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Invited paper

On Uniform Instability with Growth Rates in Banach Spaces

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ABSTRACT. The paper considers three concepts of uniform instability for evolution operators: uniform exponential instability, uniform polynomial instability and uniform h-instability. Some characterizations of these notions and connections between these concepts are given.

1. INTRODUCTION

The instability behaviors of evolution operators is a topic that has witnessed lately an impressive development in recent years.

Among the most studied concepts in this field are the exponential instability ([1], [5], [8], [9], [10], [12]), and the polynomial instability ([2], [3], [6], [11]).

As a generalization of the notions mentioned above, in a natural manner we focus on an another type of uniform instability. In fact, we consider the concept of uniform h-instability where $h : \mathbb{R}_+ \to [1, \infty)$ is a growth rate (i.e. h is bijective and nondecreasing).

In this paper we give characterizations for uniform h-instability. As particular cases we obtain necessary and sufficient conditions for uniform exponential instability and uniform polynomial instability. Connections between these concepts are emphasized.

We obtain generalizations of some instability with growth rates results as well as versions of stability with growth rates theorems for the case of instability.([2], [4], [5], [6], [7], [8], [9]).

2. UNIFORM INSTABILITY CONCEPTS

Throughout this paper we will consider X a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X, both with the norm $\|\cdot\|$. Let I be the identity operator on X and by Δ respectively T we denote the sets defined as follows:

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}, \qquad T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator on X (i.e. $\Phi(t, t) = I$ for every $t \geq 0$ and $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$ for all $(t, s, t_0) \in T$).

Let $h : \mathbb{R}_+ \to [1, \infty)$ be a growth rate.

Definition 2.1. The evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is called

(i) *strongly measurable* if for all $(s, x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|\Phi(t, s)x\|$ is measurable on $[s, \infty)$.

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(ii) with *uniform h-decay* (u.h.d.) if there exist $M > 1$ and $\omega > 0$ such that

$$Mh(t)^{\omega}\|\Phi(t,s)x\| \geq h(s)^{\omega}\|x\|,$$

for all $(t,s,x) \in \Delta \times X$.

(iii) *uniformly h-instable* (u.h.is.) if there exist $N > 1$ and $\nu > 0$ such that

$$h(t)^{\nu}\|x\| \leq Nh(s)^{\nu}\|\Phi(t,s)x\|,$$

for all $(t,s,x) \in \Delta \times X$.

In particular, for $h(t) = e^t$ and $h(t) = t+1$ we obtain the notions of *uniform exponential decay* (u.e.d.), *uniform exponential instability* (u.e.is.), respectively *uniform polynomial decay* (u.p.d.) and *uniform polynomial instability* (u.p.is.).

Remark 2.1. The evolution operator $\Phi : \Delta \to \mathcal{B}(X)$

(i) has uniform h-decay if and only if there are $M > 1$ and $\omega > 0$ such that

$$h(s)^{\omega}\|\Phi(s,t_0)x_0\| \leq Mh(t)^{\omega}\|\Phi(t,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

(ii) is uniformly h-instable if and only if there are $N > 1$ and $\nu > 0$ such that

$$h(t)^{\nu}\|\Phi(s,t_0)x_0\| \leq Nh(s)^{\nu}\|\Phi(t,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Remark 2.2. If the evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly h-instable, then it has uniform h-decay. The following example shows that the converse implication is not true.

Example 2.1. Let $h : \mathbb{R}_+ \to [1,\infty)$ with $\lim\limits_{t\to\infty} h(t) = \infty$. For $X = \mathbb{R}$ and the evolution operator

$$\Phi : \Delta \to \mathcal{B}(\mathbb{R}),\ \Phi(t,s)x = \frac{h(s)}{h(t)}x,$$

we have that Φ has u.h.d. and it is not u.h.is.

Indeed, it is easy to see that Φ has u.h.d. for $M = 2$ and $\omega = 1$.

If we suppose that Φ is u.h.is. then we have

$$h(t)^{\nu+1} \leq Nh(s)^{\nu+1},\ \text{for all } (t,s) \in \Delta.$$

For $s = 0$ and $t \to \infty$, we obtain a contradiction.

Remark 2.3. The connections between the exponential and the polynomial concepts are given in the following diagram

$$\begin{array}{ccc} u.e.is. & \Rightarrow & u.e.d. \\ \Downarrow & & \Uparrow \\ u.p.is. & \Rightarrow & u.p.d. \end{array}$$

In general, the converse implications are not true.

Example 2.2. For $X = \mathbb{R}$ and

$$\Phi : \Delta \to \mathcal{B}(\mathbb{R}),\ \Phi(t,s)x = \frac{t^2+1}{s^2+1}x,$$

we have that Φ is u.p.is. and it is not u.e.is. (see [11]).

The connections between u.h.is., u.e.is. and u.p.is. are given by

Proposition 2.1. *For every evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ the following statements are equivalent:*

(i) *Φ is uniformly h-instable;*
(ii) *the evolution operator*

$$\Phi_h : \Delta \to \mathcal{B}(X), \Phi_h(t,s) = \Phi(h^{-1}(e^t), h^{-1}(e^s))$$

is uniformly exponentially instable;
(iii) *the evolution operator*

$$\Psi_h : \Delta \to \mathcal{B}(X), \Psi_h(t,s) = \Phi(h^{-1}(t+1), h^{-1}(s+1))$$

is uniformly polynomially instable;

Proof. It results in the same manner as in the stability case. (see [4], Theorems III.1 and III.2). □

A characterization of the u.e.is. property is given by

Proposition 2.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential decay. Then Φ is uniformly exponentially instable if and only if there exist $r > 1$ and $c \in (0,1)$ such that*

$$\|x\| \leq c\|\Phi(r+s,s)x\|, \forall s \geq 0, \ \forall\, x \in X.$$

Proof. See [1]. □

3. The main results

In this section we will present some characterization theorems for the uniform h instability with growth rates.

Theorem 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-decay. The following statements are equivalent:*

(1) Φ is uniformly h-instable.
(2) there exists $L > 1$ such that $\ln \dfrac{h(t)}{h(s)}\|x\| \leq L\|\Phi(t,s)x\|$, for all $(t,s,x) \in \Delta \times X$.
(3) there are a constant $L > 1$ and a strictly nondecreasing application $\varphi : [1,\infty) \to [1,\infty)$ with $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$ such that

$$\varphi\left(\frac{h(t)}{h(s)}\right)\|x\| \leq L\|\Phi(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X. \tag{3.1}$$

(4) there are $r > e$ and $c \in (0,1)$ such that

$$\|x\| \leq c\|\Phi(h^{-1}(rs), h^{-1}(s))x\|, \text{ for all } s \geq 1 \text{ and } x \in X.$$

Proof. $(1) \Rightarrow (2)$
We suppose that Φ is u.h.is. Then using the inequality $\dfrac{\ln u}{u^\nu} \leq \dfrac{1}{\nu e}$ we obtain

$$\ln \frac{h(t)}{h(s)}\|x\| \leq N \ln \frac{h(t)}{h(s)} \cdot \left(\frac{h(t)}{h(s)}\right)^{-\nu} \|\Phi(t,s)x\| \leq \frac{N}{\nu e}\|\Phi(t,s)x\| = L\|\Phi(t,s)x\|,$$

where $L = 1 + \dfrac{N}{\nu e} > 1$, so the relation (2) is proved.
$(2) \Rightarrow (3)$ It results immediately if we consider $\varphi(t) = \ln t$.
$(3) \Rightarrow (4)$ We suppose that there are $L > 1$ and $\varphi : [1,\infty) \to [1,\infty)$ a strictly nondecreasing application with $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$ such that the inequality from relation (3.1) states for all $(t,s,x) \in \Delta \times X$.

Let $r > e$ and $s \geq 1$ with $\varphi(r) > L$. Then $c = \dfrac{L}{\varphi(r)} < 1$.
Then for $t = h^{-1}(rs)$ and $s = h^{-1}(s)$ we obtain

$$\varphi(r)\|x\| \leq L\|\Phi(h^{-1}(rs), h^{-1}(s))x\|, \ \forall s \geq 1, \ \forall x \in X, \text{ which is equivalent to } (4).$$

$(4) \Rightarrow (1)$ We suppose that there are $r > e$ and $c \in (0,1)$ such that

$$\|x\| \leq c\|\Phi(h^{-1}(rs), h^{-1}(s))x\|, \forall s \geq 1 \ \forall x \in X. \tag{3.2}$$

Let $u = \ln r$, which implies $u > 1$ and $r = e^u > e$.
Let $v \geq 0$ and $s = e^v \geq 1$. Then, from (3.2) we have

$$\|x\| \leq c\|\Phi(h^{-1}(e^{u+v}), h^{-1}(e^v))x\|,$$

that is equivalent to

$$\|x\| \leq c\|\Phi_h(u+v, v)x\|, \text{ for all } v \geq 0 \text{ and } x \in X.$$

From Theorem 2.2 we obtain that Φ_h is uniformly exponentially instable which implies from the Proposition 2.2 that Φ is uniformly h-instable, so the theorem is proved.

□

Corollary 3.1. *If $\Phi : \Delta \to \mathcal{B}(X)$ is an evolution operator which has uniform exponential decay, then the following statements are equivalent:*

(1) Φ is uniformly exponentially instable.
(2) there exists $L > 1$ with $(t-s)\|x\| \leq L\|\Phi(t,s)x\|$, for all $(t,s,x) \in \Delta \times X$.
(3) there are $L > 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly nondecreasing application with $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$ such that:

$$\varphi(t-s)\|x\| \leq L\|\Phi(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

(4) there are $r > 1$ and $c \in (0,1)$ with $\|x\| \leq c\|\Phi(r+s,s)x\|$, for all $(s,x) \in \mathbb{R}_+ \times X$.

Proof. It is immediate from Theorem 3.1 if we consider $h(t) = e^t$. □

Corollary 3.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is an evolution operator which has uniform polynomial decay, then the following statements are equivalent:*

(1) Φ is uniformly polynomially instable.
(2) there exists $L > 1$ with $\ln \dfrac{t+1}{s+1}\|x\| \leq L\|\Phi(t,s)x\|$, for all $(t,s,x) \in \Delta \times X$.
(3) there are $L > 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly nondecreasing application with $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and $\varphi(1) = 1$ such that:

$$\varphi\left(\frac{t+1}{s+1}\right)\|x\| \leq L\|\Phi(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

(4) there are $r > 1$ and $c \in (0,1) : \|x\| \leq c\|\Phi(r+s,s)x\|$, for all $(s,x) \in \mathbb{R}_+ \times X$.

Proof. It is immediate from Theorem 3.1 if we consider $h(t) = t+1$. □

In what follows, we will give three characterization theorems of Datko type for the concept of uniform instability with growth rates. In order to do this, we introduce the following classes of functions:

- $\mathcal{H}$ the set of all functions $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $H > 1$ such that $h(t+1) \leq Hh(t)$, for all $t \geq 0$.
- $\mathcal{H}_0$ the set of all functions $h : \mathbb{R}_+ \to [1,\infty)$ with the property that $h(t) \geq t+1$, for all $t \geq 0$.

- $\mathcal{H}_1$ the set of all functions $h : \mathbb{R}_+ \to [1, \infty)$ with the property that for all $\alpha < 0$, there exists $H_1 > 1$ such that $\int\limits_s^\infty h(t)^{\alpha-1}dt \leq H_1 h(s)^\alpha$, for all $s \geq 0$.
- $\mathcal{H}_2$ the set of all functions $h : \mathbb{R}_+ \to [1, \infty)$ with the property that for all $\alpha < 0$, there exists $H_2 > 1$ such that $\int\limits_s^\infty h(\tau)^\alpha d\tau \leq H_2 h(s)^\alpha$, for all $s \geq 0$.
- $\mathcal{H}_3$ the set of all functions $h : \mathbb{R}_+ \to [1, \infty)$ with the property that for all $\alpha > 0$, there exists $H_3 > 1$ such that $\int\limits_0^t h(s)^\alpha ds \leq H_3 h(t)^\alpha$, for all $t \geq 0$.

Remark 3.4. If we denote by $e(t) = e^t$ and $p(t) = t + 1$ then

- $e, p \in \mathcal{H} \cap \mathcal{H}_0$;
- $e \in \mathcal{H}_2 \cap \mathcal{H}_3 \subset \mathcal{H}_1 \cap \mathcal{H}_3$;
- $p \in \mathcal{H}_1 \setminus (\mathcal{H}_2 \cup \mathcal{H}_3)$.

Theorem 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator with uniform h-decay and $h \in \mathcal{H}_0 \cap \mathcal{H}_1$. Then Φ is uniformly h-instable if and only if there are $D > 1$ and $\alpha > 0$ with*

$$\int\limits_s^\infty \frac{h(t)^{\alpha-1}}{\|\Phi(t,s)x\|}dt \leq \frac{Dh(s)^\alpha}{\|x\|},$$

for all $s \geq 0$ and $x \in X \setminus \{0\}$.

Proof. Necessity. We suppose that Φ is u.h.is. Let $\alpha \in (0, \nu)$, where ν is given by Definition 2.1.(iii).

$$\int\limits_s^\infty \frac{h(t)^{\alpha-1}}{\|\Phi(t,s)x\|}dt \leq N \int\limits_s^\infty \frac{h(t)^{\alpha-1}}{\|x\|} \left(\frac{h(s)}{h(t)}\right)^\nu dt = \frac{Nh(s)^\nu}{\|x\|} \int\limits_s^\infty h(t)^{\alpha-\nu-1}dt \leq D\frac{h(s)^\alpha}{\|x\|},$$

where $D = NH_1$.

Sufficiency. Let $(t, s) \in \Delta$ with $h(t) > 2s$. Then

$$\frac{h(t)^\alpha}{\|\Phi(t,s)x\|} = \frac{2}{h(t)} \int\limits_{\frac{h(t)}{2}}^{h(t)} \frac{h(t)^\alpha}{\|\Phi(t,\tau)\Phi(\tau,s)x\|}d\tau \leq \frac{2M}{h(t)} \int\limits_{\frac{h(t)}{2}}^{h(t)} \frac{h(t)^\alpha}{\|\Phi(\tau,s)x\|} \left(\frac{h(t)}{h(\tau)}\right)^\omega d\tau \leq$$

$$\leq 2^{\omega+1}M \int\limits_{\frac{h(t)}{2}}^{h(t)} \left(\frac{h(t)}{h(\tau)}\right)^{\alpha-1} \frac{h(\tau)^{\alpha-1}}{\|\Phi(\tau,s)x\|}d\tau \leq 2^{\omega+\alpha}M \int\limits_s^\infty \frac{h(\tau)^{\alpha-1}}{\|\Phi(\tau,s)x\|}d\tau \leq$$

$$\leq 2^{\omega+\alpha}MD\frac{h(s)^\alpha}{\|x\|}$$

It results

$$\frac{h(t)^\alpha}{\|\Phi(t,s)x\|} \leq 2^{\omega+\alpha}MD\frac{h(s)^\alpha}{\|x\|} \tag{3.3}$$

Let $(t,s) \in \Delta$ with $h(t) < 2s$. Then we obtain

$$\frac{h(t)^\alpha}{\|\Phi(t,s)x\|} \leq \frac{Mh(t)^\alpha}{\|x\|}\left(\frac{h(t)}{h(s)}\right)^\omega = \frac{M}{\|x\|}\left(\frac{h(t)}{h(s)}\right)^{\alpha+\omega} h(s)^\alpha \leq M4^{\alpha+\omega}\frac{h(s)^\alpha}{\|x\|}$$

It results

$$\frac{h(t)^\alpha}{\|\Phi(t,s)x\|} \leq M4^{\alpha+\omega}\frac{h(s)^\alpha}{\|x\|} \tag{3.4}$$

In conclusion, from relations (3.3) and (3.4) we obtain that Φ este u.h.is., so the theorem is proved. □

Corollary 3.3. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform exponential decay. Then Φ is uniformly exponentially instable if and only if there are $D > 1$ and $\alpha > 0$ with*

$$\int_s^\infty \frac{e^{(\alpha-1)t}}{\|\Phi(t,s)x\|}dt \leq \frac{De^{\alpha s}}{\|x\|},$$

for all $(s,x) \in \mathbb{R}_+ \times X \setminus \{0\}$.

Proof. It results immediately from Theorem 3.2 taking $h(t) = e^t$. □

Corollary 3.4. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform polynomial decay. Then Φ is uniformly polynomially instable if and only if there are $D > 1$ and $\alpha > 0$ with*

$$\int_s^\infty \frac{(t+1)^{\alpha-1}}{\|\Phi(t,s)x\|}dt \leq \frac{D(s+1)^\alpha}{\|x\|},$$

for all $(s,x) \in \mathbb{R}_+ \times X \setminus \{0\}$.

Proof. It results immediately from Theorem 3.2 taking $h(t) = t+1$. □

Corollary 3.5. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform h-decay and $h \in \mathcal{H}_0 \cap \mathcal{H}_2$. Then Φ is uniformly h-instable if and only if there are $D > 1$ and $\alpha > 0$ such that*

$$\int_s^\infty \frac{h(t)^\alpha}{\|\Phi(t,s)x\|}dt \leq \frac{Dh(s)^\alpha}{\|x\|},$$

for all $(s,x) \in \mathbb{R}_+ \times X \setminus \{0\}$.

Corollary 3.6. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform exponential decay. Then Φ is uniformly exponentially instable if and only if there are $D > 1$ and $\alpha > 0$ with*

$$\int_s^\infty \frac{e^{\alpha t}}{\|\Phi(t,s)x\|}dt \leq \frac{De^{\alpha s}}{\|x\|},$$

for all $(s,x) \in \mathbb{R}_+ \times X \setminus \{0\}$.

Proof. It follows from Corollary 3.5, if we consider $h(t) = e^t$. □

Theorem 3.3. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator strongly measurable with uniform h-decay and $h \in \mathcal{H} \cap \mathcal{H}_3$. Then Φ is uniformly h-instable if and only if there are $D > 1$ and $\alpha \in (0,1)$ with*

$$\int_{t_0}^{t} \frac{\|\Phi(s,t_0)x_0\|}{h(s)^\alpha} ds \leq \frac{D\|\Phi(t,t_0)x_0\|}{h(t)^\alpha},$$

for all $(t,t_0,x_0) \in \Delta \times X$.

Proof. Necessity. We suppose that Φ is u.h.is. Let $\alpha \in (0,\nu)$ where ν in given by Remark 2.1 (ii). Then

$$\int_{t_0}^{t} \frac{\|\Phi(s,t_0)x_0\|}{h(s)^\alpha} ds \leq N \int_{t_0}^{t} \frac{\|\Phi(t,t_0)x_0\|}{h(s)^\alpha} \left(\frac{h(s)}{h(t)}\right)^\nu ds \leq \frac{N\|\Phi(t,t_0)x_0\|}{h(t)^\nu} \int_{t_0}^{t} h(s)^{\nu-\alpha} ds \leq$$
$$\leq \frac{N\|\Phi(t,t_0)x_0\|}{h(t)^\nu} H_3 h(t)^{\nu-\alpha} = \frac{NH_3\|\Phi(t,t_0)x_0\|}{h(t)^\alpha} = \frac{D\|\Phi(t,t_0)x_0\|}{h(t)^\alpha}$$

where $D = NH_3$.

Sufficiency. Let $t > s+1$ and $s > t_0$. Then we obtain

$$\frac{\|\Phi(s,t_0)x_0\|}{h(s)^\alpha} = \int_{s}^{s+1} \frac{\|\Phi(s,t_0)x_0\|}{h(s)^\alpha} d\tau \leq M \int_{s}^{s+1} \frac{h(\tau)^\omega}{h(s)^{\omega+\alpha}} \|\Phi(\tau,t_0)x_0\| d\tau \leq$$
$$\leq M \int_{s}^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\omega+\alpha} h(\tau)^{-\alpha} \|\Phi(\tau,t_0)x_0\| d\tau \leq$$
$$\leq M \int_{s}^{s+1} \left(\frac{h(s+1)}{h(s)}\right)^{\omega+\alpha} \frac{\|\Phi(\tau,t_0)x_0\|}{h(\tau)^\alpha} d\tau \leq$$
$$\leq MH^{\omega+\alpha} \int_{t_0}^{t} \frac{\|\Phi(\tau,t_0)x_0\|}{h(\tau)^\alpha} d\tau \leq DMH^{\omega+\alpha} \frac{\|\Phi(t,t_0)x_0\|}{h(t)^\alpha}.$$

It results

$$h(t)^\alpha \|\Phi(s,t_0)x_0\| \leq DMH^{\omega+\alpha} h(s)^\alpha \|\Phi(t,t_0)x_0\|. \tag{3.5}$$

If $t \in [s, s+1)$ we have

$$\|\Phi(s,t_0)x_0\| \leq M \left(\frac{h(t)}{h(s)}\right)^\omega \|\Phi(t,t_0)x_0\| \leq M \left(\frac{h(s+1)}{h(s)}\right)^\omega \|\Phi(t,t_0)x_0\| \leq$$
$$\leq MH^\omega \|\Phi(t,t_0)x_0\|.$$

It results

$$\|\Phi(s,t_0)x_0\| \leq MH^\omega \|\Phi(t,t_0)x_0\|.$$

Then

$$\frac{h(t)^\alpha}{\|\Phi(t,t_0)x_0\|} \leq MH^\omega \frac{h(t)^\alpha}{\|\Phi(s,t_0)x_0\|} = MH^\omega \left(\frac{h(t)}{h(s)}\right)^\alpha \frac{h(s)^\alpha}{\|\Phi(s,t_0)x_0\|} \leq$$
$$\leq MH^{\omega+\alpha} \frac{h(s)^\alpha}{\|\Phi(s,t_0)x_0\|}.$$

We obtained

$$h(t)^{\alpha}\|\Phi(s,t_0)x_0\| \leq MH^{\omega+\alpha}h(s)^{\alpha}\|\Phi(s,t_0)x_0\|. \tag{3.6}$$

In conclusion, from relations (3.5) and (3.6) we have that the evolution operator Φ is u.h.is. □

Corollary 3.7. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform exponential decay. Then Φ is uniformly eponentially instable if and only if there are $D > 1$ and $\alpha > 0$ with*

$$\int_{t_0}^{t} \frac{\|\Phi(s,t_0)x_0\|}{e^{\alpha s}}ds \leq \frac{D\|\Phi(t,t_0)x_0\|}{e^{\alpha t}},$$

for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. It results immediately from Theorem 3.3 for $h(t) = e^t$. □

References

[1] Biriş, L. On uniform exponential instability property of evolution operators in Banach spaces. *An. Univ. Vest Timiş. Ser. Mat. -Inform.* **47** (2009), no. 1, 3–8.

[2] Boruga, R. Majorization criteria for polynomial stability and instability of evolution operators. *Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz.* **64** (2019), Issue 1, 55–63.

[3] Boruga (Toma),R.; Megan, M. Datko criteria for uniform instability in Banach spaces. *Stud. Univ. Babeş-Bolyai Math.* **66** (2021), no. 1, 115–122.

[4] Boruga (Toma), R.; Borlea (Pătraşcu); D.I., Toth, D. M.-M. On uniform stability with growth rates in Banach spaces. *2021 IEEE 15th Internat. Symposium on Appl. Computational Intell. and Informatics (SACI)* 2021, pp. 000393-000396.

[5] Datko, R. Uniform Asymptotic Stability of Evolutionary Processes in a Banach Space. *SIAM J. Math. Anal.* 3 (1972), 428–445.

[6] Hai, P.V. Polynomial stability and polynomial instability for skew-evolution semiflows. *Results Math.* **74** (2019), no. 4, Paper No. 175, 19 pp.

[7] Megan, M., Sasu; A.L., Sasu, B. Banach function spaces and exponential instability of evolution families. *Arch. Math. (Brno)* **39** (2003), no. 4, 277 — 286.

[8] Megan, M.; Sasu, A.L.; Sasu, B. *The Asymptotic Behavior of Evolution Families*. Mirton Publishing House, 2003.

[9] Megan, M.; Sasu, A.L.; Sasu, B. Exponential stability and exponential instability for linear skew-product flows. *Math. Bohem.* **129** (2004), no. 3, 225–243.

[10] Megan, M.; Stoica, C. Exponential instability of skew-evolution semiflows in Banach spaces. *Stud. Univ. Babeş-Bolyai Math.* **53** (2008), no. 1, 17–24.

[11] Rămneanţu, M.L.; Megan, M.; Ceauşu, T. Polynomial instability of evolution operators in Banach spaces. *Carpathian J. of Math.* **29** (2013), no. 1, 77–83.

[12] Stoica, C.; Megan, M. Uniform exponential instability of evolution operators in Banach spaces. *An. Univ. Vest Timiş. Ser. Mat. -Inform.* **44** (2006), no. 2, 143-–148.

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