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Duffing equations with two-component Poisson stable coefficients

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ABSTRACT. The research considers Duffing equations with two-component Poisson stable coefficients and excitation. The existence and uniqueness of the Poisson stable solutions have been proved. A new technique for verification of the stability is developed. Numerical simulations of the coefficients, excitation as well as solution are provided.

1. INTRODUCTION AND PRELIMINARIES

The standard Duffing equation has the form [1]

$$x'' + ax' + bx + cx^3 = F_0 cos(\lambda t), \tag{1.1}$$

where a is the damping coefficient, b and c are stiffness (restoring) coefficients, F_0 is the coefficient of excitation, λ is the frequency of excitation and t is the time. The major part of papers on the equation assume that the coefficients a, b, c and F_0 are constant [2, 3]. Considering the original model one can assume mechanical reasons for variable coefficients. For instance, not constant damping and driving force [4]. Then new theoretical challenges appear. In following the suggestions, rich application opportunities appear [5, 6, 7]. The second order differential equations, despite their small dimension produce strong difficulties for study [8, 9], and are source for most modern complexity investigations. Let us remember the seminal results in [10], which were basics for the research by S. Smale [11]. Experimentally, the simulations of the Duffing equation by Y. Ueda [12], are beside those numerical observations of E. Lorenz, which had been finally completed with the exploration of sensitivity, of the fundamental concept of chaos [13]. This is why, comprehensive modifications of the Duffing equation with subsequent new methods of investigation have to be in the focus of researchers. In the present paper, we propose the model with coefficients, which are most sophisticated, if compared with previous results [2, 4, 14, 15]. They are Poisson stable functions, the pioneering concept of complexity introduced by H. Poincaré [16]. Another principal novelty of our research is the method of included intervals. It relies on the suggestions introduced in our recent research, when the method for existence of stable periodic and almost periodic oscillations in quasi-linear systems has been adapted for Poisson stable solutions in [17, 18, 19, 20, 21]. It is different than that used previously in papers [22, 23], and proceeds the approach of many studies of oscillations [24, 26, 25, 27, 28] to the case of variable coefficients and ultimate complexity of recurrence. There are many specific mathematical methods for the Duffing equation [2, 3, 5, 6, 7, 8, 9, 12, 29, 30, 31, 32, 33, 34, 35], in which various aspects of dynamics have been examined numerically and analytically. Nevertheless, one can expect that input-output results on oscillations of the general theory of differential equations [25, 26, 27, 28]

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due to their approved applications in mechanics, physics and engineering still must be applied effectively for Duffing equations, and our present study confirms that. Final our important contribution to the theory is the construction method of samples of Poisson stable functions by extending traditions, when trigonometric functions are determined as solutions of differential equations. This time, we have built Poisson stable functions as solutions of hybrid systems, which admit as a part the logistic equation. Another way for the functions construction are random processes [36]. We believe that the role of Duffing oscillators will increase in the machine learning procedures [37, 38].

The main subject of this article is the following equation,

$$x'' + p(t)x' + q(t)x + r(t)x^3 = F(t)\cos(\lambda t), \tag{1.2}$$

where $t, x \in \mathbb{R}$; λ is a real constant; $p(t)$, $q(t)$, $r(t)$ and $F(t)$ are continuous functions. We consider the equation with Poisson stable coefficients and find conditions on the functions, such that the result-dynamics is Poisson stable. The version of the Duffing equation has not been considered in literature, at all. Apparently, the reason for that is the technical difficulty of the Poisson stability identification, what has been successfully overcome in our research.

As far as we aware, F. Moon one of the first who emphasized the importance of investigation of the mechanical equations of the second order with non-periodic coefficients, such that non-periodic outputs are expected [39]. Poisson stable inputs and outputs of our research are not periodic functions, and rather chaotic [16, 40], and this meets the challenging problems of mechanics, electronics and neuroscience [20, 35, 38, 39, 40].

The following concept of recurrence is a fundamental for the present investigation.

Definition 1.1. [41]A continuous and bounded function $\psi(t) : \mathbb{R} \to \mathbb{R}^n$ is called Poisson stable, if there exists a sequence t_k, which diverges to infinity such that the sequence $\psi(t + t_k)$ converges to $\psi(t)$ as $k \to \infty$ uniformly on bounded intervals of $\mathbb{R}$.

The sequence t_k is said to be *the Poisson sequence* of the function $\psi(t)$. For convenience, in the further reasoning, instead of a *Poisson stable function* we shall use the term *Poisson function*.

For a fixed real constant $\omega > 0$, one can write that $t_k \equiv \tau_k (mod\, \omega)$, where $0 \le \tau_k < \omega$, for all $k \ge 1$. The boundedness of the sequence τ_k implies that there exists a subsequence τ_{k_l}, which converges to a number τ_ω. That is, there exist a subsequence t_{k_l} of the Poisson sequence t_k and a number τ_ω such that $t_{k_l} \to \tau_\omega (mod\, \omega)$ as $l \to \infty$. In what follows, we shall call the number τ_ω as the *Poisson shift* for the Poisson sequence t_k with respect to the ω. It is not difficult to find that for the fixed positive ω the set of all Poisson shifts, T_ω, is not empty, and it can consist of several or even infinite number elements. The number $\kappa_\omega = inf\, T_\omega$, $0 \le \kappa_\omega < \omega$, is said to be *the Poisson number* for the Poisson sequence t_k with respect to the number ω.

2. THE POISSON STABLE SOLUTION

This section presents the main result of our study. Under certain conditions, it has been rigorously proved that the Poisson solution, which is asymptotically stable, meets in the dynamics of the Duffing equation.

Let us denote $\omega = \frac{2\pi}{\lambda}$, and assume that the following conditions are satisfied.

(C1) The function $F(t)$ is Poisson. The coefficients are of two components such that $p(t) = p_0 + p_1(t)$, $q(t) = q_0 + q_1(t)$ and $r(t) = r_0 + r_1(t)$, where p_0, q_0 and r_0 are real constants, $p_1(t)$, $q_1(t)$, and $r_1(t)$ are Poisson functions;

(C2) there exists a Poisson sequence t_k common for functions $p_1(t)$, $q_1(t)$, $r_1(t)$, $F(t)$ and the Poisson number κ_ω is equal to zero;

(C3) $p_0 > 0, \quad p_0^2 - 4q_0 \leq 0.$

From the conditions (C1) and (C2), it can be easily shown that $p(t)$, $q(t)$, $r(t)$ and $F(t)$ are Poisson functions with the common Poisson sequence t_k. The two-component presentation of the coefficients is convenient to specify circumstances of stability as well as the recurrent properties of the model.

In this paper, we will make use of the norm $\|v\| = \max(\mid v_1 \mid, \mid v_2 \mid)$, for a vector $v = (v_1, v_2)$, and corresponding norm for square matrices will be utilized.

Condition (C3) implies that the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 \\ -q_0 & -p_0 \end{pmatrix}$ have negative real parts and there exist numbers $K > 1$ and $\mu > 0$ such that $\|e^{At}\| \leq Ke^{-\mu t}$ for $t \geq 0$.

We will consider the equation (1.2) provided that a solution $x(t)$ and its derivative $x'(t)$ are bounded such that $\sup_{t\in\mathbb{R}} \mid x(t) \mid < H$, $\sup_{t\in\mathbb{R}} \mid x'(t) \mid < H$, where H is a fixed positive number.

For convenience, we introduce some notations

$$\sup_{t\in\mathbb{R}} \mid p_1(t) \mid = \alpha, \sup_{t\in\mathbb{R}} \mid q_1(t) \mid = \beta, \sup_{t\in\mathbb{R}} \mid r_0 + r_1(t) \mid = \gamma, \sup_{t\in\mathbb{R}} \mid F(t) \mid = \delta.$$

Throughout the paper, the following additional conditions are required.

(C4) $\dfrac{K}{\mu}(H(\alpha+\beta) + \gamma H^3 + \delta) < H;$

(C5) $\dfrac{K}{\mu}(\alpha+\beta+3\gamma H^2) < 1.$

Theorem 2.1. *If conditions (C1)-(C5) are valid, then the Duffing equation (1.2) admits a unique asymptotically stable Poisson solution.*

Proof. Let us rewrite the equation (1.2) as the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -q(t)x_1 - p(t)x_2 - r(t)x_1^3 + F(t)cos(\lambda t). \end{aligned} \tag{2.3}$$

We will consider the system (2.3) in the matrix form

$$y' = Ay + B(t)y + C(t,y) + G(t), \tag{2.4}$$

where $y(t) = column(y_1(t), y_2(t))$,

$$A = \begin{pmatrix} 0 & 1 \\ -q_0 & -p_0 \end{pmatrix}, B(t) = \begin{pmatrix} 0 & 0 \\ -q_1(t) & -p_1(t) \end{pmatrix},$$
$$C(t,y) = \begin{pmatrix} 0 \\ -(r_0 + r_1(t))y_1^3 \end{pmatrix}, G(t) = \begin{pmatrix} 0 \\ F(t)cos(\lambda t) \end{pmatrix}.$$

Let us prove that the function $G(t)$ is Poisson stable. Since $\kappa_\omega = 0$, there exists a subsequence t_{k_l} such that $t_{k_l} \to 0 (mod\, \omega)$ as $l \to \infty$. Without loss of generality assume that $t_k \equiv \tau_k (mod\, \omega)$ and $\tau_k \to 0$ as $k \to \infty$. Fix a positive number ϵ, and a bounded interval $I \subset \mathbb{R}$. Since $\cos(\lambda t)$ is a continuous periodic function, there exists a natural number k_1 such that

$$\begin{aligned} &\mid \cos(\lambda(t+t_k)) - \cos(\lambda t) \mid = \mid \cos(\lambda t + 2m\pi + \lambda\tau_k) - \cos(\lambda t) \mid = \\ &\mid \cos(\lambda(t+\tau_k) - \cos(\lambda t) \mid \leq \frac{\epsilon}{2\delta}, \end{aligned}$$

where $m \in \mathbb{N}$, for all $t \in \mathbb{R}$ and $k > k_1$. Moreover, for the Poisson function $F(t)$ there exists a natural number k_2 such that

$$\mid F(t+t_k) - F(t) \mid \leq \frac{\epsilon}{2}$$

for all $t \in I$ and $k > k_2$. Finally, we obtain that

$$\|G(t+t_k) - G(t)\| =| F(t+t_k)\cos(\lambda(t+t_k)) - F(t)cos(\lambda t) | \leq$$
$$| F(t+t_k) || \cos(\lambda(t+t_k)) - \cos(\lambda t) | + | \cos(\lambda t) || F(t+t_k) - F(t) | \leq$$
$$\delta\frac{\epsilon}{2\delta} + \frac{\epsilon}{2} < \epsilon,$$

for all $t \in I$ and $k > \max(k_1, k_2)$. That is why, $G(t)$ is the Poisson function.

From condition (C3) it implies that a bounded on the real axis function $z(t)$ is a solution of system (2.4) if and only if it satisfies the equation

$$z(t) = \int_{-\infty}^{t} e^{A(t-s)}[B(s)z(s) + C(s, z(s)) + G(s)]ds,\ t \in \mathbb{R}. \tag{2.5}$$

Denote by U the Banach space of all Poisson functions $v(t) = column(v_1(t), v_2(t))$, with common Poisson sequence t_k such that $\|v(t)\|_0 < H$, where $\|v(t)\|_0 = \sup\limits_{\mathbb{R}} \|v(t)\|$.

Define on U the operator Φ as

$$\Phi v(t) = \int_{-\infty}^{t} e^{A(t-s)}(B(s)v(s) + C(s, v(s)) + G(s))ds,\ t \in \mathbb{R}. \tag{2.6}$$

Let us show that Φ is invariant in U. Fix a function $v(t)$ that belongs to U. We have that

$$\|\Phi v(t)\| \leq \int_{-\infty}^{t} \|e^{A(t-s)}\|(\|B(s)\|\|v(s)\| + \|C(s, v(s))\| + \|G(s)\|)ds \leq$$
$$\frac{K}{\mu}((\alpha+\beta)H + \gamma H^3 + \delta)$$

for all $t \in \mathbb{R}$. Therefore, by the condition (C4) it is true that $\|\Phi v\|_0 < H$.

Next by applying the method of included intervals [18], we will show that $\Phi v(t)$ is a Poisson function with the sequence t_k. Fix an arbitrary positive number ϵ and a closed interval $[a, b]$, $-\infty < a < b < \infty$, of the real axis. Let us choose two numbers $c < a$, and $\xi > 0$ satisfying

$$\frac{K}{\mu}(H(\alpha+\beta) + \gamma H^3 + \delta)e^{-\mu(a-c)} < \frac{\epsilon}{4}, \tag{2.7}$$

$$\frac{K}{\mu}\xi(\alpha+\beta+H+3\gamma H^2 + H^3 + 1)[1 - e^{-\mu(b-c)}] < \frac{\epsilon}{2}. \tag{2.8}$$

Since, $B(t)$ is a Poisson matrix, $G(t)$ and $r_1(t)$ are Poisson functions, and the function $v(t)$ belongs to U, for sufficiently large k the following inequalities are valid $\|B(t+t_k) - B(t)\| < \xi$, $\|G(s+t_k) - G(s)\| < \xi$, $| r_1(t+t_k) - r_1(t) | < \xi$ and $\|v(t+t_k) - v(t)\| < \xi$ for $t \in [c, b]$.

We obtain that

$$
\begin{aligned}
&\|\Phi v(t+t_k)-\Phi v(t)\| = \\
&\|\int_{-\infty}^{t} e^{A(t-s)}(B(s+t_k)v(s+t_k)+C(s+t_k,v(s+t_k))+G(s+t_k))ds - \\
&\int_{-\infty}^{t} e^{A(t-s)}(B(s)v(s)+C(s,v(s))+G(s))ds\| \le \\
&\|\int_{-\infty}^{t} e^{A(t-s)}(B(s+t_k)v(s+t_k)-B(s)v(s)+C(s+t_k,v(s+t_k))-C(s,v(s))+ \\
&G(s+t_k)-G(s))ds\| \le \int_{-\infty}^{c} \|e^{A(t-s)}\|\Big(\|B(s+t_k)v(s+t_k)-B(s)v(s)\|+ \\
&\|C(s+t_k,v(s+t_k))-C(s,v(s))\|+\|G(s+t_k)-G(s)\|\Big)ds+ \\
&\int_{c}^{t}\|e^{A(t-s)}\|\Big(\|B(s+t_k)(v(s+t_k)-v(s))\|+\|v(s)(B(s+t_k)-B(s))\|\Big)ds+ \\
&\int_{c}^{t}\|e^{A(t-s)}\|\Big(\|C(s+t_k,v(s+t_k))-C(s+t_k,v(s))\|+ \\
&\|C(s+t_k,v(s))-C(s,v(s))\|\Big)ds+\int_{c}^{t}\|e^{A(t-s)}\|\|G(s+t_k)-G(s)\|ds \le \\
&\frac{2K}{\mu}((\alpha+\beta)H+\gamma H^3+\delta)e^{-\mu(a-c)}+\frac{K}{\mu}(\xi(\alpha+\beta)+H\xi)[1-e^{-\mu(b-c)}]+ \\
&\frac{K}{\mu}(3\xi\gamma H^2+\xi H^3)[1-e^{-\mu(b-c)}]+\frac{K}{\mu}\xi[1-e^{-\mu(b-c)}]
\end{aligned}
$$

is correct for all $t\in[a,b]$. From inequalities (2.7) and (2.8) it follows that $\|\Phi v(t+t_k)-\Phi v(t)\|<\epsilon$ for $t\in[a,b]$. Therefore, the sequence $\Phi v(t+t_k)$ uniformly converges to $\Phi v(t)$ on each bounded interval of $\mathbb{R}$. That is, $\Phi v(t)$ is a Poisson function.

The function $\Phi v(t)$ is a uniformly continuous, since its derivative is a uniformly bounded on the real axis. Thus, the set U is invariant for the operator Φ.

Let us show that the operator $\Phi: U\to U$ is contractive. For any $\varphi(t),\psi(t)\in U$, one can attain that

$$
\begin{aligned}
&\|\Phi\varphi(t)-\Phi\psi(t)\| \le \\
&\int_{-\infty}^{t}\|e^{A(t-s)}\|(\|B(s)\|\|\varphi(s)-\psi(s)\|+\|C(s,\varphi(s))-C(s,\psi(s))\|)ds \le \\
&\frac{K}{\mu}\Big((\alpha+\beta)\|\varphi(t)-\psi(t)\|_0+\gamma(|\,\varphi_1^2(t)\,|+|\,\varphi_1(t)\,||\,\psi_1(t)\,|+|\,\psi_1^2(t)\,|)\|\varphi(t)-\psi(t)\|_0 < \\
&\frac{K}{\mu}(\alpha+\beta+3\gamma H^2)\|\varphi(t)-\psi(t)\|_0.
\end{aligned}
$$

Therefore, the inequality $\|\Phi\varphi-\Phi\psi\|_0<\dfrac{K}{\mu}(\alpha+\beta+3\gamma H^2)\,\|\varphi-\psi\|_0$ holds, and according to the condition (C5) the operator $\Pi: U\to U$ is contractive.

By contraction mapping theorem there exists the unique fixed point, $z(t)$, of the operator Φ, which is the unique Poisson solution of the equation (1.2).

Finally, let us discuss the asymptotic stability of the solution $z(t)$. It is true that

$$
z(t)=e^{A(t-t_0)}z(t_0)+\int_{t_0}^{t}e^{A(t-s)}(B(s)z(s)+C(s,z(s))+G(s))ds.
$$

Denote by $\bar{z}(t)$ another solution of the equation (1.2) such that

$$\bar{z}(t) = e^{A(t-t_0)}\bar{z}(t_0) + \int_{t_0}^{t} e^{A(t-s)}(B(s)\bar{z}(s) + C(s, \bar{z}(s)) + G(s))ds.$$

Making use of the relation

$$\bar{z}(t) - z(t) = e^{A(t-t_0)}((\bar{z}(t_0) - z(t_0)) + \int_{t_0}^{t} e^{A(t-s)}\Big(B(s)(\bar{z}(s) - z(s)) + C(s, \bar{z}(s)) - C(s, z(s))\Big)ds,$$

one can obtain

$$\begin{aligned} &\|\bar{z}(t) - z(t)\| \leq \|e^{A(t-t_0)}\|\|\bar{z}(t_0) - z(t_0)\| + \\ &\int_{t_0}^{t} \|e^{A(t-s)}\|(\|B(s)\|\|\bar{z}(s) - z(s)\| + \|C(s, \bar{z}(s)) - C(s, z(s))\|)ds \leq \\ &Ke^{-\mu(t-t_0)}\|\bar{z}(t_0) - z(t_0)\| + \int_{t_0}^{t} Ke^{-\mu(t-s)}\Big((\alpha + \beta)\|\bar{z}(s) - z(s)\| + \\ &\gamma(\mid \bar{z}_1^2(s) \mid + \mid \bar{z}_1(s) \mid\mid z_1(t) \mid + \mid z_1^2(s) \mid)\|\bar{z}(s) - z(s)\|\Big)ds \leq \\ &\frac{K}{\mu}(\alpha + \beta + 3\gamma H^2)\|\bar{z}(t) - z(t)\|, \end{aligned} \tag{2.9}$$

for $t \in \mathbb{R}$. With the aid of the Gronwall-Bellman Lemma, one can verify that

$$\|\bar{z}(t) - z(t)\| \leq Ke^{(K(\alpha+\beta+3\gamma H^2)-\mu)(t-t_0)}\|\bar{z}(t_0) - z(t_0)\|, \tag{2.10}$$

for all $t \geq t_0$, and condition (C5) implies that the Poisson solution, $z(t)$, is asymptotically stable solution of the equation (1.2). The theorem is proved. □

3. A NUMERICAL EXAMPLE

In [42], construction of a Poisson sequence was performed as the solution of the logistic equation

$$\lambda_{i+1} = \nu\lambda_i(1 - \lambda_i). \tag{3.11}$$

More precisely, it was proved that for each $\nu \in [3 + (2/3)^{1/2}; 4]$ there exists a Poisson solution $\{\gamma_i\}$, $i \in \mathbb{Z}$, of equation (3.11), which belongs to the interval $[0; 1]$.

The Poisson function [17], $\Theta(t)$, is a unique bounded on the real axis solution of differential equation $\Theta' = -3\Theta + \Omega(t)$, where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \gamma_i$, for $t \in [i; i+1)$, $i \in \mathbb{Z}$. In this paper we use the function notation $\Omega(t) = \Omega_{(\nu;q)}(t)$, where q denotes the length of the intervals on which the function $\Omega(t)$ is built.

Consider the following Duffing equation

$$x''(t) + (p_0 + p_1(t))x'(t) + (q_0 + q_1(t)) + (r_0 + r_1(t))x^3(t) = F(t)\cos(2t),$$

where the constants $p_0 = 4.5, q_0 = 5, r_0 = 0.006$. One can prove [17, 43] that the functions $p_1(t) = -0.2\int_{-\infty}^{t} e^{-3.5(t-s)}\Omega_{(3.9:3\pi)}ds$, $q_1(t) = 0.1(\int_{-\infty}^{t} e^{-2.5(t-s)}\Omega_{(3.9:3\pi)}ds)^2$, $r_1(t) = -0.5(\int_{-\infty}^{t} e^{-5(t-s)}\Omega_{(3.9:3\pi)}ds)^3$ and $F(t) = 2.7\int_{-\infty}^{t} e^{-3(t-s)}\Omega_{(3.9:3\pi)}ds$ are Poisson stable. Conditions (C1)-(C5) are hold with $K = 4.34$, $\mu = 2$, $\lambda = 2$ (the period $\omega = \pi$), $H = 3$, $\alpha = 0.06$, $\beta = 0.02$, $\gamma = 0.01$ and $\delta = 0.9$. Since the elements of the Poisson sequence are multiples of 3π and the period is equal to π, the Poisson number is equal to zero.

In Figures 1 and 2 the simulations of the coefficient $p(t)$ and excitation $F(t)\cos(2t))$ are shown.

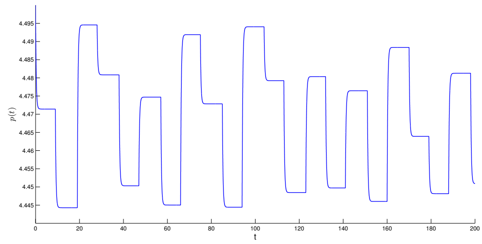

FIGURE 1. The graph of the coefficient $p(t)$.

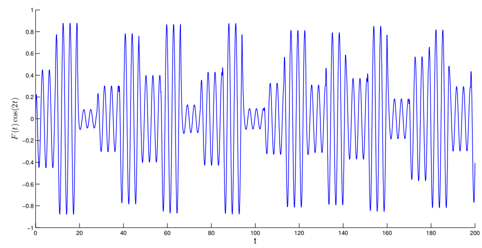

FIGURE 2. The graph of the excitation $F(t)\cos(2t))$.

According to Theorem 2.1, the equation (3.12) possesses a unique asymptotically stable Poisson solution. Since, it is impossible to determine the initial value for the Poisson solution, we will utilize the following illustration. By the asymptotic property, any solution from the domain ultimately approaches the Poisson solution, $z(t)$, of the system (3.12). That is, to visualize the behavior of the Poisson solution $z(t)$, we consider the simulation of another solution $x(t)$, with initial values $x_1(0) = x_2(0) = 0$. Applying (2.10) one can obtain that

$$\|x(t) - z(t)\| \le e^{-0.48t}\|x(0) - z(0)\| \le 2He^{-0.48t}, \tag{3.12}$$

for all $t \ge 0$. Thus, if $t > (\ln 6 + 5\ln 10)/0.48 \approx 27.72$, then $\|x(t) - z(t)\| < 10^{-5}$. The last inequality demonstrates that the difference $x(t) - z(t)$ asymptotically diminishes. Consequently, the graph of function $x(t)$ approaches the Poisson solution $z(t)$ of the system (3.12), as time increases. That is why, instead of the curve describing the Poisson solution, one can consider the graph of $x(t)$. The coordinates and trajectory of the solution $x(t)$, which asymptotically converges to the Poisson solution $z(t)$, are shown in Figures 3 and 4, respectively.

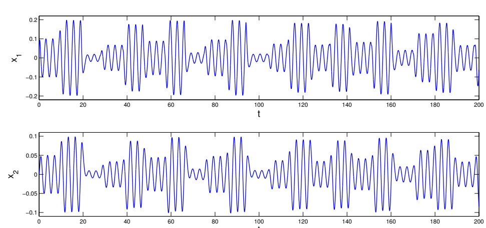

FIGURE 3. The coordinates of the solution $x(t)$ with initial values $x_1(0) = x_2(0) = 0$.

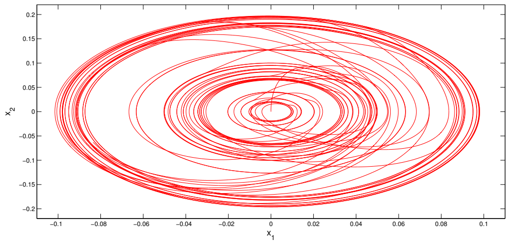

FIGURE 4. The trajectory of the solution $x(t)$.

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