CARPATHIAN J. MATH.
Volume **38** (2022), No. 3,
Pages 797-807

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2022.03.23

Dedicated to Prof. Emeritus Mihail Megan on the occasion of his 75th anniversary

Fixed point theorems for nonself generalized contractions on a large Kasahara space

ALEXANDRU-DARIUS FILIP

ABSTRACT. In this paper we give some fixed point theorems for nonself generalized contractions on a large Kasahara space, which generalize some results given by I.A. Rus and M.-A. Şerban (I.A. Rus, M.-A. Şerban, *Some fixed point theorems for nonself generalized contractions*, Miskolc Math. Notes, **17**(2016), no.2, 1021-1031) and by S. Reich and A.J. Zaslavski (S. Reich, A.J. Zaslavski, *A note on Rakotch contractions*, Fixed Point Theory, **9**(2008), no.1, 267-273) in complete metric spaces. We prove our results without using the completeness of the metric structure.

1. INTRODUCTION AND PRELIMINARIES

There are several techniques in the fixed point theory for nonself operators on a complete metric space ([7], [19], [13], [16], [15], [3], [9], [20], ...). Some results are given in the case of Kasahara spaces ([4], [5], [14]). By following the papers of S. Reich and A.J. Zaslavski [9] and I.A. Rus and M.-A. Şerban [20] we give some fixed point theorems for nonself generalized contractions on a large Kasahara space, a metric structure in which the completeness of the metric does not necessarily have to be satisfied.

In this paper we will use the notations and terminology given in [4] and [20]. We recall some notions, notations and results which will be used in the sequel of this paper.

1.1. **L-spaces.** The notion of L-space was given by M. Fréchet in 1906 (see [6]).

Let X be a nonempty set. Let $s(X) := \big\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space (denoted also by $(X, \xrightarrow{F})$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$.
(ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, we have that $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

By definition, an element $\{x_n\}_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence, $x = Lim\{x_n\}_{n\in\mathbb{N}}$ is the limit of this sequence and we also write $x_n \xrightarrow{F} x$ as $n \to \infty$.

In general, an L-space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L-spaces are: Hausdorff topological spaces, $\mathbb{R}^m_+$-metric spaces, cone-metric spaces, gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces (see: [19], [4], [7]).

Received: 28.02.2022. In revised form: 14.07.2022. Accepted: 15.07.2022
2010 *Mathematics Subject Classification.* 47H09, 47H10, 54H25.
Key words and phrases. *$\mathbb{R}^m_+$-metric space, contraction, φ-contraction, Kannan operator, diameter functional, maximal displacement functional, nonself operator, fixed point, data dependence, bijectivity.*
Corresponding author: Alexandru-Darius Filip; darius.filip@econ.ubbcluj.ro

1.2. **Matrices which converge to zero.** We denote by $\mathcal{M}_m(\mathbb{R}_+)$ the set of all $m \times m$ square matrices with positive real elements, by I_m the identity $m \times m$ matrix and by O_m the zero $m \times m$ matrix.

$S \in \mathcal{M}_m(\mathbb{R}_+)$ is said to be a matrix convergent to zero if $S^n \to O_m$ as $n \to \infty$.

A classical result in matrix analysis is the following theorem (see [21], [1]), which characterizes the matrices that converge to zero.

Theorem 1.1. *Let $S \in \mathcal{M}_m(\mathbb{R}_+)$. The following assertions are equivalent:*

(1) *S is convergent to zero;*
(2) *its spectral radius $\rho(S)$ is strictly less than 1;*
(3) *the matrix $(I_m - S)$ is nonsingular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \ldots + S^n + \ldots;$$

(4) *the matrix $(I_m - S)$ is nonsingular and $(I_m - S)^{-1}$ has nonnegative elements;*
(5) *$S^n x \to 0 \in \mathbb{R}^m$ as $n \to \infty$, for all $x \in \mathbb{R}^m$.*

Throughout this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$. Notice also that, if $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, then by $x \leq y$ we mean that $x_i \leq y_i$, for all $i \in \{1, 2, \ldots, m\}$.

1.3. **$\mathbb{R}^m_+$-metric spaces.** Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m_+$ is called a vector-valued metric on X if the following conditions are satisfied:

(1) $d(x, y) = 0 \in \mathbb{R}^m \Leftrightarrow x = y$, for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Some examples of vector-valued metrics are given bellow.

Example 1.1. *Let $X = \mathbb{R}^m$ and $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m_+$, be defined by $d(x, y) := |x - y| = \big(|x_1 - y_1|, \ldots, |x_m - y_m|\big)$, for all $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$.*

Example 1.2. *Let $X := (C[a, b])^2$ and $d : (C[a, b])^2 \times (C[a, b])^2 \to \mathbb{R}^2_+$, be defined by $d(x, y) := \big(\max\{|x_1(t) - y_1(t)| \mid t \in [a, b]\}, \max\{|x_2(t) - y_2(t)| \mid t \in [a, b]\}\big)$, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a, b])^2$.*

Example 1.3. *Let X be a nonempty set. Let $d_i : X \times X \to \mathbb{R}_+$ be metrics on X, for all $i = \overline{1, m}$. Then, the mapping $d : X \times X \to \mathbb{R}^m_+$, defined by*

$$d(x, y) := (d_1(x, y), d_2(x, y), \ldots, d_m(x, y)), \text{ for all } x, y \in X$$

is a vector-valued metric on X.

A nonempty set X endowed with a vector-valued metric $d : X \times X \to \mathbb{R}^m_+$ is called an $\mathbb{R}^m_+$-metric space and it is denoted by the pair (X, d). The notions of convergent sequence, Cauchy sequence, completeness, open and closed subset and so forth are similar to those defined for usual metric spaces.

In the context of $\mathbb{R}^m_+$-metric spaces (X, d), we define the diameter functional $\delta : P_b(X) \to \mathbb{R}^m_+$, by $\delta(A) := (\delta_{d_1}(A), \ldots, \delta_{d_m}(A))$, for all $A \in P_b(X)$.

1.4. **Large Kasahara spaces.** The notion of large Kasahara space was given by I.A. Rus in [14], as follows:

Definition 1.1. *Let X be a nonempty set, $\overset{F}{\to}$ be an L-space structure on X, $(G, +, \leq, \overset{G}{\to})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \overset{F}{\to}, d_G)$ is a large Kasahara space iff we have the following compatibility condition between $\overset{F}{\to}$ and d_G:*

(*i*) $x_n \in X$, $(x_n)_{n\in\mathbb{N}}$ *a Cauchy sequence (in some sense) with respect to* d_G *implies that* $(x_n)_{n\in\mathbb{N}}$ *converges in* $(X, \xrightarrow{F})$.

The notion of large Kasahara space which will be used in this paper, is the following:

Definition 1.2. *Let* X *be a nonempty set,* $\xrightarrow{F}$ *be an L-space structure on* X *and* $d : X \times X \to \mathbb{R}^m_+$ *be a metric on* X. *The triple* $(X, \xrightarrow{F}, d)$ *is a large Kasahara space iff we have the following compatibility conditions between* $\xrightarrow{F}$ *and* d:

(*i*) $\{x_n\}_{n\in\mathbb{N}}$ *is a fundamental sequence in* $(X, d) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ *converges in* $(X, \xrightarrow{F})$;
(*ii*) $x_n \xrightarrow{F} x^*$, $y_n \xrightarrow{F} y^*$ *and* $d(x_n, y_n) \to 0$ *as* $n \to \infty \Rightarrow x^* = y^*$.

Example 1.4 (See [10], [18], [5]). *Let* (X, ρ) *be a complete metric space and* (X, d) *be a metric space, where* $\rho, d : X \times X \to \mathbb{R}^m_+$. *We suppose that there exists a real constant* $c > 0$ *such that* $\rho(x, y) \le cd(x, y)$, *for all* $x, y \in X$. *Then,* $(X, \xrightarrow{\rho}, d)$ *is a large Kasahara space.*

Example 1.5. *We give here an example, showing that the condition* (ii) *of the Definition 1.2 is necessary.*

Let $X := \mathbb{R}^m$, $c(\mathbb{R}^m) := c_1(\mathbb{R}^m) \cup c_2(\mathbb{R}^m) \cup c_3(\mathbb{R}^m)$, *where* $c_1(\mathbb{R}^m)$ *is the set of all convergent sequences with respect to the metric* $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m_+$, *defined as in the Example 1.1 and, on* $c_1(\mathbb{R}^m)$, *we consider* $\xrightarrow{F} := \xrightarrow{d}$; $c_2(\mathbb{R}^m)$ *is the set of all subsequences* $\{x_n\}_{n\in\mathbb{N}}$ *of* $\{n\}_{n\in\mathbb{N}}$, *where* $n = (n, n, \ldots, n) \in \mathbb{N}^m$, *with* $Lim\{x_n\}_{n\in\mathbb{N}} = 0 \in \mathbb{R}^m$; $c_3(\mathbb{R}^m)$ *be the set of all subsequences* $\{y_n\}_{n\in\mathbb{N}}$ *of* $\{n + \frac{1}{n+1}\}_{n\in\mathbb{N}}$, *where* $n + \frac{1}{n+1} = (n + \frac{1}{n+1}, n + \frac{1}{n+1}, \ldots, n + \frac{1}{n+1}) \in \mathbb{R}^m$, *with* $Lim\{y_n\}_{n\in\mathbb{N}} = (1, 1, \ldots, 1) \in \mathbb{R}^m$. *Notice that* $(\mathbb{R}^m, c(\mathbb{R}^m), Lim)$ *is an L-space. But the triple* $(\mathbb{R}^m, \xrightarrow{F}, d)$ *is not a large Kasahara space. The condition* (i) *of Definition 1.2 is satisfied, but the condition* (ii) *is not. Indeed, let* $x_n := n$ *and* $y_n := n + \frac{1}{n+1}$, *for all* $n \in \mathbb{N}$. *For these two sequences we have* $x_n \xrightarrow{F} 0 \in \mathbb{R}^m$, $y_n \xrightarrow{F} (1, 1, \ldots, 1) \in \mathbb{R}^m$ *and* $d(x_n, y_n) \to 0$ *as* $n \to \infty$.

Remark 1.1. *Let* $(X, \xrightarrow{F}, d)$ *be a large Kasahara space. Then for any sequence* $\{x_n\}_{n\in\mathbb{N}^*} \subset X$ *with* $x_n \xrightarrow{d} x^*$ *as* $n \to \infty$, *we have* $x_n \xrightarrow{F} x^*$ *as* $n \to \infty$. *This implies that for any subset* $A \subset X$, *with* A *closed in* $(X, \xrightarrow{F})$, A *is closed in* (X, d).

2. The case of contractions

Let (X, d) be an $\mathbb{R}^m_+$-metric space. Let $Y \subset X$ be a closed subset and $f : Y \to X$ be a nonself operator. By definition (see [11], [19]) f is called a contraction if there exists a matrix $S \in \mathcal{M}_m(\mathbb{R}_+)$ convergent to zero, such that $d(f(x), f(y)) \le Sd(x, y)$, for all $x, y \in Y$.

We give here one of the main results of this paper, concerning the existence, uniqueness and data dependence of fixed points for nonself contractions, in the context of a large Kasahara space. The result generalizes Theorem 6, given by I.A. Rus and M.A. Şerban in [20], in the context of complete metric spaces (X, d). In our result, the completeness of the metric d does not necessarily have to be satisfied.

Theorem 2.2. *Let* $(X, \xrightarrow{F}, d)$ *be a large Kasahara space,* $Y \subset X$ *be a closed subset of* $(X, \xrightarrow{F})$ *and* $f : Y \to X$ *be an operator. We suppose that:*

(*i*) *there exists* $y_n \in Y$, *for all* $n \in \mathbb{N}^*$, *such that the set* $\{y_n \mid n \in \mathbb{N}^*\}$ *is bounded and* $f^i(y_n)$ *is defined for* $i = \overline{1, n}$, $n \in \mathbb{N}^*$;
(*ii*) f *is continuous in* $(X, \xrightarrow{F})$;
(*iii*) f *is a contraction w.r.t. the metric* d.

Then:

(1) $F_f = \{x^*\}$;
(2) $f^n(y_n) \xrightarrow{F} x^*$ *as* $n \to \infty$;
(3) $f^n(y_n) \xrightarrow{d} x^*$ *as* $n \to \infty$;
(4) $d(x, x^*) \leq (I_m - S)^{-1} d(x, f(x))$, *for all* $x \in Y$;
(5) $d(f^n(y_n), x^*) \leq S^n d(y_n, x^*)$, *for all* $n \in \mathbb{N}^*$;
(6) *if the operator* $g : Y \to X$ *is such that*
  (j) *there exists* $\eta \in (\mathbb{R}_+^*)^m$ *such that* $d(f(x), g(x)) \leq \eta$, *for all* $x \in Y$;
  (jj) $F_g \neq \varnothing$
  then $d(x^*, y^*) \leq (I_m - S)^{-1}\eta$, *for all* $y^* \in F_g$.

Proof. (1) + (2). First, we remark that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $r \in (\mathbb{R}_+^*)^m$ such that $d(y_0, y_n) \leq r$, for all $n \in \mathbb{N}^*$.

By the assumption (iii), we have the following estimations

$$d(y_0, f(y_n)) \leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \leq d(y_0, f(y_0)) + Sr, \text{ for all } n \in \mathbb{N}^*.$$

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned} d(y_0, f^i(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + Sr + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\ &\leq d(y_0, f(y_0)) + Sr + (I_m - S)^{-1} S d(y_n, f(y_n)), \text{ for all } n \in \mathbb{N}^* \end{aligned}$$

and $d(y_n, f(y_n)) \leq d(y_n, y_0) + d(y_0, f(y_n)) \leq r + Sr + d(y_0, f(y_0)) =: R \in (\mathbb{R}_+^*)^m$. So, there exists $\tilde{R} \in (\mathbb{R}_+^*)^m$, $\tilde{R} := d(y_0, f(y_0)) + Sr + (I_m - S)^{-1} SR > 0$ such that $d(y_0, f^i(y_n)) \leq \tilde{R}$, for all $n \in \mathbb{N}^*$ and $i = \overline{0, n-1}$. Thus, the set $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$.

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A)$, ..., $A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

Since the operator f is a contraction w.r.t. the metric d, there exists a matrix $S \in \mathcal{M}_m(\mathbb{R}_+)$ convergent to zero, such that $d(f(x), f(y)) \leq Sd(x, y)$, for all $x, y \in Y$. By taking the supremum over $x, y \in Y$ we get that

$$\delta(f(B)) \leq S\delta(B), \text{ for all } B \in P_b(Y).$$

By using the properties of the diameter functional δ and taking into account the assumption (iii), we have

$$\delta(A_{n+1}) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq S\delta(A_n), \text{ for all } n \in \mathbb{N}^*.$$

By mathematical induction over $n \in \mathbb{N}^*$, it follows that

$$\delta(A_{n+1}) \leq S^{n+1}\delta(A), \text{ for all } n \in \mathbb{N}^*.$$

By letting $n \to \infty$, we get that $\delta(A_{n+1}) \to 0$.

Since $f^n(y_n) \in A_n$, $f^{n-1}(y_n) \in A_{n-1} \cap A$ and $\delta(A_{n-1}) \to 0$ as $n \to \infty$, we get that $\{f^n(y_n)\}_{n \in \mathbb{N}^*}$ and $\{f^{n-1}(y_n)\}_{n \in \mathbb{N}^*}$ are fundamental sequences in (X, d).

By the condition (i) of Definition 1.2 we have that $f^{n-1}(y_n) \xrightarrow{F} u^*$ and $f^n(y_n) \xrightarrow{F} v^*$ as $n \to \infty$. On the other hand, $d(f^{n-1}(y_n), f^n(y_n)) \to 0$ as $n \to \infty$.

By the condition (ii) of Definition 1.2 we have that $u^* = v^* =: x^*$.

Since f is continuous, we have

$$f^n(y_n) = f(f^{n-1}(y_n)) \xrightarrow{F} f(x^*) \text{ as } n \to \infty.$$

So, $f(x^*) = x^*$. Hence $F_f = \{x^*\}$.

(3). By (iii), $d(f^n(y_n), x^*) = d(f^n(y_n), f^n(x^*)) \leq Sd(f^{n-1}(y_n), f^{n-1}(x^*)) \leq \ldots \leq S^n d(y_n, x^*) \to 0$ as $n \to \infty$. So, $f^n(y_n) \xrightarrow{d} x^*$ as $n \to \infty$.

(4). Let $x \in Y$. By using the triangle inequality of the metric d and the assumption (iii), we have $d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + Sd(x, x^*)$.

So, $d(x, x^*) \leq (I_m - S)^{-1} d(x, f(x))$, for all $x \in Y$.

(5). Follows from (iii).

(6). Let $y^* \in F_g$. Then, by choosing $x := y^*$ in the conclusion (4), we get

$$d(y^*, x^*) \leq (I_m - S)^{-1} d(y^*, f(y^*)) = (I_m - S)^{-1} d(g(y^*), f(y^*)).$$

By the symmetry of the metric d and by condition (j), it follows that

$$d(x^*, y^*) \leq (I_m - S)^{-1}\eta, \text{ for all } y^* \in F_g.$$

□

We give next an application for Theorem 2.2.

To do this, we define the notion of large Kasahara space group.

Definition 2.3. *Let X be a nonempty set, $(X, +)$ be a group and $(X, \to, d)$ be a large Kasahara space. By definition, the quadruple $(X, +, \to, d)$ is a large Kasahara space group if:*

(a) $s_1, s_2 \in c(X) \Rightarrow c_1 + c_2 \in c(X)$;

(b) $+ : (X, \to) \times (X, \to) \to (X, \to)$ *is continuous;*

(c) $x, y, z \in X \Rightarrow d(x + z, y + z) = d(x, y)$

Example 2.6. *Let $X := \mathbb{R}^m$ endowed with the addition. Then $(\mathbb{R}^m, +)$ is a group. Let $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m_+$ be the generalized euclidean metric on $\mathbb{R}^m$, defined as in Example 1.1. Let $\xrightarrow{d}$ be the convergence structure induced by d on $\mathbb{R}^m$. Let $\lambda \in \mathbb{R}^m_+$, $\lambda \neq 0_m := (0, 0, ..., 0) \in \mathbb{R}^m$. Let $\rho_\lambda : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m_+$, be defined by*

$$\rho_\lambda(x, y) := \begin{cases} d(x, y), & \text{if } x \neq 0_m \text{ and } y \neq 0_m \\ \lambda, & \text{if } x = 0_m \text{ or } y = 0_m \end{cases}$$

for all $x, y \in \mathbb{R}^m$. Then $(\mathbb{R}^m, +, \xrightarrow{d}, \rho_\lambda)$ is a large Kasahara space group.

The following result holds.

Theorem 2.3. *Let $(X, +, \to, d)$ be a large Kasahara space group and $f : X \to X$ be an operator. We suppose that:*

(1) $f : (X, \to) \to (X, \to)$ *is continuous;*

(2) $f : (X, d) \to (X, d)$ *is a contraction.*

Then the operator $1_X - f : X \to X$ is bijective.

Proof. In Theorem 2.2, let $Y = X$. Let $z \in X$. We shall prove that the equation $f(x) = z$ has a unique solution in X. To do this, let us consider the operator $g : X \to X$, defined by $g(x) := f(x) + z$, for all $x \in X$. We remark that the operator g satisfies the conditions of Theorem 2.2, so there exists a unique $x^* \in X$ such that $x^* = g(x^*)$. It follows that $x^* = f(x^*) + z$, which implies that the operator $1_X - f$ is bijective. □

The following problem rises.

Problem 2.1. Let $(X, +, \to, d)$ be a large Kasahara space group and $Y \in P_{cl}(X, \to)$.
We suppose that:

(1) $f : (Y, \to) \to (X, \to)$ is continuous;

(2) $f : (X, d) \to (X, d)$ is a contraction;

(3) there exists $y_n \in Y$, $n \in \mathbb{N}^*$, such that $\{y_n \mid n \in \mathbb{N}\}$ is bounded in (X, d) and $f^i(y_n)$ is defined for $i = \overline{1, n}$, $n \in \mathbb{N}^*$.

In which conditions $f : Y \to X$ is a bijective operator ?

3. The case of φ-contractions

Let $\varphi : \mathbb{R}_+^m \to \mathbb{R}_+^m$ be a function, defined by $\varphi(t) := (\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t))$, for all $t \in \mathbb{R}_+^m$, where $\varphi_i : \mathbb{R}_+^m \to \mathbb{R}_+$ are functions, for all $i = \overline{1,m}$. If φ is monotone increasing, i.e., for all $t_1, t_2 \in \mathbb{R}_+^m$, $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$ (i.e., $\varphi_i(t_1) \leq \varphi_i(t_2)$, for all $i = \overline{1,m}$), and the sequence $\varphi^n(t) \to 0 \in \mathbb{R}^m$ as $n \to \infty$ (i.e., $\varphi_i^n(t) \to 0$ as $n \to \infty$, for all $i = \overline{1,m}$), for all $t \in \mathbb{R}_+^m$, then φ is a comparison function.

If φ is a continuous comparison function satisfying $t - \varphi(t) \to +\infty^m$ as $t \to +\infty^m$ (i.e., $t_i - \varphi_i(t) \to +\infty$ as $t_i \to +\infty$, for all $i = \overline{1,m}$), then φ is called strict comparison function. In this case, we can define the function $\theta_\varphi : \mathbb{R}_+^m \to \mathbb{R}_+^m$, $\theta_\varphi(t) := (\theta_{\varphi_1}(t), \theta_{\varphi_2}(t), \ldots, \theta_{\varphi_m}(t))$, where $\theta_{\varphi_i}(t) : \mathbb{R}_+^m \to \mathbb{R}_+$ is defined by $\theta_{\varphi_i}(t) = \sup\{s_i \in \mathbb{R}_+ \mid s_i - \varphi_i(s) \leq t_i\}$, for all $i = \overline{1,m}$ and $t \in \mathbb{R}_+^m$. The function θ_φ is increasing and has the property $\theta_\varphi(t) \to 0 \in \mathbb{R}^m$ as $t \to 0 \in \mathbb{R}^m$ (i.e., $\theta_{\varphi_i}(t) \to 0$ as $t \to 0 \in \mathbb{R}^m$, for all $i = \overline{1,m}$). We will use the function θ_φ to study the data dependence of the fixed points.

If φ is a comparison function satisfying

$$\sum_{n\in\mathbb{N}} \varphi^n(t) < +\infty^m \text{ (i.e., } \sum_{n\in\mathbb{N}} \varphi_i^n(t) < +\infty \text{ for all } i = \overline{1,m}),$$

for all $t \in \mathbb{R}_+^m$, then φ is called strong comparison function.

More consideration on comparison functions are given in [2] and [11].

The following theorem generalizes Theorem 1 and Theorem 3 given by I.A. Rus and M.-A. Şerban in [20], for φ-contractions in the context of complete metric spaces, and Theorem 1, given by S. Reich and A.J. Zaslavski in [9], for Rakotch contractions, in the same context. Notice that in our result, the completion of the metric space is not used at all.

Theorem 3.4. *Let $(X, \xrightarrow{F}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \xrightarrow{F})$ and $f : Y \to X$ be an operator. We suppose that:*

- (i) *there exists $y_n \in Y$, for all $n \in \mathbb{N}^*$, such that the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded and $f^i(y_n)$ is defined for $i = \overline{1,n}$, $n \in \mathbb{N}^*$;*
- (ii) *f is continuous in $(X, \xrightarrow{F})$;*
- (iii) *f is a φ-contraction w.r.t. the metric d, where φ is a strict and strong comparison function.*

Then:

- (1) *$F_f = \{x^*\}$;*
- (2) *$f^n(y_n) \xrightarrow{F} x^*$ as $n \to \infty$;*
- (3) *$f^n(y_n) \xrightarrow{d} x^*$ as $n \to \infty$;*
- (4) *$d(f^n(y_n), x^*) \leq \varphi(d(y_n, x^*))$, for all $n \in \mathbb{N}^*$;*
- (5) *$d(x, x^*) \leq \theta_\varphi(d(x, f(x)))$, for all $x \in Y$;*
- (6) *if $g : Y \to X$ is such that*
  - (j) *there exists $\eta \in (\mathbb{R}_+^*)^m$ such that $d(f(x), g(x)) \leq \eta$, for all $x \in Y$;*
  - (jj) *$F_g \neq \varnothing$*

  then $d(x^*, y^*) \leq \theta_\varphi(\eta)$, for all $y^* \in F_g$.

Proof. (1) + (2). First, we remark that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $r \in (\mathbb{R}_+^*)^m$ such that $d(y_0, y_n) \leq r$, for all $n \in \mathbb{N}^*$.

Since $\varphi : \mathbb{R}_+^m \to \mathbb{R}_+^m$ is a comparison function, we have $\varphi(t) \leq t$, for all $t \in \mathbb{R}_+^m$. By (iii) we have the following estimations

$$\begin{aligned} d(y_0, f(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \leq d(y_0, f(y_0)) + \varphi(d(y_0, y_n)) \\ &\leq d(y_0, f(y_0)) + r, \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned} d(y_0, f^i(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\ &\leq d(y_0, f(y_0)) + r + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\ &\leq d(y_0, f(y_0)) + r + \sum_{j=1}^{\infty} \varphi^j(d(y_n, f(y_n))), \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

Since φ is a strong comparison function, $\sum_{j=1}^{\infty} \varphi^j(d(y_n, f(y_n))) < +\infty^m$. So, there exists a number $\Phi \in \mathbb{R}_+^m$ such that $d(y_0, f^i(y_n)) \leq d(y_0, f(y_0)) + r + \Phi$, for all $n \in \mathbb{N}^*$. Thus, the set $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$.

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A), \ldots, A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

Since the operator f is a φ-contraction w.r.t. the metric d, we have $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in Y$. By taking the supremum over $x, y \in Y$ in the contraction condition, we get that

$$\delta(f(B)) \leq \varphi(\delta(B)), \text{ for all } B \in P_b(Y).$$

By using the properties of the diameter functional δ and by taking into account the assumption (iii), we have

$$\delta(A_{n+1}) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq \varphi(\delta(A_n)), \text{ for all } n \in \mathbb{N}^*.$$

By mathematical induction over $n \in \mathbb{N}^*$, it follows that

$$\delta(A_{n+1}) \leq \varphi^{n+1}(\delta(A)), \text{ for all } n \in \mathbb{N}^*.$$

By letting $n \to \infty$, we get that $\delta(A_{n+1}) \to 0$.

Since $f^n(y_n) \in A_n$, $f^{n-1}(y_n) \in A_{n-1} \cap A$ and $\delta(A_{n-1}) \to 0$ as $n \to \infty$, we get that $\{f^n(y_n)\}_{n \in \mathbb{N}^*}$ and $\{f^{n-1}(y_n)\}_{n \in \mathbb{N}^*}$ are fundamental sequences in (X, d).

By the condition (i) of Definition 1.2 we have that $f^{n-1}(y_n) \xrightarrow{F} u^*$ and $f^n(y_n) \xrightarrow{F} v^*$ as $n \to \infty$. On the other hand, $d(f^{n-1}(y_n), f^n(y_n)) \to 0$ as $n \to \infty$.

By the condition (ii) of Definition 1.2 we have that $u^* = v^* =: x^*$.

Since f is continuous, we have

$$f^n(y_n) = f(f^{n-1}(y_n)) \xrightarrow{F} f(x^*) \text{ as } n \to \infty.$$

So, $f(x^*) = x^*$. Hence $F_f = \{x^*\}$.

(3) + (4). By using the assumption (iii), we have $d(f^n(y_n), x^*) = d(f^n(y_n), f^n(x^*)) \leq \varphi(d(f^{n-1}(y_n), f^{n-1}(x^*))) \leq \ldots \leq \varphi^n(d(y_n, x^*))$, for all $n \in \mathbb{N}^*$. Since $\varphi : \mathbb{R}_+^m \to \mathbb{R}_+^m$ is a comparison function, we have that $\varphi^n(d(y_n, x^*)) \to 0$ as $n \to \infty$. So, $f^n(y_n) \xrightarrow{d} x^*$ as $n \to \infty$. On the other hand, φ is an increasing function on $\mathbb{R}_+^m$ and $\varphi(t) \leq t$, for all $t \in \mathbb{R}_+^m$, with equality when $t = 0$. By mathematical induction over $n \in \mathbb{N}^*$ we have

$$d(f^n(y_n), x^*) \leq \varphi^n(d(y_n, x^*)) \leq \ldots \leq \varphi(d(y_n, x^*)), \text{ for all } n \in \mathbb{N}^*.$$

(5). Since $d(x,x^*) \le d(x,f(x)) + d(f(x),x^*) \le d(x,f(x)) + \varphi(d(x,x^*))$, for all $x \in Y$, it follows that $d(x,x^*) - \varphi(d(x,x^*)) \le d(x,f(x))$, for all $x \in Y$. We have next

$$d(x,x^*) \le \sup_{d(x,x^*)-\varphi(d(x,x^*)) \le d(x,f(x))} d(x,x^*) = \theta_\varphi(d(x,f(x))), \text{ for all } x \in Y.$$

(6). Let $y^* \in F_g$. Then, from the conclusion (5) we have

$$d(x^*,y^*) = d(y^*,x^*) \le \theta_\varphi(d(y^*,f(y^*))) = \theta_\varphi(d(g(y^*),f(y^*))) \le \theta_\varphi(\eta).$$

□

4. The case of Kannan operators

We consider the following particular matrix set:

$$\mathcal{M}_m^\Delta(\mathbb{R}_+) := \left\{ \begin{pmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1m} \\ 0 & s_{22} & s_{23} & \dots & s_{2m} \\ 0 & 0 & s_{33} & \dots & s_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & s_{mm} \end{pmatrix} \in \mathcal{M}_m(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} s_{ii} < \frac{1}{2} \right\}.$$

Lemma 4.1. *Let $S \in \mathcal{M}_m^\Delta(\mathbb{R}_+)$. Then S and $(I_m - S)^{-1}S$ are convergent to zero.*

Proof. Since the eigenvalues of the matrices S and $(I_m - S)^{-1}S$ are in the open unit disk, the conclusion follows from Theorem 1.1. □

We define now the notion of nonself Kannan operator, which will be used throughout this section.

Definition 4.4. *Let (X,d) be a metric space, where $d : X \times X \to \mathbb{R}_+^m$. Let $Y \in P(X)$. The operator $f : Y \to X$ is a Kannan operator, if there exists a matrix $S \in \mathcal{M}_m^\Delta(\mathbb{R}_+)$ convergent to zero, such that*

$$d(f(x),f(y)) \le S[d(x,f(x)) + d(y,f(y))], \text{ for all } x,y \in Y.$$

In the proof of the main result of this section, we will use the maximal displacement functional.

Definition 4.5. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous nonself operator. The functional $E_f : P(Y) \to [\mathbb{R}_+ \cup \{+\infty\}]^m$ defined by*

$$E_f(A) := \sup\{d(x,f(x)) \mid x \in A\}, \text{ for all } A \in P(Y),$$

is called the maximal displacement functional corresponding to f.

The maximal displacement functional has the following properties:
(i) $A, B \in P(Y)$, $A \subset B$ imply $E_f(A) \le E_f(B)$;
(ii) $E_f(A) = E_f(\overline{A})$, for all $A \in P(Y)$.

Lemma 4.2. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous Kannan operator. Then:*

(1) *$E_f(f(A)) \le (I_m - S)^{-1}SE_f(A)$, for all $A \in Y$ with $f(A) \subset Y$;*
(2) *$E_f(f(A) \cap Y) \le (I_m - S)^{-1}SE_f(A)$, for all $A \in Y$ with $f(A) \cap Y \ne \varnothing$.*

Proof. (1). From the definition of the maximal displacement functional corresponding to f we have that $E_f(f(A)) = \sup\{d(x,f(x)) \mid x \in f(A)\} = \sup\{d(f(u),f^2(u)) \mid u \in A\}$. Since f is a Kannan operator, we have $d(f(u),f^2(u)) \le S[d(u,f(u)) + d(f(u),f^2(u))]$, for all $u \in A$. It follows that $d(f(u),f^2(u)) \le (I_m - S)^{-1}Sd(u,f(u))$, for all $u \in A$. Hence $E_f(f(A)) \le (I_m - S)^{-1}S\sup\{d(u,f(u)) \mid u \in A\} = (I_m - S)^{-1}SE_f(A)$, for all $A \in Y$ with $f(A) \subset Y$.

(2). Taking into consideration that f is a Kannan operator, we have that $E_f(f(A) \cap Y) = \sup\{d(x, f(x)) \mid x \in f(A) \cap Y\} = \sup\{d(f(u), f^2(u)) \mid u \in A,\ f(u) \in Y\} \leq (I_m - S)^{-1} S \sup\{d(u, f(u)) \mid u \in A\} = (I_m - S)^{-1} S E_f(A)$, for all $A \in Y$ with $f(A) \cap Y \neq \varnothing$. $\square$

The following result generalizes Theorem 4 given by I.A. Rus and M.-A. Şerban in [20] for Kannan operators in the context of complete metric spaces. Our result is proved in the context of a large Kasahara space, without imposing the completion condition upon the metric structure.

Theorem 4.5. *Let $(X, \xrightarrow{F}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \xrightarrow{F})$ and $f : Y \to X$ be an operator. We suppose that:*

- (i) *there exists $y_n \in Y$, for all $n \in \mathbb{N}^*$, such that the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded and $f^i(y_n)$ is defined for $i = \overline{1, n}$, $n \in \mathbb{N}^*$;*
- (ii) *f is continuous in $(X, \xrightarrow{F})$;*
- (iii) *f is a Kannan operator w.r.t. the metric d;*
- (iv) *$E_f(Y) < +\infty^m$.*

Then:

1. *$F_f = \{x^*\}$;*
2. *$f^n(y_n) \xrightarrow{F} x^*$ as $n \to \infty$;*
3. *$f^n(y_n) \xrightarrow{d} x^*$ as $n \to \infty$;*
4. *$d(x, x^*) \leq (I_m + S) d(x, f(x))$, for all $x \in Y$;*
5. *$d(f^{n-1}(x_n), x^*) \leq (I_m + S) \left[(I_m - S)^{-1} S\right]^{n-1} d(x_n, f(x_n))$, for all $n \in \mathbb{N}^*$;*
6. *if $g : Y \to X$ is such that*
   - (j) *there exists $\eta \in (\mathbb{R}_+^*)^m$ such that $d(f(x), g(x)) \leq \eta$, for all $x \in Y$;*
   - (jj) *$F_g \neq \varnothing$*

   then $d(x^*, y^*) \leq (I_m + S)\eta$, for all $y^* \in F_g$.

Proof. (1) + (2). First, we remark that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is a bounded set. Indeed, since the set $\{y_n \mid n \in \mathbb{N}^*\}$ is bounded, for a given $y_0 \in Y$ there exists a constant $r \in (\mathbb{R}_+^*)^m$ such that $d(y_0, y_n) \leq r$, for all $n \in \mathbb{N}^*$.

By (iii) we have the following estimations

$$\begin{aligned} d(y_0, f(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \\ &\leq d(y_0, f(y_0)) + S[d(y_0, f(y_0)) + d(y_n, f(y_n))] \\ &\leq (I_m + S) d(y_0, f(y_0)) + S d(y_n, y_0) + S d(y_0, f(y_n)), \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

It follows that $d(y_0, f(y_n)) \leq (I_m - S)^{-1}(I_m + S) d(y_0, f(y_0)) + (I_m - S)^{-1} S r$, for all $n \in \mathbb{N}^*$. Let us denote $\Lambda := (I_m - S)^{-1} S \in \mathcal{M}_m(\mathbb{R}_+)$. So, there exists a constant $R := (I_m - S)^{-1}(I_m + S) d(y_0, f(y_0)) + \Lambda r \in (\mathbb{R}_+^*)^m$ such that $d(y_0, f(y_n)) \leq R$, for all $n \in \mathbb{N}^*$.

On the other hand, for any $i \geq 2$, we have

$$\begin{aligned}
d(y_0, f^i(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)) \\
&\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n)) \\
&\leq d(y_0, f(y_0)) + S[d(y_0, f(y_0)) + d(y_n, f(y_n))] + \sum_{j=1}^{i-1} d(f^j(y_n), f^{j+1}(y_n)) \\
&\leq (I_m + S)d(y_0, f(y_0)) + S[d(y_n, y_0) + d(y_0, f(y_n))] + \sum_{j=1}^{i-1} \Lambda^j d(y_n, f(y_n)) \\
&\leq (I_m + S)d(y_0, f(y_0)) + Sr + Sd(y_0, f(y_n)) \\
&\qquad + (I_m - \Lambda)^{-1}\Lambda[d(y_n, y_0) + d(y_0, f(y_n))] \\
&\leq (I_m + S)d(y_0, f(y_0)) + \left[S + (I_m - \Lambda)^{-1}\Lambda\right](r + R), \text{ for all } n \in \mathbb{N}^*.
\end{aligned}$$

Thus, the set $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\}$ is bounded.

Let $A \in P_b(Y)$ be such that $\{f^i(y_n) \mid i = \overline{0, n-1},\ n \in \mathbb{N}^*\} \subset A$.

Let $A_1 := f(A)$, $A_2 := f(A_1 \cap A), \ldots, A_{n+1} := f(A_n \cap A)$, for all $n \in \mathbb{N}^*$. By this construction, we obtain the sequence of sets $\{A_n\}_{n \in \mathbb{N}^*}$ with the following properties: $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}^*$, and $f^n(y_n) \in A_n$, for all $n \in \mathbb{N}^*$.

By the definitions of the diameter functional δ and maximal displacement functional E_f and taking into account the properties stated in Lemma 4.2, we have

$$\begin{aligned}
\delta(A_{n+1}) = \delta(f(A_n \cap A)) &\leq 2SE_f(A_n \cap A) = 2SE_f(f(A_{n-1} \cap A) \cap A) \\
&\leq 2S\Lambda E_f(A_{n-1} \cap A) \leq \ldots \leq 2S\Lambda^n E_f(A) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By following the proof of Theorem 2.2, the conclusions follow.

(3). Follows from the proof of (5).

(4). Let $x \in Y$. By using the assumption (iii), we have

$d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + S[d(x, f(x)) + d(x^*, f(x^*))]$.

It follows that $d(x, x^*) \leq (I_m + S)d(x, f(x))$, for all $x \in Y$.

(5). Let $x := f^{n-1}(x_n)$ in the conclusion (4). Then we have

$$\begin{aligned}
&d(f^{n-1}(x_n), x^*) \leq (I_m + S)d(f^{n-1}(x_n), f^n(x_n)) \leq (I_m + S)\Lambda d(f^{n-2}(x_n), f^{n-1}(x_n)) \\
&\quad \leq \ldots \leq (I_m + S)\Lambda^{n-1} d(x_n, f(x_n)), \text{ for all } n \in \mathbb{N}^*, \text{ where } \Lambda := (I_m - S)^{-1}S.
\end{aligned}$$

(6). Let $y^* \in F_g$. By letting $x := y^*$ in the conclusion (4) we have $d(y^*, x^*) \leq (I_m + S)d(y^*, f(y^*))$. Hence $d(x^*, y^*) \leq (I_m + S)d(g(y^*), f(y^*)) \leq (I_m + S)\eta$, for all $y^* \in F_g$. $\square$

5. Some open problems

The above considerations give rise to the following problems.

Problem 5.1. Let (X, d) be a metric space, where $d : X \times X \to \mathbb{R}^m_+$. Let $Y \subset X$ be a nonempty closed subset and $\{y_n\}_{n \in \mathbb{N}} \subset Y$ be a bounded sequence. For which generalized contractions ([11], [19], [2], [8]) $f : Y \to X$, the following implication holds: $f^i(y_n)$ is defined for $i = \overline{1, n}$, $n \in \mathbb{N}^*$ implies that the set $\{f^i(y_n) \mid i = \overline{1, n},\ n \in \mathbb{N}^*\}$ is bounded ?

Problem 5.2. If f is a solution of Problem 5.1. in which conditions we have that:

(i) $F_f = \{x^*\}$;

(ii) $f^n(y_n) \to x^*$ as $n \to \infty$?

Problem 5.3. Let $(X, \xrightarrow{F}, d)$ be a large Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$. Let $Y \subset X$ be a closed subset of $(X, \xrightarrow{F})$ and $f : Y \to X$ be an operator. We suppose that:

(i) there exists a bounded sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$ such that $f^i(y_n)$ is defined for $i = \overline{1,n}$, $n \in \mathbb{N}^*$;

(ii) f is continuous in $(X, \xrightarrow{F})$.

The problem is to find those contractions f, satisfying the above conditions, for which we have that:

(1) $F_f = \{x^*\}$;

(2) $f^n(y_n) \xrightarrow{F} x^*$ as $n \to \infty$.

REFERENCES

[1] Allaire, G.; Kaber, S.M., *Numerical Linear Algebra*, Springer, New York, USA, 2008.

[2] Berinde, V., *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin Heidelberg, 2007.

[3] Chiş-Novac, A.; Precup, R.; Rus, I.A. Data dependence of fixed points for non-self generalized contractions. *Fixed Point Theory* **10** (2009), 73–87.

[4] Filip, A.-D., *Fixed Point Theory in Kasahara Spaces*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2015.

[5] Filip, A.-D. and Rus, I.A., Fixed point theory for nonself generalized contractions in Kasahara spaces, *An. Univ. Vest Timiş. Ser. Mat.-Inform.*, **57** (2019), 66–76.

[6] Fréchet, M., *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.

[7] Granas, A. and Dugundji, J., *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[8] Hardy, G.E. and Rogers, T.D., A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.*, **16**(1973), 201–206.

[9] Reich, S. and Zaslavski, A.J., A note on Rakotch contractions, *Fixed Point Theory*, **9**(2008), no.1, 267-273.

[10] Rus, I.A., Basic problem for Maia's theorem, *Sem. Fixed Point Theory*, 1981, 112-115.

[11] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[12] Rus, I.A., Picard operators and applications, *Sci. Math. Jpn.*, **58**(2003), 191-219.

[13] Rus, I.A., *Fixed Point Structure Theory*, Cluj University Press, 2006.

[14] Rus, I.A., Kasahara spaces, *Sci. Math. Jpn.*, **72**(2010), no. 1, 101-110.

[15] Rus, I.A., Five open problems in fixed point theory in terms of fixed point structures (I): single valued operators, *Proceedings of the* 10th *IC-FPTA*, 39-60, July 9-18 Cluj-Napoca, Romania, House of the Book of Science, Cluj-Napoca, 2013.

[16] Rus, I.A., The generalized retraction methods in fixed point theory for nonself operators, *Fixed Point Theory*, **15**(2014), no.2, 559-578.

[17] Rus, I.A., Some variants of contraction principle, generalizations and applications, *Studia Univ. Babeş-Bolyai Math.*, **61**(2016), 343-358.

[18] Rus, I.A., Mureşan, A. and Mureşan, V., Weakly Picard operators on a set with two metrics, *Fixed Point Theory*, **6**(2005), no.2, 323-331.

[19] Rus, I.A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*, Cluj University Press, 2008.

[20] Rus, I.A. and Şerban, M.-A., Some fixed point theorems for nonself generalized contractions, *Miskolc Math. Notes*, **17**(2016), no.2, 1021-1031.

[21] Varga, R.S., *Matrix Iterative Analysis*, vol. 27 of Springer Series in Computational Mathematics, Springer, Berlin, Germany, 2000.

BABEŞ-BOLYAI UNIVERSITY, FACULTY OF ECONOMICS AND BUSINESS ADMINISTRATION,
DEPARTMENT OF STATISTICS-FORECASTS-MATHEMATICS,
TEODOR MIHALI STREET, NO. 58-60, 400591 CLUJ-NAPOCA, ROMANIA
Email address: darius.filip@econ.ubbcluj.ro