mixed affine probability measure is defined by

$$(3.26) \quad d\Phi_{\varphi,n-j}^{-n}(L, K) = \frac{1}{\Phi_{\varphi,n-j}^{-n}(L, K)} d\varphi_{\varphi,n-j}^{-n}(L, K).$$

Lemma 3.1. ([24]) (The Orlicz affine Minkowski inequality for mixed affine quermassintegrals) If $\varphi \in \Phi$, $0 < j \leq n$ and $K, L \in \mathcal{K}_o^n$, then

$$\left(\frac{\Phi_{\varphi,n-j}(K, L)}{\Phi_{n-j}(K)} \right)^{-n} \geq \varphi \left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j} \right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Theorem 3.2. (The affine Orlicz log-Minkowski inequality) If $L, K \in \mathcal{K}_o^n$, $0 < j \leq n$ and $\varphi \in \Phi$, then

$$(3.27) \quad \int_{G_{n,j}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\Phi_{\varphi,n-j}^{-n}(L, K) \geq \ln \left(\frac{\Phi_{n-j}^n(L)}{\Phi_{\varphi,n-j}^n(L, K)} \right) \geq \ln \left(\varphi \left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right)^{1/j} \right) \right).$$

If φ is strictly convex, each equality holds if and only if L and K are homothetic.

Proof. From (3.22) and (3.25), we have

$$\begin{aligned}
 & \int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L) \\
 (3.28) \quad &= \int_{G_{n,j}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{\varphi, n-j}^{-n}(L, K).
 \end{aligned}$$

Note the following equality

$$\Phi_{\varphi, n-j}^{-n}(L, K) = \int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \left(\frac{\omega_n \text{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi).$$

From Lebesgue’s dominated convergence theorem, we obtain

$$\int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L) \rightarrow \Phi_{\varphi, n-j}^{-n}(L, K),$$

as $q \rightarrow \infty$, and

$$\begin{aligned}
 & \int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L) \\
 & \rightarrow \int_{G_{n,j}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{\varphi, n-j}^{-n}(L, K),
 \end{aligned}$$

as $q \rightarrow \infty$.

Given the function $g_{L,K}(q) : [1, \infty] \rightarrow \mathbb{R}$, is defined by

$$(3.29) \quad g_{L,K}(q) = \frac{1}{\Phi_{\varphi, n-j}^{-n}(L, K)} \int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L).$$

From (3.29), we obtain

$$\begin{aligned}
 (3.30) \quad & \frac{dg_{L,K}(q)}{dq} = \frac{n}{(q+n)^2 \Phi_{\varphi, n-j}^{-n}(L, K)} \times \\
 & \times \int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L).
 \end{aligned}$$

and

$$(3.31) \quad \lim_{q \rightarrow \infty} g_{L,K}(q) = 1.$$

From (3.29), (3.30) and (3.31), we have

$$\begin{aligned}
 \lim_{q \rightarrow \infty} \ln(g_{L,K}(q))^{q+n} &= -(q+n)^2 \lim_{q \rightarrow \infty} \frac{1}{g_{L,K}(q)} \frac{dg_{L,K}(q)}{dq} \\
 &= -\frac{n}{\Phi_{\varphi,n-j}^{-n}(L,K)} \times \\
 &\quad \times \lim_{q \rightarrow \infty} \frac{\int_{G_{n,j}} \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L)}{g_{L,K}(q)} \\
 &= -\frac{n}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \times \\
 &\quad \times \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L).
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\exp \left(-\frac{n}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L) \right) \\
 &= \lim_{q \rightarrow \infty} (g_{L,K})^{q+n} \\
 (3.32) \quad &= \lim_{q \rightarrow \infty} \left(\frac{1}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L) \right)^{q+n}.
 \end{aligned}$$

On the other hand, from Hölder’s inequality

$$\begin{aligned}
 &\left(\int_{G_{n,j}} \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L) \right)^{(q+n)/q} \left(\int_{G_{n,j}} d\varphi_{n-j}^{-n}(L) \right)^{-n/q} \\
 (3.33) \quad &\leq \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} d\varphi_{n-j}^{-n}(L) = \Phi_{\varphi,n-j}^{-n}(L,K).
 \end{aligned}$$

If φ is strictly convex, from the equality of Hölder’s inequality, it follows that the equality in (3.33) holds if and only if $K|\xi$ and $L|\xi$ are homothetic. This yields that equality in (3.33) holds if and only if K and L are homothetic. Hence

$$\left(\frac{1}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L) \right)^{q+n} \leq \left(\frac{\Phi_{n-j}^{-n}(L)}{\Phi_{\varphi,n-j}^{-n}(L,K)} \right)^{-n}.$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Therefore

$$\begin{aligned}
 &\exp \left(-\frac{n}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi,K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L) \right) \\
 &\leq \left(\frac{\Phi_{n-j}^{-n}(L)}{\Phi_{\varphi,n-j}^{-n}(L,K)} \right)^n.
 \end{aligned}$$

Hence

$$\frac{1}{\Phi_{\varphi,n-j}^n(L, K)} \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\varphi_{n-j}^{-n}(L) \geq \ln \left(\frac{\Phi_{\varphi,n-j}^{-n}(L, K)}{\Phi_{n-j}^{-n}(L)} \right).$$

That is

$$(3.34) \quad \int_{G_{n,j}} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\Phi_{\varphi,n-j}^{-n}(L, K) \geq \ln \left(\frac{\Phi_{\varphi,n-j}^{-n}(L, K)}{\Phi_{n-j}^{-n}(L)} \right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. The completes proof of the first inequality in (3.27).

Furthermore, by using the Orlicz affine Minkowski inequality for mixed affine quermassintegrals in Lemma 3.1, we obtain

$$(3.35) \quad \int_{G_{n,j}} \ln \left(\frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\text{vol}_j(L|\xi)} \right) d\Phi_{\varphi,n-j}^{-n}(L, K) \geq \ln \left(\varphi \left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right)^{1/j} \right) \right).$$

If φ is strictly convex, from the equality of Orlicz affine Minkowski inequality, the equality in (3.35) holds if and only if K and L are homothetic.

This completes the proof. □

When $\varphi(t) = t$, (3.27) becomes the affine log-Minkowski inequality (1.11) stated in the introduction. When $\varphi(t) = t^p$ and $p \geq 1$, (3.27) becomes the L_p -affine log-Minkowski inequality (1.7) stated in the introduction.

Moreover, when $j = n$, some new Orlicz log-Minkowski inequalities for the Orlicz mixed volumes are derived. Here we omit the details.

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