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On the crossing numbers of the join products of five graphs on six vertices with discrete graphs

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ABSTRACT. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. In the paper, the crossing number of the join product $G^* + D_n$ for the connected graph G^* on six vertices consisting of one path on four vertices P_4 and two leaves adjacent with the same outer vertex of the path P_4 is given, where D_n consists of n isolated vertices. Finally, by adding some edges to the graph G^*, we obtain the crossing numbers of the join products of other four graphs with D_n.

1. INTRODUCTION

The issue of reducing the number of crossings is interesting in a lot of areas. One of the most popular areas is the implementation of the VLSI layout, which has revolutionized circuit design and has had a strong effect on parallel calculations. Crossing numbers have also been studied to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of clarity of the graphic drawings, some reduction of an edge crossing is probably the most important. Therefore, examining the number of crossings of simple graphs is a classic but very challenging problem. Garey and Johnson [8] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem.

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing* see Klešč [16]). One can easily verify that a drawing with the minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no two edges cross more than once, no edge crosses itself, and also no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by [18], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

A survey of the exact values of the crossing numbers for several families of graphs can be found by Clancy *et al.* [6]. The purpose of this paper is to extend the known results concerning this topic. Some parts of proofs will be based on Kleitman's result [14] on the crossing numbers for some complete bipartite graphs $K_{m,n}$. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad \min\{m,n\} \le 6. \tag{1.1}$$

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The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices, and let P_n be the *path* on n vertices. The exact values for the crossing numbers of $G + D_n$ for all graphs G of order at most four are given by Klešč and Schrötter [21]. Also, the crossing numbers of the graphs $G + D_n$ are known for a lot of graphs G of order five and six [1, 5, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22, 23, 26, 27, 29, 30, 33, 34, 35, 36]. In all these cases, the graph G is connected and contains usually at least one cycle. The crossing numbers of the join product $G + D_n$ are known only for some disconnected graphs [4, 24, 25, 31, 32].

The methods in the paper mostly use the multiple combinatorial properties of the cyclic permutations. Let $G^* = (V(G^*), E(G^*))$ be the connected graph of order six consisting of one path P_4 and two leaves adjacent with the same outer vertex of the path P_4, and let also $V(G^*) = \{v_1, v_2, \dots, v_6\}$. As $|E(G^*)| < |V(G^*)|$, we were unable to determine the crossing number of the join product $G^* + D_n$ using the methods used in [18] and [22]. The crossing number of $G^* + D_n$ equal to $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ is determined in Theorem 3.1 with the proof that is strongly based on various cases of a fixation mainly thanks to Lemma 3.2. The crossing numbers of $G_k + D_n$ for four other graphs G_k on six vertices are given in Corollary 4.1 by adding new edges to the graph G^*. The paper concludes by giving the crossing numbers of the join products of G^*, G_1, and G_2 with the paths P_n in Corollary 4.2 with the same values such as for the discrete graphs D_n. Certain parts of proofs can be also simplified with the help of software COGA which generates all cyclic permutations of six elements and its description can be found in Berežný and Buša [3]. In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. Cyclic permutations and possible drawings of G^*

We consider the join product of the graph G^* with the discrete graph D_n, which yields that the graph $G^* + D_n$ (sometimes used notation $G^* + nK_1$) consists of just one copy of G^* and n vertices $t_1, t_2, \dots, t_n$. Here, each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of the graph G^*. Let T^i, $1 \le i \le n$, denote the subgraph induced by the six edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{6,n}$ and

$$G^* + D_n = G^* \cup K_{6,n} = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.2}$$

Let D be a good drawing of the graph $G^* + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [9] or Woodall [37]. We use the notation (123456) if the counter-clockwise order the edges is incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. We emphasize that a rotation is a cyclic permutation; that is, (123456), (234561), (345612), (456123), (561234), and (612345) denote the same rotation. Thus, $6!/6 = 120$ different $\mathrm{rot}_D(t_i)$ can appear in a drawing of the graph $G^* + D_n$. By $\overline{\mathrm{rot}}_D(t_i)$, we understand the inverse permutation of $\mathrm{rot}_D(t_i)$. In the given drawing D, all subgraphs T^i, $i = 1, \dots, n$ of the graph $G^* + D_n$ are divided into three mutually disjoint subsets depending on how many times the edges of the subgraph T^i cross the edges of G^* in D. For $i = 1, \dots, n$, let $T^i \in R_D$ if $\mathrm{cr}_D(G^*, T^i) = 0$, and $T^i \in S_D$ if $\mathrm{cr}_D(G^*, T^i) = 1$. Every other subgraph T^i crosses the edges of G^* at least twice in D.

Moreover, let F^i denote the subgraph $G^* \cup T^i$ for some $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given subdrawing of G^* in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$. Note that if D is a good drawing of $G^* + D_n$ with the empty set R_D, then $\sum_{i=1}^{n} \mathrm{cr}_D(G^*, T^i) \geq n$ enforces at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings in D provided by

$$\mathrm{cr}_D(G^* + D_n) \geq \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(G^*, K_{6,n}) \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

According to the expected result of the main Theorem 3.1, this leads to a consideration of the nonempty set R_D in all good drawings of $G^* + D_n$. Moreover, we can redraw a crossing of two edges of G^* to get a new subdrawing of G^* induced by D (with vertex notation in a different order) with fewer edge crossings in three cases presented in Fig. 1, and so, the proof of Lemma 2.1 can be omitted.

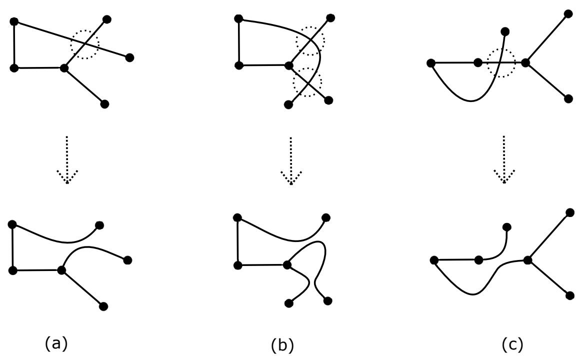

FIGURE 1. Elimination of a crossing in G^*.

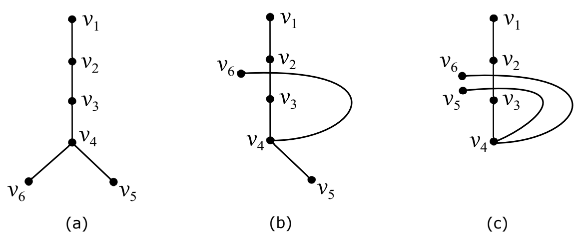

FIGURE 2. One planar drawing of G^* and two drawings of G^* with $\mathrm{cr}_D(G^*) \geq 1$.

Lemma 2.1. *In an effort to obtain a drawing D of the join product of the graph G^* with the discrete graph D_n with the smallest numbers of crossings, either the edges of G^* do not cross each other or it is not possible to eliminate any crossing on the edges of G^* by redrawing it. Moreover, the subdrawing of G^* induced by D is isomorphic to one of the three drawings depicted in Fig. 2.*

Remark that we would obtain a drawing of another graph in an effort to use some elimination for both subdrawings of G^* in Fig. 2(b) and (c). Assume a good drawing D of the graph $G^* + D_n$ in which the edges of G^* do not cross each other. In this case, without loss of generality, let the vertices of G^* be denoted in such a way as shown in Fig. 2(a). Our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G^*. Since there is unique subdrawing of $F^i \setminus \{v_2, v_3, v_4\}$ represented by the edge rotation (156) on the vertex t_i, we have three, two and two possibilities how to obtain the subdrawing of F^i depending on which region the edges t_iv_4, t_iv_3 and t_iv_2 are placed in, respectively. These $3 \times 2 \times 2 = 12$ possibilities under our consideration can be denoted by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ with corresponding indexes and we will call them by the *configurations* of corresponding subdrawings of the subgraph $F^i = G^* \cup T^i$ in D. The configuration of F^i is of type $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ in the drawing D, if there is a region of the subdrawing $D(F^i)$ with five, four, or at most three vertices of G^* on its boundary, respectively. For our purposes, it does not matter which of the regions is unbounded, and so we can assume that the drawings are as shown in Fig. 3.

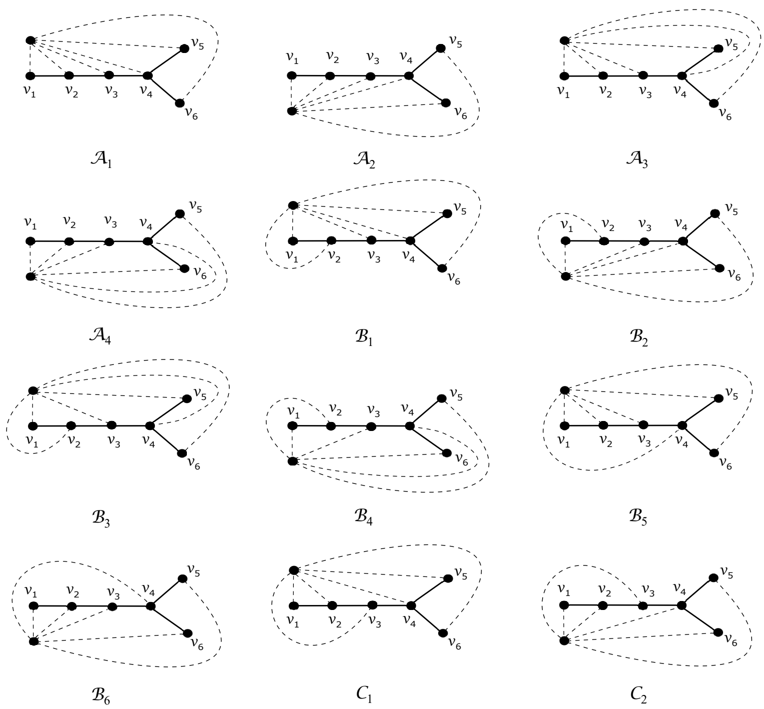

FIGURE 3. Twelve possible drawings of F^i with a configuration from $\mathcal{M}$.

In the rest of the paper, we represent each cyclic permutation by the permutation with 1 in the first position. The resulting descriptions of the twelve mentioned configurations are collected in Table 1. Clearly, in a fixed drawing D of $G^* + D_n$, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{C}_1, \mathcal{C}_2\}$ need not appear. We denote by $\mathcal{M}_D$ the set of all configurations that exist in the drawing D belonging to the set $\mathcal{M}$.

$\mathrm{conf}(F^i)$	$\mathrm{rot}_D(t_i)$	$\mathrm{conf}(F^i)$	$\mathrm{rot}_D(t_i)$
$\mathcal{A}_1$	(123456)	$\mathcal{A}_2$	(156432)
$\mathcal{A}_3$	(123546)	$\mathcal{A}_4$	(154632)
$\mathcal{B}_1$	(134562)	$\mathcal{B}_2$	(125643)
$\mathcal{B}_3$	(135462)	$\mathcal{B}_4$	(125463)
$\mathcal{B}_5$	(123564)	$\mathcal{B}_6$	(145632)
$\mathcal{C}_1$	(124563)	$\mathcal{C}_2$	(135642)

TABLE 1. The corresponding rotations of t_i for $F^i = G^* \cup T^i$, where $T^i \in R_D$.

Let $\mathcal{X}$, $\mathcal{Y}$ be the configurations from $\mathcal{M}_D$. We briefly denote by $\mathrm{cr}_D(\mathcal{X},\mathcal{Y})$ the number of crossings in D between T^i and T^j for two different $T^i, T^j \in R_D$ such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X},\mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X},\mathcal{Y})\}$ over all good drawings of the graph $G^* + D_n$ with $\mathcal{X},\mathcal{Y} \in \mathcal{M}_D$. Our aim shall be to establish $\mathrm{cr}(\mathcal{X},\mathcal{Y})$ for all pairs $\mathcal{X},\mathcal{Y} \in \mathcal{M}$. In particular, the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations (123456) and (156432), respectively. Since the minimum number of interchanges of adjacent elements of (123456) required to produce cyclic permutation $\overline{(156432)} = (123465)$ is one, any subgraph T^j with the configuration $\mathcal{A}_2$ of F^j crosses the edges of any T^i with the configuration $\mathcal{A}_1$ of F^i at least once, i.e., $\mathrm{cr}(\mathcal{A}_1,\mathcal{A}_2) \geq 1$. Details have been worked out by Woodall [37]. The similar reason gives remaining values in the symmetric Table 2, where $\mathcal{X}_p$ and $\mathcal{Y}_q$ are configurations of two different subgraphs F^i and F^j with $\mathcal{X},\mathcal{Y} \in \{\mathcal{A},\mathcal{B},\mathcal{C}\}$.

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathcal{A}_1$	6	1	5	2	5	2	4	3	4	3	4	3
$\mathcal{A}_2$	1	6	2	5	2	5	3	4	3	4	3	4
$\mathcal{A}_3$	5	2	6	3	6	3	5	4	5	5	6	4
$\mathcal{A}_4$	2	5	3	6	3	6	4	5	5	5	4	6
$\mathcal{B}_1$	5	2	6	3	6	6	5	5	6	4	6	4
$\mathcal{B}_2$	2	5	3	6	6	6	5	5	4	6	4	6
$\mathcal{B}_3$	4	3	5	4	5	5	6	6	5	6	6	5
$\mathcal{B}_4$	3	4	4	5	5	5	6	6	6	5	5	6
$\mathcal{B}_5$	4	3	5	5	6	4	5	6	6	6	6	5
$\mathcal{B}_6$	3	4	5	5	4	6	6	5	6	6	5	6
$\mathcal{C}_1$	4	3	6	4	6	4	6	5	6	5	6	6
$\mathcal{C}_2$	3	4	4	6	4	6	5	6	5	6	6	6

TABLE 2. The necessary number of crossings between T^i and T^j for the configurations $\mathcal{X}_p$, $\mathcal{Y}_q$.

3. THE CROSSING NUMBER OF $G^* + D_n$

Two vertices t_i and t_j of $G^* + D_n$ are *antipodal* in a drawing of $G^* + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipode-free* if it has no antipodal vertices. In the rest of the paper, each considered drawing of the graph $G^* + D_n$ will be assumed antipode-free. In the proof of the main Theorem 3.1, the following lemmas related to some restricted subdrawings of the graph $G^* + D_n$ are required.

Lemma 3.2. *For $n > 2$, let D be a good and antipode-free drawing of $G^* + D_n$ satisfying $|R_D| > \lceil \frac{n}{2} \rceil$, $|S_D| < \lfloor \frac{n}{2} \rfloor$, and $2|R_D| + |S_D| > \lceil \frac{3n}{2} \rceil$. Let T^{n-1}, $T^n \in R_D$ be two different subgraphs with $\mathrm{cr}_D(T^{n-1} \cup T^n) \geq 2$. Let α, β, γ be integers fulfilling the assumptions $\alpha + \beta + \gamma = 18$, $7 \leq \alpha \leq 10$, $4 \leq \beta \leq 6$, and $4 \leq \gamma \leq 5$. If the conditions*

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq \alpha \qquad \text{for any } T^i \in R_D \setminus \{T^{n-1}, T^n\}, \tag{3.3}$$

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq \beta \qquad \text{for any } T^i \in S_D, \tag{3.4}$$

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq \gamma \qquad \text{for any } T^i \notin R_D \cup S_D \tag{3.5}$$

hold, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings in D.

Proof. For easier reading, let $r = |R_D|$ and $s = |S_D|$. By the assumption of Lemma 3.2, $r \geq \lceil \frac{n}{2} \rceil + 1$, $-s \geq 1 - \lfloor \frac{n}{2} \rfloor$, and $2r + s \geq \lceil \frac{3n}{2} \rceil + 1$. The number of T^i that cross the graph G^* at least twice is equal to $n - r - s$. If the conditions (3.3), (3.4) and (3.5) hold, then by fixing of the graph $G^* \cup T^{n-1} \cup T^n$ we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + \alpha(r-2) + \beta s + \gamma(n-r-s) + 2 \geq 6\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + \Big\lfloor \frac{n}{2} \Big\rfloor,$$

where all possible values of the integers α, β, and γ are given in Table 3. Of course, there are a lot of possibilities using a software, e.g., Matlab to verify the last inequality in all mentioned cases.

α	7	8	8	9	9	10
β	6	5	6	4	5	4
γ	5	5	4	5	4	4

TABLE 3. Possibilities for $\alpha + \beta + \gamma = 18$, $7 \leq \alpha \leq 10$, $4 \leq \beta \leq 6$, and $4 \leq \gamma \leq 5$.

□

Lemma 3.3. *For $n > 2$, let D be a good and antipode-free drawing of $G^* + D_n$ with $\mathrm{cr}_D(G^*) = 0$ and with the vertex notation of G^* in such a way as shown in Fig. 2(a). If T^{n-1}, $T^n \in R_D$ are two subgraphs such that F^{n-1} and F^n have different configurations from any of the sets $\{\mathcal{A}_1, \mathcal{B}_6\}$ and $\{\mathcal{A}_2, \mathcal{B}_5\}$, then*

$$\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 4 \qquad \text{for any } T^j \in S_D,$$

i.e.,

$$\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^j) \geq 5 \qquad \text{for any } T^j \in S_D.$$

Proof. Let us assume the configurations $\mathcal{A}_1$ of F^n and $\mathcal{B}_6$ of F^{n-1}. The unique subdrawing $D(F^n)$ of the subgraph F^n contains just six regions with the vertex t_n on their boundaries. If we consider a subgraph $T^j \in S_D$ satisfying the restriction $\mathrm{cr}_D(T^n, T^j) = 1$, then the corresponding vertex t_j can only be placed in the region with the five vertices v_1, v_2, v_3, v_4, and v_6 of G^* on its boundary. Using this knowledge, the edges t_jv_1, t_jv_3, t_jv_4, and t_jv_6 produce no crossings on edges of F^n, and the edge t_jv_2 either does not cross any edge of F^n or crosses the edge t_nv_1. If the edges of F^n are not crossed by t_jv_2 and also t_jv_5 produces two crossings on edges of F^n, then $\mathrm{rot}_D(t_j) = (164532)$. If t_jv_2 crosses t_nv_1 and also t_jv_5 crosses v_4v_6 of G^*, then $\mathrm{rot}_D(t_j) = (126543)$. Since the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_j) = (164532)$ required to produce cyclic permutation $\overline{\mathrm{rot}}_D(t_{n-1}) = (123654)$ is four, the subgraph T^j crosses the edges of T^{n-1} at least four times. For the second case of $\mathrm{rot}_D(t_j) = (126543)$, the subdrawing of F^{n-1} enforces at least three crossings on the edges T^j provided by t_jv_5 must cross v_4v_6 of G^*. For both such subcases, we obtain the desired result $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 3 + 1 = 4$.

We can apply the same idea for the case, if there is a $T^j \in S_D$ with $\mathrm{cr}_D(T^{n-1}, T^j) = 1$ (but only with one possible subdrawing of T^j in which t_jv_5 and t_jv_6 crosses $t_{n-1}v_4$ and v_3v_4, respectively). It remains to consider the case where $\mathrm{cr}_D(T^n, T^j) \geq 2$ and $\mathrm{cr}_D(T^{n-1}, T^j) \geq 2$, which yields that $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 2 + 2 = 4$ trivially holds for any such $T^j \in S_D$. Due to the symmetry, the proof proceeds in the similar way for the second pair of configurations $\{\mathcal{A}_2, \mathcal{B}_5\}$, and this completes the proof of Lemma 3.3. □

We have to emphasize that we cannot generalize Lemma 3.3 for all pairs of different configurations from $\mathcal{M}_D$ with the number of crossings between T^{n-1} and T^n equal to three in Table 2. E.g., if we consider the configurations $\mathcal{A}_1$ of F^n and $\mathcal{C}_2$ of F^{n-1}, then the reader can easily find a subdrawing of $G^* \cup T^{n-1} \cup T^n \cup T^j$ in which $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^j) = 4$ with $T^j \in S_D$.

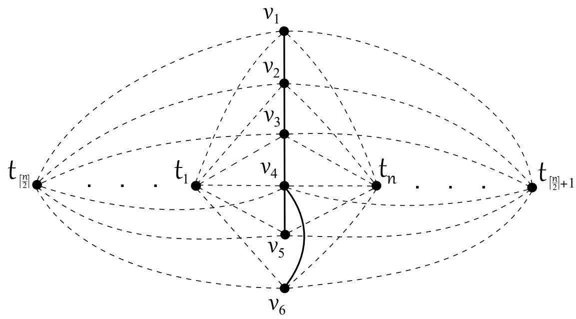

FIGURE 4. The good drawing of $G^* + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings.

Lemma 3.4. $\mathrm{cr}(G^* + D_1) = 0$ *and* $\mathrm{cr}(G^* + D_2) = 1$.

Proof. The graph $G^* + D_1$ is planar, hence $\mathrm{cr}(G^* + D_1) = 0$. Fig. 4 offers the subdrawing of $G^* + D_2$ with 1 crossing, and so $\mathrm{cr}(G^* + D_2) \leq 1$. The graph $G^* + D_2$ contains a subgraph isomorphic to $K_{3,3}$, and it is well-known by [14] that $\mathrm{cr}(K_{3,3}) = 1$. As $\mathrm{cr}(G^* + D_2) \geq \mathrm{cr}(K_{3,3}) = 1$, the proof of Lemma 3.4 is done. □

Theorem 3.1. $\mathrm{cr}(G^* + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Proof. By Lemma 3.4, the result is true for $n = 1$ and $n = 2$. In Fig. 4, the edges of $K_{6,n}$ cross each other

$$6\binom{\lceil \frac{n}{2} \rceil}{2} + 6\binom{\lfloor \frac{n}{2} \rfloor}{2} = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

times, each subgraph T^i, $i = 1, \dots, \lceil \frac{n}{2} \rceil$ on the left side does not cross the edges of G^* and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \dots, n$ on the right side crosses the edges of G^* exactly once. Thus, $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings appear among the edges of the graph $G^* + D_n$ in this drawing. To prove the reverse inequality by induction on n, suppose now that there is a good drawing D of $G^* + D_n$ with

$$\mathrm{cr}_D(G^* + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \quad \text{for some } n \geq 3, \tag{3.6}$$

and that

$$\mathrm{cr}(G^* + D_m) = 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \quad \text{for any integer } m < n. \tag{3.7}$$

If we use the notation $r = |R_D|$ and $s = |S_D|$, then $\mathrm{cr}_D(K_{6,n}) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ by (1.1) together with (3.6) force the following relation with respect to the edge crossings of G^* in D:

$$\mathrm{cr}_D(G^*) + \sum_{T^i \in R_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \in S_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \notin R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) < \left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$\mathrm{cr}_D(G^*) + 0r + 1s + 2(n - r - s) < \left\lfloor \frac{n}{2} \right\rfloor. \tag{3.8}$$

The mentioned inequality (3.8) subsequently enforces $r > \lceil \frac{n}{2} \rceil$, $s < \lfloor \frac{n}{2} \rfloor$, and $2r+s > \lceil \frac{3n}{2} \rceil$. As the set R_D is nonempty, we deal with the possibilities of obtaining a subgraph $T^i \in R_D$, and a contradiction with the assumption (3.6) will be reached in all considered subcases:

Case 1: $\mathrm{cr}_D(G^*) = 0$. Without loss of generality, we consider the subdrawing of G^* induced by D given in Fig. 2(a). Thus, we will deal with the configurations belonging to the nonempty set $\mathcal{M}_D$. Let us first show that the considered drawing D must be antipode-free. As a contradiction, suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using positive values in Table 2, one can easily to verify that $\{T^{n-1}, T^n\} \not\subseteq R_D$, i.e., $\mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \geq 1$. Again by (1.1), we know that $\mathrm{cr}_D(K_{6,3}) \geq 6$, which yields that any T^k, $k = 1, 2, \dots, n-2$, crosses the edges of $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G^* + D_n) = \mathrm{cr}_D(G^* + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n) +$$

$$+\mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 6(n-2) + 1 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

The obtained contradiction with the assumption (3.6) does not allow the existence of two antipodal vertices, that is, D is antipode-free.

(1) $\{\mathcal{A}_1, \mathcal{A}_2\} \subseteq \mathcal{M}_D$. Without lost of generality, let us assume two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then, $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq 7$ is fulfilling for any $T^i \in R_D$ with $i \notin \{n-1, n\}$ by summing the values in all columns in the first two rows of Table 2. Moreover, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 5$ holds for any subgraph $T^j \notin R_D$ provided by the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_{n-1})$ required to produce the cyclic permutation $\mathrm{rot}_D(t_n)$ is five. As $\mathrm{cr}_D(T^{n-1} \cup T^n) \geq 1$, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7(r-2) + 6s + 6(n-r-s) + 1 =$$

$$= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + r - 13 \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n +$$

$$+ \left\lceil \frac{n}{2} \right\rceil + 1 - 13 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

This contradicts the assumption (3.6), and so, suppose that $\{\mathcal{A}_1, \mathcal{A}_2\} \not\subseteq \mathcal{M}_D$.

(2) $\{\mathcal{A}_1, \mathcal{A}_4\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2, \mathcal{A}_3\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_1, \mathcal{B}_2\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2, \mathcal{B}_1\} \subseteq \mathcal{M}_D$. Let us consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{A}_4$, respectively. Then, the condition (3.3) for $\alpha = 8$ holds summing the values in two considered rows for eleven possible columns of Table 2. Since $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 4$ for any $T^j \notin R_D$ according to the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_{n-1})$ required to produce $\mathrm{rot}_D(t_n)$ is four, both conditions (3.4) and (3.5) for $\beta = 5$ and $\gamma = 5$ are trivially fulfilling. If F^{n-1} and F^n have different configurations from the set $\{\mathcal{A}_2, \mathcal{A}_3\}$ or $\{\mathcal{A}_1, \mathcal{B}_2\}$ or $\{\mathcal{A}_2, \mathcal{B}_1\}$, the same argument is applied. In the next part, we can suppose $\{\mathcal{A}_1, \mathcal{A}_4\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_2, \mathcal{A}_3\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_1, \mathcal{B}_2\} \not\subseteq \mathcal{M}_D$, and $\{\mathcal{A}_2, \mathcal{B}_1\} \not\subseteq \mathcal{M}_D$, that is, there are at least three crossings on the edges of $T^i \cup T^j$ for any two different subgraphs $T^i, T^j \in R_D$.

(3) $\{\mathcal{A}_1, \mathcal{B}_6\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2, \mathcal{B}_5\} \subseteq \mathcal{M}_D$. If $T^{n-1}, T^n \in R_D$ are two different subgraphs such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{B}_6$, respectively, then all three conditions (3.3), (3.4), and (3.5) also hold for $\alpha = 8$ using values of Table 2, $\beta = 5$ by Lemma 3.3, and $\gamma = 5$ provided by the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_{n-1})$ required to produce $\mathrm{rot}_D(t_n)$ is three. Due to the symmetry, the same argument can be applied for the case $\{\mathcal{A}_2, \mathcal{B}_5\} \subseteq \mathcal{M}_D$. In addition, suppose that $\{\mathcal{A}_1, \mathcal{B}_6\} \not\subseteq \mathcal{M}_D$ and $\{\mathcal{A}_2, \mathcal{B}_5\} \not\subseteq \mathcal{M}_D$.

To finish the proof of this case, suppose that $\{\mathcal{A}_1, \mathcal{C}_2\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_1, \mathcal{B}_4\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_3, \mathcal{A}_4\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_3, \mathcal{B}_2\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_4, \mathcal{B}_1\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2, \mathcal{C}_1\} \subseteq \mathcal{M}_D$ or

$\{\mathcal{A}_2,\mathcal{B}_3\} \subseteq \mathcal{M}_D$. If $T^{n-1}, T^n \in R_D$ are two subgraphs such that F^{n-1} and F^n have different configurations from one of the subsets mentioned above, then the conditions (3.3), (3.4), and (3.5) are true for $\alpha = 9$ using values of Table 2, and $\beta = 4$, $\gamma = 5$ due to the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_{n-1})$ required to produce $\mathrm{rot}_D(t_n)$ is three. As Lemma 3.2 contradicts (3.6) of D, let us also assume $\{\mathcal{A}_1,\mathcal{C}_2\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_1,\mathcal{B}_4\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_3,\mathcal{A}_4\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_3,\mathcal{B}_2\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_4,\mathcal{B}_1\} \not\subseteq \mathcal{M}_D$, $\{\mathcal{A}_2,\mathcal{C}_1\} \not\subseteq \mathcal{M}_D$, and $\{\mathcal{A}_2,\mathcal{B}_3\} \not\subseteq \mathcal{M}_D$.

(4) $\mathcal{C}_p \in \mathcal{M}_D$ for $p \in \{1,2\}$. In the rest of the paper, let $T^n \in R_D$ with the configuration $\mathcal{C}_p$ of $F^n = G^* \cup T^n$ for some $p \in \{1,2\}$. Since there are at most three vertices of G^* on its boundary in each region of $D(F^n)$, the edges of F^n must be crossed at least three times by any subgraph $T^j \notin R_D$. By fixing the subgraph F^n, we have

$$\mathrm{cr}_D(G^*+D_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4(r-1) + 3s + 3(n-r-s) =$$

$$= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n + r - 4 \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n + \Big\lceil\frac{n}{2}\Big\rceil + 1 - 4 \geq$$

$$\geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + \Big\lfloor\frac{n}{2}\Big\rfloor.$$

This also contradicts the assumption (3.6) of D, and therefore, let $\mathcal{C}_p \notin \mathcal{M}_D$ for $p = 1,2$.

Now, let us turn to a subcase with $\{\mathcal{A}_1,\mathcal{B}_3\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_1,\mathcal{B}_5\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2,\mathcal{B}_4\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_2,\mathcal{B}_6\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_3,\mathcal{B}_4\} \subseteq \mathcal{M}_D$ or $\{\mathcal{A}_4,\mathcal{B}_3\} \subseteq \mathcal{M}_D$ or $\{\mathcal{B}_1,\mathcal{B}_6\} \subseteq \mathcal{M}_D$ or $\{\mathcal{B}_2,\mathcal{B}_5\} \subseteq \mathcal{M}_D$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{B}_3$, respectively. Then, $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^i) \geq 10$ holds for any $T^i \in R_D$ with $i \notin \{n-1,n\}$ using values of Table 2. Moreover, the edges of $T^{n-1} \cup T^n$ are crossed at least twice by any subgraph $T^j \notin R_D$ according to the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_{n-1})$ required to produce $\mathrm{rot}_D(t_n)$ is two. As $\mathrm{cr}_D(T^{n-1} \cup T^n) \geq 4$, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\mathrm{cr}_D(G^*+D_n) \geq 6\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor + 10(r-2) + 3s + 4(n-r-s) + 4 =$$

$$= 6\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor + 4n + 6r - s - 16 \geq 6\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor +$$

$$+4n + 6\left(\Big\lceil\frac{n}{2}\Big\rceil + 1\right) + 1 - \Big\lfloor\frac{n}{2}\Big\rfloor - 16 \geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + \Big\lfloor\frac{n}{2}\Big\rfloor.$$

This also confirms a contradiction with (3.6) in D. Finally, let us also consider that the subsets of configurations mentioned above are not included in $\mathcal{M}_D$, that is, there are at least five crossings on the edges of $T^i \cup T^j$ for any two different subgraphs $T^i, T^j \in R_D$.

(5) $\mathcal{A}_p \in \mathcal{M}_D$ for $p \in \{1,\dots,4\}$ or $\mathcal{B}_q \in \mathcal{M}_D$ for $q \in \{1,\dots,6\}$. In the rest of the paper, let $T^n \in R_D$ with the configuration $\mathcal{A}_p$ or $\mathcal{B}_q$ of $F^n = G^* \cup T^n$ for some $p \in \{1,\dots,4\}$ or $q \in \{1,\dots,6\}$. Therewith, the antipode-free property of D forces that the edges of F^n are crossed at least two and three times by any subgraph $T^j \in S_D$ and $T^j \notin R_D \cup S_D$, respectively. By fixing the subgraph F^n, we have

$$\mathrm{cr}_D(G^*+D_n) \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 5(r-1) + 2s + 3(n-r-s) + 0 =$$

$$= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n + 2r - s - 5 \geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n+$$

$$+2\left(\left\lceil\frac{n}{2}\right\rceil+1\right)+1-\left\lfloor\frac{n}{2}\right\rfloor-5\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

Case 2. $\mathrm{cr}_D(G^*) \geq 1$, and we consider the subdrawing of G^* induced by D given in Fig. 2(b). Again, for subgraphs $T^i \in R_D$, we establish all possible rotations $\mathrm{rot}_D(t_i)$ which could appear in D. Clearly, there is only one subdrawing of $F^i \setminus \{v_2, v_4\}$ and can be represented by the subrotation (1536). We have just four possibilities of getting a subdrawing of $F^i = G^* \cup T^i$ depending on which of the two regions the edges t_iv_2 and t_iv_4 can be placed in. Thus, there are four different cyclic permutations for $\mathrm{rot}_D(t_i)$ with $\mathrm{cr}_D(G^*, T^i) = 0$, namely, the cyclic permutations (125436), (124536), (154362), and (145362). For any two different subgraphs $T^i,\ T^j \in R_D$, the edges of T^i are crossed by T^j at least four times because the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce cyclic permutation $\overline{\mathrm{rot}}_D(t_j)$ is at least four. In the rest of the paper, let $T^n \in R_D$. Since there are at most three vertices of G^* on its boundary in each region of $D(F^n)$, the edges of F^n must be crossed at least three times by any subgraph $T^k \notin R_D$. By fixing the subgraph F^n, we have

$$\mathrm{cr}_D(G^*+D_n)\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4(r-1)+3(n-r)+1=6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+$$

$$+3n+r-3\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+\left\lceil\frac{n}{2}\right\rceil+1-3\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

Finally, if we assume the subdrawing of the graph G^* induced by D given in Fig. 2(c), the same process as in the previous case can be applied (but only with two possible rotations (124356) and (143562) for subgraphs $T^i \in R_D$).

Thus, it was shown that there is no good drawing D of the graph $G^* + D_n$ with less than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$ crossings, and proof of Theorem 3.1 is done. □

4. FOUR OTHER GRAPHS

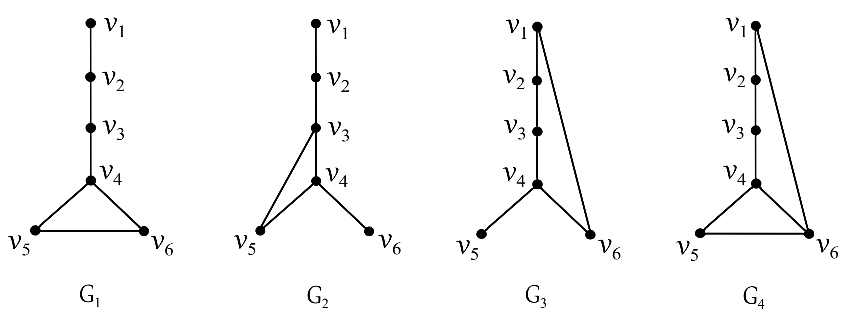

FIGURE 5. Four graphs G_1, G_2, G_3, and G_4 by adding new edges to the graph G^*.

In Fig. 5, let G_1 be the graph obtained from the planar drawing of G^* in Fig. 2(a) by adding the edge v_5v_6, i.e., $G_1 = G^* \cup \{v_5v_6\}$. Similarly, let $G_2 = G^* \cup \{v_3v_5\}$, $G_3 = G^* \cup \{v_1v_6\}$, and $G_4 = G^* \cup \{v_1v_6, v_5v_6\}$. Since we can add these edges v_1v_6, v_3v_5, and v_5v_6 to the graph G^* without additional crossings in at least one Fig. 4 or 6, the good drawings of $G_k + D_n$ with exactly $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$ crossings are obtained for all $k = 1, 2, 3, 4$.

Corollary 4.1. $\mathrm{cr}(G_k + D_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$ *for* $k = 1, 2, 3, 4$ *and* $n \geq 1$.

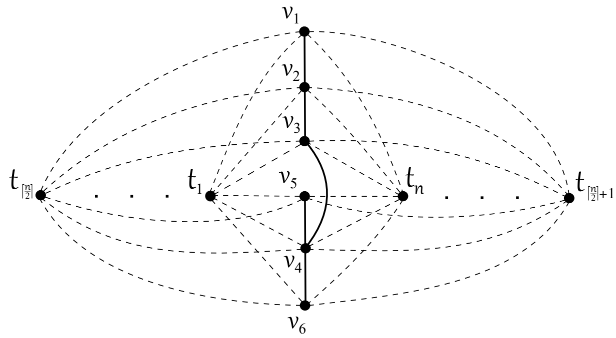

FIGURE 6. The good drawing of $G^* + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings.

Remark that the crossing number of the graph $G_4 + D_n$ was obtained in [20] without using the vertex rotation. Moreover, into both drawings in Fig. 4 and 6, it is possible to add $n-1$ edges which form the path P_n, $n \geq 2$ on the vertices of D_n without additional crossing. The same holds for the graph G_1 and G_2 in Fig. 4 and 6, respectively. Thus, the next results are also obvious.

Corollary 4.2. $\mathrm{cr}(G^* + P_n) = \mathrm{cr}(G_1 + P_n) = \mathrm{cr}(G_2 + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 2$.

5. CONCLUSIONS

A lot of attention began to be focused to the crossing number of crossing-critical graphs. A graph G is k-crossing-critical if its crossing number is at least k, but if we remove any edge of G, its crossing number drops below k. Some necessary conditions for k-crossing-criticality of graphs were described by Barát and Tóth [2], and Richter and Thomassen [28]. At this moment, we cannot extend the behavior of the graphs $G^* + D_n$ from the point of view of k-crossing-criticality, because we do not yet know the crossing numbers of $H + D_n$ for disconnected subgraphs H obtained by removing one edge from the graph G^*.

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REFERENCES

[1] Asano, K. The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$. *J. Graph Theory* **10** (1986), no. 1, 1–8.

[2] Barát, J.; Tóth, G. Improvement on the Crossing Number of Crossing-Critical Graphs. *Discrete Comput. Geom.* **67** (2022), no. 2, 595-604.

[3] Berežný, Š.; Buša, Jr., J. Algorithm of the Cyclic-Order Graph Program (Implementation and Usage). *J. Math. Model. and Geometry* **7** (2019), no. 3, 1–8.

[4] Berežný, Š.; Staš, M. Cyclic permutations and crossing numbers of join products of two symmetric graphs of order six. *Carpathian J. Math.* **35** (2019), no. 2, 137–146.

[5] Berežný, Š.; Staš, M. On the crossing number of join of the wheel on six vertices with the discrete graph. *Carpathian J. Math.* **36** (2020), no. 3, 381–390.

[6] Clancy, K.; Haythorpe, M.; Newcombe, A. A survey of graphs with known or bounded crossing numbers. *Australasian J. Combin.* **78** (2020), no. 2, 209–296.

[7] Ding, Z.; Huang, Y. The crossing numbers of join of some graphs with n isolated vertices. *Discuss. Math. Graph Theory* **38** (2018), no. 4, 899–909.

[8] Garey, M. R.; Johnson, D. S. Crossing number is NP-complete. *SIAM J. Algebraic. Discrete Methods* **4** (1983), no. 3, 312–316.

[9] Hernández-Vélez, C.; Medina, C.; Salazar, G. The optimal drawing of $K_{5,n}$. *Electron. J. Combin.* **21** (2014), no. 4, ♯ P4.1, 29 pp.

[10] Ho, P. T. The crossing number of $K_{1,m,n}$. *Discrete Math.* **308** (2008), no. 24, 5996–6002.

[11] Ho, P. T. The crossing number of $K_{1,1,3,n}$. *Ars Combinatoria* **99** (2011), 461–471.

[12] Ho, P. T. The Crossing Number of $K_{2,4,n}$. *Ars Combinatoria* **109** (2013), 527–537.
[13] Huang, Y.; Zhao, T. The crossing number of $K_{1,4,n}$. *Discrete Math.* **308** (2008), no. 9, 1634–1638.
[14] Kleitman, D. J. The crossing number of $K_{5,n}$. *J. Combinatorial Theory* **9** (1970), 315–323.
[15] Klešč, M. On the crossing numbers of products of stars and graphs of order five. *Graphs and Combinatorics* **17** (2001), no. 2, 289–294.
[16] Klešč, M. The join of graphs and crossing numbers. *Electron. Notes in Discrete Math.* **28** (2007), 349–355.
[17] Klešč, M. On the Crossing Numbers of Cartesian Products of Stars and Graphs on Five Vertices. *Combinatorial Algorithms*, Springer, LNCS **5874** (2009), 324–333.
[18] Klešč, M. The crossing numbers of join of the special graph on six vertices with path and cycle. *Discrete Math.* **310** (2010), no. 9, 1475–1481.
[19] Klešč, M.; Draženská, E. The crossing numbers of products of the graph $K_{2,2,2}$ with stars. *Carpathian J. Math.* **24** (2008), no. 3, 327–331.
[20] Klešč, M.; Kravecová, D.; Petrillová, J. The crossing numbers of join of special graphs. *Electrical Engineering and Informatics 2*: Proceeding of the Faculty of Electrical Engineering and Informatics of the Technical University of Košice, (2011), 522–527.
[21] Klešč, M.; Schrötter, Š. The crossing numbers of join products of paths with graphs of order four. *Discuss. Math. Graph Theory* **31** (2011), no. 2, 321–331.
[22] Klešč, M.; Schrötter, Š. The crossing numbers of join of paths and cycles with two graphs of order five. *Combinatorial Algorithms* Springer, LNCS, **7125** (2012), 160–167.
[23] Klešč, M.; Schrötter, Š. On the crossing numbers of Cartesian products of stars and graphs of order six. *Discuss. Math. Graph Theory* **33** (2013), no. 3, 583–597.
[24] Klešč, M.; Staš, M. Cyclic permutations in determining crossing numbers. *Discuss. Math. Graph Theory* **42** (2022), no. 4, 1163–1183.
[25] Klešč, M.; Staš, M.; Petrillová, J. The crossing numbers of join of special disconnected graph on five vertices with discrete graphs. *Graphs and Combinatorics* **38** (2022), no. 2, 35.
[26] Mei, H.; Huang, Y. The Crossing Number of $K_{1,5,n}$. *Int. J. Math. Combin.* **1** (2007), no. 1, 33–44.
[27] Ouyang, Z.; Wang, J.; Huang, Y. The crossing number of join of the generalized Petersen graph $P(3,1)$ with path and cycle. *Discuss. Math. Graph Theory* **38** (2018), no. 2, 351–370.
[28] Richter, R. B; Thomassen, C. Minimal graphs with crossing number at least k. *J. Comb. Theory* Ser. B **58** (1993), no. 2, 217–224.
[29] Staš, M. On the Crossing Numbers of the Joining of a Specific Graph on Six Vertices with the Discrete Graph. *Symmetry* **12** (2020), no. 1, 135.
[30] Staš, M. Determining Crossing Numbers of the Join Products of Two Specific Graphs of Order Six With the Discrete Graph. *Filomat* **34** (2020), no. 9, 2829–2846.
[31] Staš, M. On the crossing number of join product of the discrete graph with special graphs of order five. *Electron. J. Graph Theory Appl.* **8** (2020), no. 2, 339–351.
[32] Staš, M. On the crossing numbers of join products of five graphs of order six with the discrete graph. *Opuscula Math.* **40** (2020), no. 3, 383–397.
[33] Staš, M. On the crossing numbers of the join products of six graphs of order six with paths and cycles, *Symmetry* **13** (2021), no. 12, 2441.
[34] Staš, M. On the Crossing Numbers of Join Products of Four Graphs of Order Six With the Discrete Graph. *Azerbaijan Journal of Mathematics* **12** (2022), no. 1, 80–97.
[35] Su, Z.; Klešč, M. Crossing Numbers of $K_{1,1,4,n}$ and $K_{1,1,4}\Box T$. *Ars Combinatoria* **148** (2020), 137–148.
[36] Wang, Y.; Huang, Y. The crossing number of Cartesian product of 5-wheel with any tree. *Discuss. Math. Graph Theory* **41** (2021), no. 1, 183–197.
[37] Woodall, D. R. Cyclic-order graphs and Zarankiewicz's crossing number conjecture. *J. Graph Theory* **17** (1993), no. 6, 657–671.

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