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In memoriam Prof. Charles E. Chidume (1947-2021)

Approximating fixed points results for demicontractive mappings could be derived from their quasi-nonexpansive counterparts

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ABSTRACT. We prove that the convergence theorems for Mann iteration used for approximating the fixed points of demicontractive mappings in Hilbert spaces can be derived from the corresponding convergence theorems in the class of quasi-nonexpansive mappings. Our derivation is based on the embedding of demicontractive mappings in the class of quasi-nonexpansive mappings by means of the averaged operator $T_\lambda = (1-\lambda)I + \lambda T$: if T is k-demicontractive, then, for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive.

In this way we design a very simple and unifying technique of proof for various well known results in the iterative approximation of fixed points of demicontractive mappings.

We illustrate this embedding technique for the case of two classical convergence results in the class of demicontractive mappings: [Măruşter, Şt. The solution by iteration of nonlinear equations in Hilbert spaces. *Proc. Amer. Math. Soc.* **63** (1977), no. 1, 69–73] and [Hicks, T. L.; Kubicek, J. D. On the Mann iteration process in a Hilbert space. *J. Math. Anal. Appl.* **59** (1977), no. 3, 498–504].

1. INTRODUCTION

Let H be a real Hilbert space with norm and inner product denoted as usually by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $C \subset H$ be a closed and convex set and $T : C \to C$ be a self mapping. Denote by

$$Fix\,(T) = \{x \in C : Tx = x\}$$

the set of fixed points of T. The mapping T is called:

1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C. \tag{1.1}$$

2) *quasi-nonexpansive* if $Fix\,(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C \text{ and } y \in Fix\,(T). \tag{1.2}$$

3) *demicontractive* ([48], [56]) if $Fix\,(T) \neq \emptyset$ and there exists a positive number $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2, \tag{1.3}$$

for all $x \in C$ and $y \in Fix\,(T)$. We also say that T is k-demicontractive.

4) *k-strictly pseudocontractive* of the Browder-Petryshyn type ([24]) if there exists $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2, \forall x, y \in C. \tag{1.4}$$

5) *hemicontractive* if (1.3) holds with $k = 1$.

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6) *pseudocontractive* if (1.4) holds with $k = 1$.

It is easily seen that any nonexpansive mapping with $Fix\,(T) \neq \emptyset$ is quasi-nonexpansive and that any quasi-nonexpansive mapping is demicontractive, but the reverses may not be true, as illustrated by the next example.

Example 1.1. *Let H be the real line with the usual norm and $C = [0,1]$. Define T on C by $Tx = \frac{7}{8}$, if $0 \leq x < 1$ and $T1 = \frac{1}{4}$. Then, $Fix\,(T) = \left\{\frac{7}{8}\right\}$, T is demicontractive but T is neither quasi-nonexpansive nor nonexpansive.*

Indeed, with $y = \frac{7}{8}$ and any $x \in [0,1)$, inequality (1.3) *becomes:*

$$|Tx - y|^2 = 0 \leq |x - y|^2 + k|x - Tx|^2,$$

and obviously holds, for any $k > 0$.

It remains to check (1.3) *for the case $x = 1$, which yields*

$$\left|\frac{1}{4} - \frac{7}{8}\right|^2 \leq \left|1 - \frac{7}{8}\right|^2 + k\left|1 - \frac{1}{4}\right|^2,$$

and which holds true for any $k \geq \frac{2}{3}$.

Hence T is $\frac{2}{3}$-demicontractive. To show that T is not quasi-nonexpansive, take $x = 1$ and $y = \frac{7}{8}$ in (1.2)*, to get $\frac{5}{8} \leq \frac{1}{8}$, a contradiction. Hence T is not quasi-nonexpansive.*

T is also not nonexpansive: take the same values $x = 1$ and $y = \frac{7}{8}$ in (1.1) *to rich the above contradiction.*

Remark 1.1. *The function T in Example 1.1 can be found in* [3]*, Example 2.1, to illustrate (α, β)-nonexpansive mappings introduced and studied there. This fact indicates that the class of demicontractive mappings and that of (α, β)-nonexpansive mappings have nonempty intersection.*

Condition (1.2) of quasi-nonexpansiveness was introduced in 1916 by Tricomi [72] for real functions, and was relaunched fifty years later by Diaz and Metcalf [40], [41], and then studied by Dotson [42], [43], Senter and Dotson [64] and many others, for mappings in Hilbert and Banach spaces.

On the other hand, any nonexpansive mapping is k-strictly pseudocontractive of the Browder-Petryshyn type and hence pseudocontractive, but the reverses are not generally valid.

Moreover, if we take $y \in Fix\,(T)$ in (1.4), we see that any k-strictly pseudocontractive mapping of the Browder-Petryshyn type is k-demicontractive, but the reverse is not more true either.

Example 1.2. *Let T be defined as in Example 1.1. We know that T is demicontractive and would like to prove that it is not strictly pseudocontractive. Assume the contrary, that is, there exists $k < 1$ such that* (1.4) *holds for any $x, y \in [0,1]$. By taking $x \in [0,1)$ and $y = 1$ in* (1.4) *we get*

$$\left(\frac{5}{8}\right)^2 \leq (x-1)^2 + k\left(x - 1 - \frac{5}{8}\right)^2, \; x \in [0,1),$$

from which, by letting $x \to 1$ we obtain $k \geq 1$, a contradiction.

Hence T is not strictly pseudocontractive of the Browder-Petryshyn type but it is pseudocontractive, as (1.4) *holds with $k = 1$.*

The next example provides a demicontractive mapping (which is also quasi-nonexpansive) but which is neither nonexpansive nor pseudocontractive.

Example 1.3 ([34]). *Let H be the real line and $C = [0, 1]$. Define T on C by $Tx = \dfrac{2}{3}x \sin \dfrac{1}{x}$ if $x \neq 0$ and $T0 = 0$. Then, $Fix\,(T) = \{0\}$, T is demicontractive (and also quasi-nonexpansive) but T is not nonexpansive and is not pseudocontractive. Indeed, for $x \in C$ and $y = 0$,*

$$|Tx - 0|^2 = |Tx|^2 = \left|\frac{2}{3}x \sin(1/x)\right|^2 \leq \left|\frac{2}{3}x\right|^2 \leq |x - 0|^2 \leq |x - 0|^2 + k|Tx - x|^2,$$

for any $k < 1$. Hence (1.3) *is satisfied. To see that T is not nonexpansive, just take $x = \dfrac{2}{\pi}$ and $y = \dfrac{2}{3\pi}$ to get*

$$|Tx - Ty| = \frac{16}{9\pi} > \frac{4}{3\pi} = |x - y|.$$

With the same values of x and y, we have

$$|x - y|^2 + |x - Tx - y + Ty|^2 = \frac{160}{81\pi^2} < \frac{256}{81\pi^2} = |Tx - Ty|^2,$$

which shows that T is not pseudocontractive.

As demicontractive mappings constitute one of the most general classes of nonexpansive type mappings with important applications for which the fixed points can be obtained by iterative schemes, there was and still is a great interest in studying their properties, see [2], [4], [27]-[38], [44]-[61], [65]-[79] and most of the references therein.

In the current paper we are focusing only on some early developments related to iterative approximation of fixed points of demicontractive mappings and quasi-nonexpansive mappings in order to illustrate our imbedding technique.

2. Demicontractive mappings

Demicontractive mappings form one of the most general classes of nonexpansive type mappings for which the fixed points could be approximated by means of iterative schemes.

They were introduced independently by Măruşter [55], [56] and Hicks and Kubicek [48]. Hicks and Kubicek introduced the notion of demicontractive mappings given by (1.3) and also labeled them with the current name in 1977, while Măruşter introduced the demicontractive mappings in a different way, with a different name, first in the setting of $\mathbb{R}^n$ in 1973 [55] and then in 1977 [56] in the case of a real Hilbert space.

We start by presenting first Măruşter's definition as it appeared earlier than the one given by Hicks and Kubicek.

Definition 2.1 (Măruşter [56]). *Let H be a real Hilbert space and C a closed convex subset of H. A mapping $T : C \to C$ such that $Fix\,(T) \neq \emptyset$ is said to satisfy condition (A) if there exists $\lambda > 0$ such that*

$$\langle x - Tx, x - x^* \rangle \geq \lambda \|Tx - x\|^2, \forall x \in C, x^* \in Fix\,(T). \tag{2.5}$$

The fact that (1.3) and (2.5) define the same class of mappings in the setting of a Hilbert space was observed more than two decades later, by Moore [58], a student of Professor Charles E. Chidume (who was at ICTP Trieste at that time, see [10]).

The proof of this fact is based on the next identity, obtained routinely from the properties that link the inner product and the norm in a Hilbert space:

$$\|x - x^*\|^2 + k\|x - Tx\|^2 - \|Tx - x^*\|^2 = 2\langle x - x^*, x - Tx\rangle - (1 - k)\|x - Tx\|^2,$$

which shows that (2.5):

$$\|x-x^*\|^2+k\|Tx-x\|^2-\|Tx-x^*\|^2\geq 0, \forall x\in C, x^*\in Fix\,(T),$$

and (1.3)

$$\langle x-Tx, x-x^*\rangle \geq \lambda\|Tx-x\|^2, \forall x\in C, x^*\in Fix\,(T)$$

are indeed equivalent, with $\lambda=\dfrac{1-k}{2}>0$.

In order to state Măruşter's demicontractive mapping principle ([56] and Hicks and Kubicek demicontractive mapping principle ([48]), we need the following notion apparently due to Browder [25].

Let C be a nonempty closed convex subset of a real Hilbert space H and $T:C\to C$ mapping. Then T is said to be *demiclosed at* 0 on C if, for a sequence $\{x_n\}$ in C converging weakly to $u\in C$ and such that $\|Tx_n\|\to 0$, as $n\to\infty$, one has $Tu=0$.

We can now state the first two results from literature related to the approximation of fixed point points of demicontractive mappings, i.e., Hicks and Kubicek demicontractive mapping principle ([48]) and Măruşter's demicontractive mapping principle ([56], respectively.

Theorem 2.1 (Hicks and Kubicek [48]). *Suppose C is a closed convex subset of a Hilbert space H. Suppose $T:C\to C$ such that:*

(a) $Fix\,(T)\neq\emptyset$.

(b) T is demicontractive with contraction coefficient k.

(c) If any sequence $\{x_n\}$ converges weakly to x and $(I-T)x_n$ converges strongly to 0 then $(I-T)(x)=0$.

Then for $v_1\in C$ and $d_n\to d$, $0<d<1-k$, the iteration process defined by

$$v_{n+1}=(1-d_n)v_n+d_nTv_n, \tag{2.6}$$

converges weakly to a fixed point of T.

Theorem 2.2 (Măruşter [56]). *Let $T:C\to C$ be a nonlinear mapping, where C is a closed convex subset of a Hilbert space H. Suppose that T satisfies condition (A), $I-T$ is demiclosed at 0 and the sequence $\{x_k\}$ generated by*

$$x_{k+1}=(1-t_k)x_k+t_kTx_k\ (x_0\in C) \tag{2.7}$$

with $0<a\leq t_k\leq b<2\lambda$ belongs to C.

Then $\{x_k\}$ converges weakly to an element of $Fix\,(T)$.

Remark 2.2. *1. Despite the fact they were discovered independently, the statements of Hicks and Kubicek demicontractive principle and Măruşter mapping principle are very similar.*

2. However, note that Theorem 2.2 is slightly more general than Theorem 2.1, due to the stronger assumptions on the parameter sequence $\{d_n\}$ involved in the Mann iterative scheme (2.6) *in comparison to the assumptions on the parameter sequence $\{t_k\}$ involved in the Mann iterative scheme* (2.7).

3. If we look at the relationship between the two constants appearing in Theorem 2.2 and Theorem 2.1, i.e.,

$$\lambda=\frac{1-k}{2},$$

one can see that in fact the parameter sequences $\{d_n\}$ and (2.6) *satisfy exactly the same boundedness condition.*

It is the extraordinary merit of Professor Charles E. Chidume (1947-2021) to identify and value in the 80s the high potential of demicontractive mappings in the metric fixed point theory and extend the concepts and results from Hilbert spaces to Banach spaces.

Indeed, he was the first one who cited Măruşter's paper [56], in a paper from 1984 [28], in which he extended the notion of demicontractive mappings in the sense of Măruşter to banach spaces, while the Hicks and Kubicek paper [48] was cited by Chidume three years later, in 1987 [29]. Note also that the first paper to cite Hicks and Kubicek paper [48], according to MathScinet, is from 1984 [39].

There was a long standing interest and there is still a great interest of researchers for the study of demicontractive mappings. If we search in MathScinet for the word "demicontractive" in the "title" of publications we find 116 records (as by 20 June 2022), while if we are searching for the same word but in the "review text", we find 136 records (as by 20 June 2022).

The distribution per years of the later list for the last 8 years is also suggestive: 2022 (4 papers); 2021 (17); 2020 (25); 2019 (21); 2018 (15); 2017 (10); 2016 (4); 2015 (5); etc.

Starting from this interest for the study of demicontractive mappings, our aim in this paper is to provide a unified proof for Theorem 2.2 and Theorem 2.1 and some other general related results.

To this end, we use an idea extracted from the technique of enriching contractive type mappings, introduced by the author [9], [11], in combination with the classical techniques of proof due to Dotson [42] and Senter and Dotson [64], respectively.

3. Quasi-nonexpansive and enriched quasi-nonexpansive mappings

In a recent paper [10], the author used the technique of enriching contractive mappings to introduce the so called class of *enriched nonexpansive mappings* in Hilbert spaces, for which there were established results on the existence and approximation of fixed points. The same technique has been then extended to prove results on the existence and approximation of fixed points of enriched nonexpansive mappings in the setting of a Banach space [11].

The main idea of the enriching process is originating in a paper by Krasnoselskij [51]. To present it, we recall the concept of asymptotic regularity which was introduced formally in 1966 by Browder and Petryshyn ([24], Definition 1, page 572) in connection with the study of fixed points of nonexpansive mappings.

A mapping T of a Banach space X into itself is said to be *asymptotically regular on* X if for each x in X, $T^{n+1}x - T^n x \to 0$ strongly in X as $n \to \infty$.

Krasnoselskij [51] noted that if T is a nonexpansive mapping (which, in general, is not asymptotically regular) and

$$T_\lambda := (1-\lambda)I + \lambda T,\ \lambda \in [0,1]$$

is the averaged mapping associated to T, then for any $\lambda \in (0,1)$:

(i) T_λ is nonexpansive;
(ii) $Fix\,(T) = Fix(T_\lambda)$;
(iii) T_λ is asymptotically regular.

From the point of view of fixed point theory, (ii) and (iii) show that the averaged operator T_λ is **richer** than the original operator T, while (i) shows that they share the same set of fixed points. This is a real benefit for the iterative schemes: one can approximate the fixed point of T by an iterative scheme associated to the richer mapping T_λ.

For example, for a nonexpansive mapping T with $Fix\,(T) \neq \emptyset$, the Picard iteration associated to T, i.e., $\{x_n = T^n x\}$ does not converge in general, or even if it converges, its limit is not a fixed point of T, while the Picard iteration associated to T_λ, $\{y_n = T_\lambda^n x\}$ converges to a fixed point of T under appropriate assumptions like the ones in the seminal paper by Krasnoselskij [51].

Starting from these facts, the author [10] used T_λ instead of T in the definition of a nonexpansive mappings and thus obtained a larger class of mappings, called *enriched nonexpansive*. The definition is as follows.

Let $(X, \|\cdot\|)$ be a linear normed space. A self mapping $T : X \to X$ is said to be an *enriched nonexpansive mapping* (or *b-enriched nonexpansive mapping*) if there exists $b \in [0, +\infty)$ such that

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in X. \tag{3.8}$$

Note that condition (3.8) is equivalent to

$$\|T_\lambda x - T_\lambda y\| \leq \|x-y\|, \forall x, y \in X,$$

that is, to the nonexpansiveness of T_λ.

It is easily seen that any nonexpansive mapping T is enriched nonexpansive (it is a 0-enriched nonexpansive mapping, i.e., it satisfies (3.8) with $b = 0$).

Note also that the above inclusion is strict as there exist enriched nonexpansive mappings which are not nonexpansive, see Example 2.1 in [10].

It turned out later that, in the setting of a Hilbert space, the class of enriched nonexpansive mappings coincides with that of k-strictly pseudocontractive of the Browder-Petryshyn type [8]. This was the reason to also consider enriched nonexpansive mappings in a Banach space [11], where the two classes of mappings do not more coincide.

After a systematic study of other classes of enriched mappings, i.e., enriched contractions [15], enriched Kannan mappings [16], enriched Ćirić-Reich-Rus contractions [17], enriched Chatterjea mappings [18] etc., the concept of unsaturated and saturated class of contractive mappings emmerged naturally, see [19].

Definition 3.2. *Let $(X, \|\cdot\|)$ be a linear normed space and let $\mathcal{C}$ be a subset of the family of all self mappings of X. A mapping $T : X \to X$ is said to be $\mathcal{C}$-*enriched *or* enriched with respect to $\mathcal{C}$ *if there exists $\lambda \in (0, 1]$ such $T_\lambda \in \mathcal{C}$.*

We denote by $\mathcal{C}^e$ the set of all enriched mappings with respect to $\mathcal{C}$.

Remark 3.3. *From Definition 3.2 it immediately follows that $\mathcal{C} \subseteq \mathcal{C}^e$ (since $T \equiv T_1$).*

Definition 3.3. *Let X be a linear vector space and let $\mathcal{C}$ be a subset of the family of all self mappings of X and let $\mathcal{C}^e$ be the set of all enriched mappings with respect to $\mathcal{C}$.*

If $\mathcal{C} = \mathcal{C}^e$, we say that $\mathcal{C}$ is a saturated class of mappings, *otherwise $\mathcal{C}$ is said to be* unsaturated.

Remark 3.4. *Note that $\mathcal{C}$ is unsaturated if and only if the inclusion $\mathcal{C} \subset \mathcal{C}^e$ is strict.*

Example 3.4.

The following are known ([19]*) to be unsaturated classes of contractive mappings: nonexpansive mappings, Banach contractions; Kannan mappings, Ćirić-Reich-Rus contractions, Chatterjea mappings, almost contractions etc., while k-strictly pseudocontractive of the Browder-Petryshyn type and demicontractive mappings are both saturated classes of contractive mappings.*

The reason behind this situation is that, in a Hilbert space, the class of k-strictly pseudocontractive mappings of the Browder-Petryshyn type coincides with the class of enriched nonexpansive mappings, while the class of demicontractive mappings coincides with that of enriched quasi-nonexpansive mappings, see [9] and [19] for more details.

The next lemma illustrates this fact for the case of demicontractive mappings.

Lemma 3.1. *Let H be a real Hilbert space, $C \subset H$ be a closed and convex set. If $T : C \to C$ is k-demicontractive, then the Krasnoselskij perturbation T_λ of T is $(1 + k/\lambda - 1/\lambda)$-demicontractive.*

Proof. By hypothesis, we have $Fix\,(T) \neq \emptyset$ and there exists $k < 1$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2,$$

for all $x \in C$ and $y \in Fix\,(T)$, which is equivalent to

$$\langle Tx - x, x - y\rangle \leq \frac{k-1}{2} \cdot \|x - Tx\|^2, x \in C,\ y \in Fix\,(T).$$

Then, for all $x \in C$ and $y \in Fix\,(T)$, we have

$$\|T_\lambda x - y\|^2 = \|\lambda(Tx - x) + x - y\|^2 = \|x - y\|^2 + 2\lambda\langle Tx - x, x - y\rangle$$
$$+\lambda^2\|Tx - x\|^2 \leq \|x - y\|^2 + (\lambda^2 + \lambda k - \lambda)\|Tx - x\|^2$$
$$= \|x - y\|^2 + \frac{\lambda^2 + \lambda k - \lambda}{\lambda^2} \cdot \|T_\lambda x - x\|^2,\ x \in C,\ y \in Fix\,(T),$$

which shows that T_λ is $(1 + k/\lambda - 1/\lambda)$-demicontractive, since for $k, \lambda \in (0, 1)$,

$$0 < 1 + \frac{k}{\lambda} - \frac{1}{\lambda} < 1.$$

□

The next lemma is of particular importance in the present paper and will be essential in the proofs of our main results.

Lemma 3.2. *Let H be a real Hilbert space, $C \subset H$ be a closed and convex set. If $T : C \to C$ is k-demicontractive, then for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive.*

Proof. From the proof of Lemma 3.1, we have that, for all $x \in C$ and $y \in Fix\,(T)$

$$\|T_\lambda x - y\|^2 \leq \|x - y\|^2 + \frac{\lambda^2 + \lambda k - \lambda}{\lambda^2} \cdot \|T_\lambda x - x\|^2,\ x \in C,\ y \in Fix\,(T).$$

Hence, if $\lambda^2 + \lambda k - \lambda < 0$, that is, $\lambda < 1 - k$, then the above inequality implies

$$\|T_\lambda x - y\|^2 \leq \|x - y\|^2,\ x \in C,\ y \in Fix\,(T),$$

i.e., T_λ is quasi-nonexpansive. □

4. Main results

In order to prove our main results in this section, we need some more definitions and auxiliary results, mainly taken from [42] and [64]. Throughout the rest of this paper, H will be a Hilbert space and C a nonempty subset of H.

According to [64], a mapping $T : C \to C$ with $Fix\,(T) \neq \emptyset$ is said to satisfy *Condition I*, if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with the properties $f(0) = 0$ and $f(r) > r$, for $r > 0$, such that

$$\|x - Tx\| \geq f(d(x, Fix\,(T)), \forall x \in C, \tag{4.9}$$

where

$$d(x, Fix\,(T)) = \inf\{\|x - z\| : z \in Fix\,(T)\}$$

is the distance between the point x and the set $Fix\,(T)$.

Remark 4.5. *Note that a condition of the form* (4.9) *is usually referred as retraction-displacement condition, see the recent papers* [22], [23] *and* [63], *where the authors studied the fixed point equation $x = Tx$ in terms of a retraction-displacement condition and have given various examples, corresponding to Picard, Krasnoselskii, Mann and Halpern iterative algorithms.*

Lemma 4.3. *(Lemma 1,* [64]*) Suppose C is a closed bounded subset of a Banach space X and let $T : C \to C$ be a mapping satisfying $Fix\,(T) \neq \emptyset$. If $I - T$ maps closed bounded subsets of C onto closed subsets of X then T satisfies Condition I.*

A simpler form of Condition I, called Condition II, has been also introduced in [64] and corresponds to the particular case $f(t) = \alpha t$, with $\alpha > 0$ a real number, in Condition I. Condition II is equivalent to the retraction-displacement conditions obtained in [22] for various fixed point algorithms and in connection with some particular contractive conditions.

For example, if T is a contraction with contraction coefficient $c \in (0,1)$, then we get $f(t) = (1-c)t$ (in the case of Condition I) and $\alpha = 1 - c$ (in the case of Condition II).

As announced in Introduction, to prove our main results in the present paper we combine Lemma 3.2 with Theorem 8 [42] and Theorem 2 [64], respectively. Their statements are given in following.

Theorem 4.3 (Dotson [43]). *Suppose H is a real Hilbert space, C is a closed convex subset of H, $T : C \to C$ is quasi-nonexpansive on C and has at least one fixed point in C, and $I - T$ is demiclosed. Then for arbitrary $v_1 \in C$, the sequence*

$$v_{n+1} = (1-t_n)v_n + t_n T v_n,\ n \geq 1,$$

with $t_n \in [a,b]$ for all n and $0 < a < b < 1$, converges weakly to a fixed point of T.

Theorem 4.4 (Senter and Dotson [64]). *Suppose E is a uniformly convex Banach space, C is a closed, convex subset of E and T is a quasi-nonexpansive mapping of C into C. If T satisfies Condition I, where $Fix\,(T)$ is the fixed point set of T in C, then for arbitrary $x_1 \in C$, the sequence*

$$x_{n+1} = (1-t_n)x_n + t_n T x_n,\ n \geq 1,$$

with $t_n \in [a,b]$ for all n and $0 < a < b < 1$, converges strongly to a member of $Fix\,(T)$.

Now we can state and prove the main results of this paper: a unified weak convergence theorem (Theorem 4.5) and a unified strong convergence theorem (Theorem 4.6) for Mann iteration in the class of demicontractive mappings.

Theorem 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a k-demicontractive mapping. Suppose $I - T$ is demiclosed. Then, for $\lambda_n \in [a,b]$, with $0 < a < b < 1$, and for any given $x_0 \in C$, the Mann iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n,\ n \geq 0, \tag{4.10}$$

converges weakly to a fixed point of T.

Proof. Since T is k-demicontractive, by Lemma 3.2 it follows that the averaged mapping T_μ given by

$$T_\mu x = (1-\mu)x + \mu T x$$

is quasi-nonexpansive, for any $\mu < 1 - k$.

Now, by Theorem 4.3 it follows that, for any sequence $\{t_n\}$ satisfying

$$0 < a \leq t_n \leq b < 1,$$

the iterative process $\{x_n\}$ defined by $x_0 \in C$ arbitrary, and

$$x_{n+1} = (1-t_n)x_n + t_n T_\mu x_n,\ n \geq 0, \tag{4.11}$$

converges strongly to a fixed point of T_μ.

But $Fix\,(T) = Fix\,(T_\mu)$, for any $\mu \in (0,1)$, and

$$(1-t_n)x_n + t_n T_\mu x_n = (1-\mu t_n)x_n + \mu t_n T x_n,$$

which, by using (4.11) and denoting $\lambda_n = \mu t_n$, proves that, indeed, $\{x_n\}$ defined by (4.10) converges weakly to a fixed point of T.

Since $0 < \mu < 1 - k$ and $\{t_n\}$ satisfies $0 < a \leq t_n \leq b < 1$, it follows that $\lambda_n \in [a_1, b_1]$, with $a_1 = \mu a$ and $b_1 = (1-k)b$.

□

Remark 4.6. *Theorem 4.5 unifies both Theorem 2.1 (Hicks and Kubicek) and Theorem 2.2 (Măruşter), which are obtained as immediate corollaries.*

A more general result is given by the next unifying strong convergence theorem.

Theorem 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space X and let $T : C \to C$ be a k-demicontractive mapping. Suppose T satisfies Condition I. Then $Fix\,(T) \neq \emptyset$ and for $\lambda_n \in [a, b]$, with $0 < a < b < 1$, and for any given $x_0 \in C$, the Mann iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n,\, n \geq 0,$$

converges strongly to a fixed point of T.

Proof. We proceed similarly to the proof of Theorem 4.5 but this time the conclusion follows by Lemma 3.2 and Theorem 4.4.

□

Remark 4.7.

Although Theorem 4.4 is valid in a uniformly convex Banach space, see Senter and Dotson [64], *our Theorem 4.6 is stated in a Hilbert space only, because Lemma 3.2 was established in this setting here. By using the duality pairing instead of the inner product from Hilbert spaces, Chidume* [28] *introduced and studied the class of demicontractive mappings in Banach spaces.*

It is therefore a challenge to extend Lemma 3.2 to Banach spaces and build a similar unifying technique of proof in this setting for the class of demicontractive mappings.

5. Conclusions

1. In this paper we have shown that the convergence theorems for Mann iteration used for approximating the fixed points of demicontractive mappings in Hilbert spaces could be derived from the corresponding convergence theorems in the class of quasi-nonexpansive mappings.

2. Our derivation is based on an imbedding technique described by Lemma 3.2, which essentially shows that if T is k-demicontractive, then for any $\lambda \in (0, 1-k)$, T_λ is quasi-nonexpansive. A similar technique works for k-strict pseudocontractions, which can be embedded in the class of nonexpansive mappings in Hilbert spaces, first exploited by Browder and Petryshyn [24], [59], and also used much later by Zhou [80] in the case of nonself mappings. **Open problem:** Is this derivation valid in Banach spaces, too ?

3. In this way we obtained a unifying technique of proof for various well known results in the fixed point theory of demicontractive mappings that has been illustrated for the case of the first two classical convergence results in the class of demicontractive mappings in literature: Măruşter [56] and Hicks and Kubicek [48].

4. We also note that if T is k-demicontractive, then T_λ is $(1 + k/\lambda - 1/\lambda)$-demicontractive, which means that demicontractive mappings constitute a *saturated* class of contractive mappings, see [16] for definitions and more details. In fact, our technique of embedding demicontractive mappings in the class of quasi-nonexpansive mappings is esssentially based on the fact that, in a Hilbert space, demicontractive mappings are *enriched* quasi-nonexpansive mappings, see [8], [9], [11].

5. Similar results could be established for other iterative schemes used for approximating the fixed points of demicontractive mappings in Hilbert and Banach spaces. Some authors, see for example Marino and Xu [54], use the term "quasi k-strict pseudocontraction" to designate a demicontractive mapping.

Also, in the same context, it would be possible to extend some results established for k-strict pseudo-contractions to the more general class of k-demicontractive mappings.

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