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# Aggregation Function Constructed from Copula

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ABSTRACT. In this work, we show that a slight change in the Sklar's formula tremendously affects the class of aggregation functions it represents. While the original formula can only be used to construct $d$-increasing aggregation functions, this new formula can be used to construct any continuous aggregation function excepted possibly those belong to the boundary of this set. In particular, all continuous aggregation functions can be approximated by aggregation functions in this form. This shows that it is sufficient to only consider aggregation functions in this form for most cases. Construction examples via this method are also given.

## 1. INTRODUCTION

In order to get information over a sheer volume of data existed nowadays, we are forced to aggregate data. For instance, we might use the average to represent a set of data. Other aggregation functions might also be used depending on the situation. A natural question is whether we can classify aggregation functions or, more importantly, whether we can use this classification to construct an appropriate aggregation function. After all, the more choices we have, the better it is to find an appropriate one.

Over the years, several construction methods have been introduced. This includes, for example, the constructions based on partial information such as constraints [7, 11], forms [20, 15, 16], properties [25, 32], or values on subsets of domains such as diagonal sections and curved sections [2, 8, 17, 19, 21, 29, 34, 43]. Another example is the transformations of aggregation functions such as flippings [14, 24], polynomial transformations [3, 6, 10, 33, 40, 42, 41], compositions [18, 31] and others [27, 28, 30]. Penalty-based constructions also gain interest in recent years [1, 4, 5, 13, 12, 23, 35, 36, 39, 42, 38].

We are interested in Sklar's copula-based method [37]. Originally, Sklar used copulas to construct multivariate distribution functions. Specifically, he demonstrates that any multivariate distribution function can be constructed as a composition of a copula and its marginal distribution functions. Following his idea, several copula construction methods have been proposed in literature.

Mathematically, multivariate distribution functions can be considered as a special type of aggregation functions. Thus, it is natural to question whether the Sklar's method can be extended to aggregation functions. This has been shown to be true for bivariate 2-increasing aggregation functions [18]. We also believe that the proof can also be extended to the multivariate k-increasing aggregation functions. This two-step method simplifies the selection process of aggregation functions. The abundance of copulas constructed over the years is also helpful in constructing aggregation functions. Therefore, we want to adjust this method for a reasonable larger class of aggregation functions.

In this work, we make a slight change to the Sklar's formula. This can be viewed as a mixture of transformation methods and composition methods. Nevertheless, this new formula allows us to construct aggregation functions belonged to a dense class in the

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space of aggregation functions including strictly increasing aggregation functions. In fact, this method only fails to construct aggregation functions lying in the boundary. Therefore, it should be widely adaptable.

In the next section, we will review basic terminologies used throughout this work. In Section 3, we will present our construction along with some examples. Its properties will be presented in Section 4, however.

## 2. PRELIMINARIES

For simplicity, we will assume that the data is numerical with values in the unit interval $\mathbb{I} = [0,1]$. An element of $\mathbb{I}^k$ will be denoted by the vector notation $\vec{x} = (x_1, \ldots, x_k)$ with special cases for the zero vector $\vec{0} = (0, \ldots, 0)$ and the vector $\vec{1} = (1, \ldots, 1)$. Denote also $\vec{e}_i = (e_{i1}, \ldots e_{ik})$ and $\vec{e}_{-i} = \vec{1} - \vec{e}_i$ where $e_{ij} = 1$ if $i = j$ and $e_{ij} = 0$ otherwise. Comparisons between two vectors will be done component-wise. For example, $\vec{x} < \vec{y}$ if and only if $x_i < y_i$ for all $i$. The notation $\vec{x} \lneq \vec{y}$ will stand for $\vec{x} \leq \vec{y}$ but $\vec{x} \neq \vec{y}$.

**Definition 2.1.** A function $A : \mathbb{I}^k \to \mathbb{I}$ is called an *aggregation function* if it is nondecreasing with $A\left(\vec{0}\right) = 0$ and $A\left(\vec{1}\right) = 1$. The volume $V_A$ of $A$ is defined by

$$V_A\left(\left(\vec{a}, \vec{b}\right]\right) = \sum_{\vec{x} \in \prod_{i=1}^k \{a_i, b_i\}} (-1)^{N(\vec{x}, \vec{a})} A(\vec{x})$$

for all interval $\left(\vec{a}, \vec{b}\right] \subseteq \mathbb{I}^k$. Here, $N(\vec{x}, \vec{a})$ is the number of $i$ such that $x_i = a_i$. A *k-variate* aggregation function is $k$-increasing if its volume is always non-negative. A *semi-copula* is an aggregation function $S : \mathbb{I}^k \to \mathbb{I}$ such that

$$S(x\vec{e}_i + \vec{e}_{-i}) = x \tag{2.1}$$

for all $x \in \mathbb{I}$ and all $i = 1, \ldots, k$. A $k$-increasing semi-copula is also called a *copula*. Equivalently, a copula is a $k$-increasing (aggregation) function $C$ such that $W \leq C \leq M$ pointwisely where

$$W(\vec{x}) = \max\left(0, \sum_{i=1}^k x_i - k + 1\right), \text{ and}$$
$$M(\vec{x}) = \min(x_1, \ldots, x_k)$$

for all $\vec{x} \in \mathbb{I}^k$.

It can also be shown that any semi-copula $S$ satisfies $L \leq S \leq M$ where

$$L(\vec{x}) = \begin{cases} x, & \vec{x} = x\vec{e}_i + \vec{e}_{-i}; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the average function Mean defined by

$$\mathrm{Mean}(x_1, \ldots, x_k) = \frac{1}{k}\sum_{i=1}^k x_i$$

for all $(x_1, \ldots, x_k) \in \mathbb{I}^k$ is an aggregation function which is not a semi-copula. Note that $M$ is a copula while $W$ and $L$ are semi-copulas. Therefore, semi-copulas are not necessary continuous. Copulas, on the other hand, are always continuous. In fact, a copula is 1-Lipschitz, that is,

$$|C(\vec{x}) - C(\vec{y})| \leq \sum_{i=1}^k |x_i - y_i|$$

for any copula $C$ and any $\vec{x}, \vec{y} \in \mathbb{I}^k$. It follows that the first-order partial derivatives of a copula exist with values lying in $\mathbb{I}$ almost everywhere.

The important of copulas immerses from Sklar Theorem. Given any $k$-variate copula $C$ and any univariate distribution functions $F_1, \ldots, F_k$, it can be easily proved that the function $F$ defined by

$$F(\vec{x}) = C(F_1(x_1), \ldots, F_k(x_k)) \tag{2.2}$$

for all $\vec{x} \in \mathbb{R}^k$ is a multivariate distribution function with marginals $F_1, \ldots, F_k$. Sklar Theorem [37] states that the converse of the above statement holds, that is, any multivariate distribution functions can be constructed via equation (2.2). This result allows us to model multivariate distribution functions in two steps – first model the marginals separately, then connect these marginals together using copulas. This method could also be applied to aggregation functions. In 2008, Durante et al. [18] show that this is possible for bivariate 2-increasing aggregation functions. A straightforward extension can also be done for $k$-increasing aggregation functions.

In this work, we will show that this two-step method is also possible for almost all aggregation functions if we were to replace copulas with linear combinations of copulas and the product function. Even though this is only a slight change comparing to equation (2.2) and the work of [18], its effect is surprising. This new formula can be used to construct all aggregation functions excepted possibly those that belong to the (geometric) boundary of this convex set. Therefore, we may also used copulas already constructed in literature to construct aggregation functions.

We will divide our results into two sections. The next section presents our construction method along with construction examples. Construction properties will be presented in Section 4.

## 3. CONSTRUCTION METHOD

In this section, we will present our construction method. We will defer the discussion of its properties, nevertheless, to the next section. Recall that a *scaling function* is simply a nondecreasing function on $\mathbb{I}$ sending 1 to 1.

$$\vec{A}(\vec{x}) = (A_1(x_1), \ldots, A_k(x_k))$$

for all $\vec{x} \in \mathbb{I}^k$ where $A_1, \ldots, A_k$ are scaling functions.

Denote $\Pi$ the product function and let $F \circ G$ stand for the composite function of $F$ and $G$. Given a scaling vector $\vec{A}$ and a constant $c \in \mathbb{R}$, we may construct a function $A$ by letting

$$A = (1-c)\, C \circ \vec{A} + c\Pi \circ \vec{A} \tag{3.3}$$

where the copula $C$ is chosen so that $\partial_i C \geq c(\partial_i C - \partial_i \Pi)$ a.e. on $\mathrm{Range}\left(\vec{A}\right)$ for all $i = 1, \ldots, k$ and

$$C \circ \vec{A}\left(\vec{0}\right) = -\left(\frac{c}{1-c}\right) \prod_{i=1}^{k} A_i(0).$$

This two conditions are also necessary for $A$ to be an aggregation function. Notice that for the copula $C$ to exists, the above equation must statisfies the Fréchet-Hoeffding bounds. That is, $C \circ \vec{A}\left(\vec{0}\right) \geq W \circ \vec{A}\left(\vec{0}\right)$. This implies $c \notin (0,1)$ when $A_i(0) > 0$ for all $i$.

Also, we only require the values of $C$ on $\mathrm{Range}\left(\vec{A}\right)$. Thus, we actually require subcopulas. Since any subcopula can be extended to a copula, the difference is subtle. Still, using only $\mathrm{Range}\left(\vec{A}\right)$ can increase a range of parameter(s) in (subcopula) family, see for instance, Example 3.1.

It will be proved in the next section that $A$ is indeed an aggregation function. Since there are already abundant families of copulas being constructed in literature, this method will be very useful in practice.

At first glance, this might seem to be a rather trivial extension from the case $c = 0$ in [18]. Its effect, nevertheless, is tremendous. Consider the case of quadratic aggregation functions, for example. Recall that any quadratic bivariate aggregation function is a convex combination of $\pi_{ij}$ and $\zeta_{ij}$, where $i, j \in \{1, 2\}$, defined by

$$\begin{aligned} \pi_{ij}\left(\vec{x}\right) &= x_i x_j, \\ \zeta_{ij}\left(\vec{x}\right) &= x_i + x_j - x_i x_j \end{aligned} \tag{3.4}$$

for all $\vec{x} = (x_1, x_2) \in \mathbb{I}^2$ [40]. Since there are 6 of such $\pi_{ij}$ and $\zeta_{ij}$, the set of quadratic bivariate aggregation functions can be represent by a octahedron with $\pi_{ij}$ and $\zeta_{ij}$ as its vertices. (See Figure (1).) Now, it can be easily proved that any aggregation function of the form

$$A = a_1\pi_{11} + a_2\pi_{22} + b_1\pi_{12} + b_2\zeta_{12} + c_1\zeta_{11} + c_2\zeta_{22}$$

where $b_1 < b_2$ is not 2-increasing. Thus, the set of such $A$ correspond to to the lower half of the octahedron in Figure 1 on page 386. Therefore, the case $c = 0$ in equation (3.3) can only represent half of quadratic bivariate aggregation functions. On the contrary, only 2 sides of the boundary of the octahedron can not be written in the form (3.3). This means the formula (3.3) can be used to constructed most of quadratic bivariate aggregation functions. See Corollary 4.1 for the proof of this fact. In general, we also know that any aggregation function can be approximated by aggregation functions constructed by the formula (3.3), see Theorem 4.2.

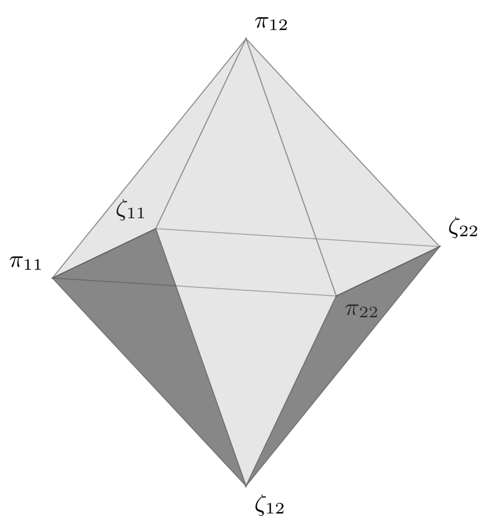


FIGURE 1. The set of bivariate quadratic aggregation functions presented as a solid octahedron, and its elements that can not be written in the form (3.3) (dark gray color).

In general, this construction method can produced almost all aggregation functions excepted possibly those that belong to the boundary of the set of aggregation functions.

This implies that it is sufficient to only consider aggregation functions in this form for most cases. This is the beauty of formula (3.3).

Next, we will provide examples for our construction method. The focus will be on bivariate aggregation functions. Construction of multivariate aggregation functions can be done analogously.

The first example provides a simple situation where we varies the scaling functions but fixed the copulas while the second example focuses on the usage of different copulas to construct distinct aggregation functions. It is also possible to vary both scaling functions and copulas using the idea from these examples.

**Example 3.1.** Assume that we know how the data should be combined together. For example, via the function $C_\theta$ defined by

$$C_\theta\left(u,v\right) = uv + \theta uv\left(1-u\right)\left(1-v\right)$$

for all $u, v \in [a, 1]$. Note that $C_\theta$ is the Eyraud-Farlie–Gumbel–Morgenstern (EFGM) copula when its domain is $\mathbb{I}^2$ and $-1 \leq \theta \leq 1$. Notice, however, that $(1-c)\, C_\theta + c\Pi = C_{(1-c)\theta}$ for any constant $c$. Thus, we might as well assume $c = 0$ but increase the possible range of $\theta$ to the whole real line. This means $C_\theta$ does not have to be a copula in this example.

The value $u\left(1-u\right)$ can be interpreted as the fuzziness of $u$ and similarly for $v\left(1-v\right)$. So $\theta uv\left(1-u\right)\left(1-v\right)$ represents overall fuzziness of the data $(u, v)$. Thus, $C_\theta\left(u, v\right)$ can be viewed as a combination between the truthful level and the fuzziness level of $(u, v)$.

Sometimes data have to be scaled before combined. In this case, the aggregation $A\left(u, v\right)$ of $(u, v)$ can be written as

$$A\left(u,v\right) = C_\theta\left(F\left(u\right), G\left(v\right)\right)$$

for some scaling functions $F : \mathbb{I} \to \mathbb{I}$ and $G : \mathbb{I} \to \mathbb{I}$. We will assume that $F\left(0\right) = G\left(0\right)$ for simplicity and denote this common value by $a$. To guarantee that $A$ is nondecreasing, we must have $\partial_1 C_\theta \geq 0$ and $\partial_2 C_\theta \geq 0$ on $[a, 1]^2$. Direct computation yields

$$1-\theta \geq 0, \text{ and } 1+\theta\left(1-2a\right) \geq 0$$

which is guaranteed, e.g., if we choose $\theta = -\left(1-a\right)^{-2}$. Note that this value of $\theta$ does not actually depend on the shape of $F$ or $G$ but only through the fact that $\mathrm{Range}\left(F\right), \mathrm{Range}\left(G\right) \subseteq [a, 1]$. In other words, there are several choices of $F$ and $G$ to choose from. For instance, we could choose $F$ to be one of the following functions.

(1) An exponential scaling of the form

$$F\left(u\right) = ae^{-u \ln a}$$

for all $u \in I_a$.

(2) A sigmoid function such as

$$F\left(u\right) = a + \alpha \tanh\left(u\right)$$

or

$$F\left(u\right) = a + \alpha \frac{u}{\sqrt{1+u^2}}$$

for all $u \in I_a$.

(3) A polynomial function of the form

$$F\left(u\right) = a + \alpha \int_a^u x^{k_0}\left(1-x\right)^{k_{l+1}} \prod_{i=1}^{l}\left(x-t_i\right)^{2k_i}\, dx \tag{3.5}$$

for all $u \in I_a$ where $a < t_i < 1$ are all distinct and $k_i$ are positive integers.

Here, $\alpha$ is simply a scaling constant forcing $F\left(1\right) = 1$.

**Example 3.2.** In this example, we will consider a construction of aggregation functions with fixed scaling functions. For instance, we may use the scaling function of the $\mathrm{Mean} : \mathbb{I} \to \mathbb{I}$ which is equal to

$$F(x) = \frac{1}{2}(1+x)$$

for all $x \in \mathbb{I}$. Now, we may consider an aggregation function $A : \mathbb{I}^2 \to \mathbb{I}$ in the form

$$A(x,y) = (1-c)\,C(F(x), F(y)) + cF(x)\,F(y)$$

for all $x, y \in \mathbb{I}$. Here, the constant $c$ and the copula $C$ must be chosen so that $0 = (1-c)\,C\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{4}c$,

$$\partial_1 C(u,v) \geq c\,(\partial_1 C(u,v) - v) \tag{3.6}$$

and

$$\partial_2 C(u,v) \geq c\,(\partial_2 C(u,v) - u)$$

for all $u, v \in \mathrm{Range}(F) = \left[\frac{1}{2}, 1\right]$. Otherwise, $A$ would not be an aggregation function. There are several choices of copulas to choose from. One choice would be the Ali–Mikhail–Haq copula $C_\theta$ given by

$$C_\theta(u,v) = \frac{uv}{1+\theta(1-u)(1-v)}$$

for all $u, v \in \mathbb{I}$ where $-1 \leq \theta \leq 1$. Since we only require $C_\theta$ to be a subcopula with domain $\left[\frac{1}{2}, 1\right]^2$, we can extend the range of $\theta$. Direct computation show that $-4 < \theta \leq 1$ in this case. Then

$$C_\theta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4+\theta}$$

so that

$$c = -\frac{\frac{1}{4+\theta}}{\frac{1}{4} - \frac{1}{4+\theta}} = -\frac{4}{4+\theta-4} = -\frac{4}{\theta}.$$

Also,

$$\partial_1 C_\theta(u,v) = \frac{v + \theta v(1-v)}{(1+\theta(1-u)(1-v))^2}$$

which implies

$$\partial_1 C_\theta(u,v) - v = \frac{\theta v(1-v)\left(u^2 - (1-u)^2(1+\theta(1-v))\right)}{(1+\theta(1-u)(1-v))^2}.$$

Since $C_\theta$ is symmetric, it is sufficient to satisfy inequality (3.6). For $\theta < 0$, $\partial_1 C_\theta(u,v) - v < 0$ and hence

$$\begin{aligned}
&\max\left\{\frac{\partial_1 C_\theta(u,v)}{\partial_1 C_\theta(u,v) - v} \,:\, \frac{1}{2} \le u, v \le 1\right\} \\
&= \max\left\{\frac{1+\theta(1-v)}{\theta(1-v)\left(u^2 - (1-u)^2(1+\theta(1-v))\right)} \,:\, \frac{1}{2} \le u, v \le 1\right\} \\
&= \max\left\{\frac{-t}{(1-t)\left(u^2 - (1-u)^2 t\right)} \,:\, \begin{array}{c} \frac{1}{2} \le u \le 1, \\ 1+\frac{\theta}{2} \le t \le 1 \end{array}\right\} \\
&= -\min\left\{\frac{t}{(1-t)\left(u^2 - (1-u)^2 t\right)} \,:\, \begin{array}{c} \frac{1}{2} \le u \le 1, \\ 1+\frac{\theta}{2} \le t \le 1 \end{array}\right\} \\
&= -\min\left\{\frac{t}{(1-t)} \,:\, 1+\frac{\theta}{2} \le t \le 1\right\} \\
&= -\frac{1+\frac{\theta}{2}}{1-\left(1+\frac{\theta}{2}\right)} \\
&= 1+\frac{2}{\theta}.
\end{aligned}$$

This forces

$$c = -\frac{4}{\theta} \ge 1 + \frac{2}{\theta}$$

which is equivalent to $\theta \ge -6$.

For $\theta > 0$, $\partial_1 C_\theta(u,v) - v > 0$ if and only if $1+\theta(1-v) \le \frac{u^2}{(1-u)^2}$. Write $t = 1+\theta(1-v)$. Now,

$$\begin{aligned}
&\min\left\{\frac{t}{(1-t)\left(u^2 - (1-u)^2 t\right)} \,:\, \begin{array}{c} \frac{1}{2} \le u \le 1, \\ 1 \le t \le 1+\frac{\theta}{2} \\ t \le \frac{u^2}{(1-u)^2} \end{array}\right\} \\
&= \min\left\{\frac{t}{(t-1)\left(u^2 - (1-u)^2 t\right)} \,:\, \begin{array}{c} \frac{\sqrt{t}}{\sqrt{t}+1} \le u \le 1, \\ 1 \le t \le 1+\frac{\theta}{2} \end{array}\right\} \\
&= \min\left\{\frac{t}{(t-1)} \,:\, 1 \le t \le 1+\frac{\theta}{2}\right\} \\
&= 1+\frac{2}{\theta}
\end{aligned}$$

and

$$\max\left\{\frac{t}{(t-1)\left(u^2-(1-u)^2t\right)}\ :\ \begin{array}{c}\frac{1}{2}\le u\le 1,\\ 1\le t\le 1+\frac{\theta}{2}\\ t\ge\frac{u^2}{(1-u)^2}\end{array}\right\}$$

$$=-\min\left\{\frac{t}{(t-1)\left((1-u)^2t-u^2\right)}\ :\ \begin{array}{c}\frac{1}{2}\le u\le\frac{\sqrt{t}}{\sqrt{t}+1},\\ 1\le t\le 1+\frac{\theta}{2}\end{array}\right\}$$

$$=-\min\left\{\frac{4t}{(t-1)^2}\ :\ 1\le t\le 1+\frac{\theta}{2}\right\}$$

$$=-\frac{4\left(1+\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2}$$

$$=-\frac{1}{\theta^2}-\frac{1}{2\theta}.$$

Therefore, we must have

$$1+\frac{2}{\theta}\ge-\frac{4}{\theta}\ge-\frac{1}{\theta^2}-\frac{1}{2\theta}$$

which is always true for $\theta>0$. Hence

$$\begin{aligned}A(x,y)&=\left(1+\frac{4}{\theta}\right)C(F(x),F(y))-\frac{4}{\theta}F(x)F(y)\\&=\frac{\left(1+\frac{4}{\theta}\right)(1+x)(1+y)}{4+\theta(1-x)(1-y)}-\frac{1}{\theta}(1+x)(1+y)\\&=\frac{1}{\theta}\left(\frac{4+\theta}{4+\theta(1-x)(1-y)}-1\right)(1+x)(1+y)\\&=\frac{1}{\theta}\left(\frac{\theta-\theta(1-x)(1-y)}{4+\theta(1-x)(1-y)}\right)(1+x)(1+y)\\&=\frac{(1+x)(1+y)(1-(1-x)(1-y))}{4+\theta(1-x)(1-y)}\end{aligned}$$

is an aggregation function whenever $-4<\theta\le 1$.

Similar constructions can also be done with other families of copulas.

## 4. Construction properties

In this section, we will discuss important properties of our construction. First, we will recall the definition of scaling functions.

**Definition 4.2.** The *ith scaling* associated with an aggregation function $A:\mathbb{I}^k\to\mathbb{I}$ is a function $A_i:\mathbb{I}\to\mathbb{I}$ defined by

$$A_i(x)=A(x\vec{e}_i+\vec{e}_{-i})$$

for all $x\in\mathbb{I}$. Denote also $\vec{A}:\mathbb{I}^k\to\mathbb{I}^k$ where

$$\vec{A}(\vec{x})=(A_1(x_1),\ldots,A_k(x_k))$$

for all $\vec{x}\in\mathbb{I}^k$. The function $\vec{A}$ is called the scaling vector of $A$.

Note that an aggregation function is a semi-copula if and only if all of its associated scaling functions are the identity function.

**Theorem 4.1.** *Let $\vec{A} : \mathbb{I}^k \to \mathbb{I}^k$ be a continuous scaling vector, $c$ be a constant and $C$ be a copula. If $\partial_i C \geq c (\partial_i C - \partial_i \Pi)$ a.e. on $\operatorname{Range}\left(\vec{A}\right)$ for all $i = 1, \ldots, k$ and that*

$$C \circ \vec{A}\left(\vec{0}\right) = -\left(\frac{c}{1-c}\right) \prod_{i=1}^{k} A_i(0),$$

*then the function $A$ given by (3.3) is a continuous aggregation function with $\vec{A}$ as its associated scaling vector. Moreover, any $k$-times continuously differentiable aggregation function with a strictly increasing scaling vector can be constructed this way.*

*Proof.* Let $S = (1-c)\, C + c\Pi$. Since $\partial_i C$ exists, $\partial_i S = (1-c)\, \partial_i C + c\partial_i \Pi$ also exists and we have $\partial_i S \geq 0$ on $\operatorname{Range}\left(\vec{A}\right)$. Thus, $S$ is nondecreasing which also implies $A$ is nondecreasing. Clearly, $A\left(\vec{1}\right) = S\left(\vec{1}\right) = 1$ while

$$A\left(\vec{0}\right) = (1-c)\, C \circ \vec{A}\left(\vec{0}\right) + c\Pi \circ \vec{A}\left(\vec{0}\right) = 0.$$

Therefore, $A$ is an aggregation function.

Next, assume that $A : \mathbb{I}^k \to \mathbb{I}$ is a $k$-times continuously differentiable aggregation function with a strictly increasing associated scaling vector $\vec{A} : \mathbb{I}^k \to \mathbb{I}^k$. Then its associated scaling $A_i$ are also continuously differentiable and strictly increasing. Therefore, their inverse exists, strictly increasing, and also continuously differentiable.

Denote $\vec{a} = \vec{A}\left(\vec{0}\right)$. Then we have $\operatorname{Range}\left(\vec{A}\right) = \left[\vec{a}, \vec{1}\right]$. Define $S(x_1, \ldots, x_k) = A\left(A_1^{-1}(x_1), \ldots, A_k^{-1}(x_k)\right)$ for all $(x_1, \ldots, x_k) \in \left[\vec{a}, \vec{1}\right]$. By chain rule, its mixed partial derivative $\partial_1 \cdots \partial_k S$ also exists and continuous. Notice that $S$ is nondecreasing with

$$\begin{aligned} S_i(u) &= S(u\vec{e}_i + \vec{e}_{-i}) \\ &= A\left(A_i^{-1}(u)\, \vec{e}_i + \vec{e}_{-i}\right) \\ &= A_i\left(A_i^{-1}(u)\, \vec{e}_i + \vec{e}_{-i}\right) \\ &= u \end{aligned}$$

for all $u \geq a_i$. We will recursively extended $S$ per coordinate as follows. First, we set $T_1 = S$ if $a_1 = 0$ otherwise, set

$$T_1(\vec{x}) = \begin{cases} \frac{x_1}{a_1} S(a_1, x_2, \ldots, x_k), & x_1 \leq a_1; \\ S(x_1, x_2, \ldots, x_k), & \text{otherwise;} \end{cases}$$

whenever $\vec{x} \in \mathbb{I}^k$ with $x_i \geq a_i$ for all $i \neq 1$. Then we have $T_1$ is an extension of $S$ which is $k$-times continuously differentiable almost everywhere (excepted possibly when $x_1 = a_1$). Also,

$$T_1(u\vec{e}_i + \vec{e}_{-i}) = u$$

whenever either $u \in \mathbb{I}$ and $i = 1$ or $u \geq a_i$ and $i > 1$. Now, we extends $T_i : \mathbb{I}^i \times \prod_{j=i+1}^{k} [a_j, 1] \to \mathbb{I}$ to $T_{i+1} : \mathbb{I}^{i+1} \times \prod_{j=i+1}^{k} [a_j, 1] \to \mathbb{I}$, whenever $i < k$, by setting $T_{i+1} = T_i$ if $a_{i+1} = 0$; otherwise, set

$$T_{i+1}(\vec{x}) = \begin{cases} \frac{x_{i+1}}{a_{i+1}} S(x_1, \ldots, x_i, a_{i+1}, x_{i+2}, \ldots, x_k), & x_{i+1} \leq a_{i+1}; \\ S(x_1, x_2, \ldots, x_k), & \text{otherwise;} \end{cases}$$

for all $\vec{x} \in \mathbb{I}^k$ with $x_j \geq a_j$ for all $j > i+1$. Again, $T_{i+1}$ is $k$-times continuously differentiable almost everywhere (excepted possibly when $x_{i+1} = a_{i+1}$). Now, we have $T = T_k$

is an extension of $S$ which is $k$-times continuously differentiable almost everywhere (excepted possibly when $x_i = a_i$ for some $i$). Moreover,

$$T\left(u\vec{e}_i + \vec{e}_{-i}\right) = u$$

for all $u \in \mathbb{I}$ and all $i$. In particular,

$$T\left(\vec{x}\right) = 0$$

whenever $x_i = 0$ for some $i$. We can then define

$$c = \min\left(\left(\min_{\forall i, u_i \neq a_i} \partial_1 \cdots \partial_k T\left(\vec{u}\right)\right), 0\right)$$

and

$$\begin{aligned} C\left(\vec{u}\right) &= \frac{1}{1-c} \int\limits_{[\vec{0},\vec{u}]} \left(\partial_1 \cdots \partial_k T - c\right) \\ &= \frac{1}{1-c}\left(V_T\left([0,\vec{u}]\right) - cV_\Pi\left([0,\vec{u}]\right)\right) \\ &= \frac{1}{1-c}\left(T\left(\vec{u}\right) - c\Pi\left(\vec{u}\right)\right) \end{aligned}$$

for all $\vec{u} \in \mathbb{I}^k$. Since $C$ is an integration of a nonnegative function, it must be $k$-increasing. Since both $T$ and $\Pi$ satisfy equation (2.1), so is $C$. Therefore, $C$ is a copula. Notice that, we actually have

$$C\left(\vec{u}\right) = \frac{1}{1-c}\left(S\left(\vec{u}\right) - c\Pi\left(\vec{u}\right)\right)$$

whenever $\vec{u} \in \mathrm{Range}\left(\vec{A}\right)$. The constant $c$ can also be written in term of $S$. Therefore, the choice of extension $T$ does not actually matter. Also,

$$\begin{aligned} C\left(\vec{A}\left(\vec{0}\right)\right) &= \frac{1}{1-c}\left(S\left(\vec{A}\left(\vec{0}\right)\right) - c\Pi\left(\vec{A}\left(\vec{0}\right)\right)\right) \\ &= \frac{1}{1-c}\left(A\left(\vec{0}\right) - c\prod_{i=1}^{k} A_i\left(0\right)\right) \\ &= -\left(\frac{c}{1-c}\right)\prod_{i=1}^{k} A_i\left(0\right). \end{aligned}$$

By construction, formula (3.3) must hold as desired. □

**Remark 4.1.** Note that the constant $c$ in the above proof is not unique. It can actually be replaced by any number less than $(\min \partial_1 \cdots \partial_1 T) \wedge 0$.

As mentioned before, any quadratic aggregation function can be written in the form of (3.3). Its proof is a simple consequence of the above theorem.

**Corollary 4.1.** *Any bivariate quadratic aggregation function can be written in the form of* (3.3) *for some copula $C$ and some constant $c$ unless it is either a convex combination of $\pi_{11}, \zeta_{11}$, and $\zeta_{12}$ or a convex combination of $\pi_{22}, \zeta_{22}$, and $\zeta_{12}$.*

*Proof.* Since a quadratic bivariate function is continuously twice differentiable, we only have to consider the monotonicity condition. Let

$$A = a_1\pi_{11} + a_2\pi_{22} + b_1\pi_{12} + b_2\zeta_{12} + c_1\zeta_{11} + c_2\zeta_{22}$$

be a quadratic bivariate aggregation function. Then

$$\begin{aligned} A_i'(x) &= 2c_i + b_1 + 2(a_i - c_i)x \\ &\geq 0. \end{aligned}$$

since $A$ is nondecreasing. If $A_i'(x) = 0$ for some $x \in \mathbb{I}$ , then we must actually have $A_i'(x) = 0$ on either $[0, x]$ or $[x, 1]$ which further implies $A_i'(x) = 0$ on $\mathbb{I}^2$ due to the monotonicity of linear functions. Since all coefficients are nonnegative, the latter is equivalent to $a_i = c_i = b_1 = 0$. Therefore, exactly one of the following cases occurs:

- $\vec{A}$ is strictly increasing,
- $a_1 = c_1 = b_1 = 0$,
- $a_2 = c_2 = b_1 = 0$.

If $\vec{A}$ is strictly increasing, we are immediately done. Otherwise, $A$ is either a convex combination of $\pi_{11}, \zeta_{11}$, and $\zeta_{12}$ or a convex combination of $\pi_{22}, \zeta_{22}$, and $\zeta_{12}$ as desired. □

In corollary 4.1, we see that only quadratic aggregation functions on the boundary of the set can not be written in the form (3.3). This situation also holds in general – only aggregation functions on the boundary of the set can not be written in the form (3.3).

**Theorem 4.2.** *The set of aggregation functions constructed via* (3.3) *is dense in the space of continuous aggregation functions. Furthermore, it contains all the interior of the space.*

*Proof.* Denote $\mathcal{A}$ the set of continuous aggregation functions, $\mathcal{B}$ the set of aggregation functions in the form (3.3), and $\mathcal{S}$ the set of smooth strictly increasing aggregation functions, all with the same dimension $d$. We already know $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{A}$. If we can show that $\mathcal{S}$ is dense in $\mathcal{A}$, then so is $\mathcal{B}$. To do this, we will used the Bernstein polynomial. Let $A \in \mathcal{A}$, and define

$$A_n(\vec{x}) = \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} A\left(\frac{1}{n}\cdot\vec{k}\right) \prod_{i=1}^{d} B(n, k_i, x_i)$$

for all $\vec{x} \in \mathbb{I}^d$ where $B(n, k, x) = \binom{n}{k} x^k (1-x)^{n-k}$.
It is well-known that $A_n$ converges to $A$ uniformly. Moreover, $A_n$ are smooth since they are polynomial, $A_n\left(\vec{0}\right) = A\left(\vec{0}\right) \prod_{i=1}^{d} B(n, 0, 0) = 0$, and $A_n\left(\vec{1}\right) = A\left(\vec{1}\right) \prod_{i=1}^{d} B(n, n, 1) = 1$. Thus, it remains to show that $A_n$ are nondecreasing. For this, notice that

$$\begin{aligned} \partial_j A_n(\vec{x}) &= \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} \left( A\left(\frac{1}{n}\cdot\vec{k}\right) \prod_{i=1, i\neq j}^{d} B(n, k_i, x_i) \right) \partial_j B(n, k_i, x_i) \\ &= n \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} \left( A\left(\frac{1}{n}\cdot\vec{k}\right) \prod_{i=1, i\neq j}^{d} B(n, k_i, x_i) \right) \left(B(n-1, k_j - 1, x_j) - B(n-1, k_j, x_j)\right) \\ &= n \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} A\left(\frac{1}{n}\cdot\vec{k} + \frac{1}{n}\cdot\vec{e}_j\right) B(n-1, k_j, x_j) \prod_{i=1, i\neq j}^{d} B(n, k_i, x_i) \\ &\quad - n \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} A\left(\frac{1}{n}\cdot\vec{k}\right) B(n-1, k_j, x_j) \prod_{i=1, i\neq j}^{d} B(n, k_i, x_i) \\ &= n \sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}} \left( A\left(\frac{1}{n}\cdot\vec{k} + \frac{1}{n}\cdot\vec{e}_j\right) - A\left(\frac{1}{n}\cdot\vec{k}\right) \right) B(n-1, k_j, x_j) \prod_{i=1, i\neq j}^{d} B(n, k_i, x_i) \end{aligned}$$

with the convention that $B(n-1,k,x)=0$ when $k<0$ or $k>n-1$. Since $A$ is nondecreasing, the last line must also be nonnegative. Hence, $A_n$ are nondecreasing. Set $S_n=\left(1-\frac{1}{n}\right)A_n+\frac{1}{n}\Pi$. Then $S_n\in\mathcal{S}$ are also smooth strictly increasing aggregation functions and $S_n\to A$ uniformly to $A$. This finishes the proof of the first statement.

For the second statement, notice that the set $\mathcal{S}$ is also convex. Now, the fact that $\mathcal{S}$ is a convex dense subset of a closed convex set $\mathcal{A}$ lying in a Banach space implies that $\mathcal{S}$ must contain all the interior points of $\mathcal{A}$. Since $\mathcal{S}\subseteq\mathcal{B}$, the same also holds for $\mathcal{B}$. Therefore, aggregation functions that can not be represented as (3.3) must lie on the boundary. □

Last, we will end this section by showing that our construction method can also be used to construct aggregation functions with special properties. Recall that an aggregation function $A$ is idempotent if $A\left(x\cdot\vec{1}\right)=x$ for all $x\in\mathbb{I}$, and it is invariant if $A(\vec{x})$ does not depends on the order of $x_i$'s that appear in $\vec{x}$. In particular, all scaling functions associated to an invariant aggregation function must be the same.

**Corollary 4.2.** *The set of idempotent invariant aggregation functions in the form* (3.3) *is dense in the set of idempotent invariant (continuous) aggregation functions. In this case, the diagonal function $\delta_C$ of the copula $C$ in* (3.3) *must be given by*

$$\delta_C(a(x))=\frac{1}{1-c}\left(x-ca^d(x)\right) \tag{4.7}$$

*for all $x\in\mathbb{I}$ where $a:\mathbb{I}\to\mathbb{I}$ is its associated scaling function and $d$ is its dimension.*

*Proof.* First, we will prove the second statement. Let $A$ be an idempotent invariant aggregation function in the form (3.3) and $a:\mathbb{I}\to\mathbb{I}$ be its associated scaling function. Then we must have

$$\begin{aligned}x&=A\left(x\cdot\vec{1}\right)\\&=(1-c)\,C\circ\vec{A}\left(x\cdot\vec{1}\right)+c\Pi\circ\vec{A}\left(x\cdot\vec{1}\right)\\&=(1-c)\,\delta_C(a(x))+ca^d(x)\end{aligned}$$

which yields the result via some algebraic manipulations.

Now, for the first statement, let $A$ be an idempotent invariant (continuous) aggregation function. Again, set

$$A_n(\vec{x})=\sum_{\vec{k}=\vec{0}}^{n\cdot\vec{1}}A\left(\frac{1}{n}\cdot\vec{k}\right)\prod_{i=1}^{d}B(n,k_i,x_i)$$

for all $\vec{x}\in\mathbb{I}^d$. Then $A_n$ are smooth invariant aggregation functions and $A_n\to A$ uniformly to $A$. Set $B_n=\left(1-\frac{1}{n}\right)A_n+\frac{1}{n}\Pi$. Then $B_n$ are also smooth invariant aggregation functions, $B_n\to A$ uniformly, and $\vec{B}_n$ are strictly increasing as well. Therefore, its diagonal function $b_n=\delta_{B_n}$ must also be smooth and strictly increasing and $b_n$ converges uniformly to the identity function on $\mathbb{I}$. Now, $b_n$ are univariate distribution functions, so its associated quantile $b_n^{-1}$ must also converge uniformly to the identity function on $\mathbb{I}$. Clearly, $b_n^{-1}$ are smooth and strictly increasing.

Set

$$D_n(\vec{x})=B_n\left(b_n^{-1}(x_1),\ldots,b_n^{-1}(x_d)\right)$$

for all $\vec{x}\in\mathbb{I}^d$. Then $D_n$ are idempotent invariant aggregation functions converges uniformly to $A$. Since $D_n$ are also smooth and strictly increasing, they must be in the form (3.3). □

It should be mentioned that Corollary 4.2 only states that all diagonal functions must be in the form of (4.7) but not the converse. After all, the range of possible number $c$ will depend on the scaling function $a$. This fact will be summarized in the following result when the scaling function is continuously differentiable.

**Theorem 4.3.** *Let $a : \mathbb{I} \to \mathbb{I}$ be a continuously differentiable strictly increasing scaling function. Denote $\beta_d(x) = \frac{a(x)-x}{a(x)(1-a^{d-1}(x))}$, $\gamma_d(x) = \frac{1}{da^{d-1}(x)a'(x)}$, and $\eta_d(x) = \frac{da'(x)-1}{da'(x)(1-a^{d-1}(x))}$. Let $\delta$ be a function that satisfies (4.7).*

*(1) If $a(0) > 0$, then $\delta$ can be extended to diagonal function if and only if either*

$$c \geq \left(\sup_{0<x<1} \beta_d(x)\right) \vee \left(\sup_{0<x<1} \gamma_d(x)\right) \vee \left(\sup_{0<x<1} \eta_d(x)\right)$$

*or*

$$c \leq \left(\inf_{0<x<1} \eta_d(x)\right) \wedge 0.$$

*(2) If $a(0) = 0$, then $\delta$ is a diagonal function if and only if*

$$c \leq \left(\inf_{0<x<1} \beta_d(x)\right) \wedge \left(\inf_{0<x<1} \gamma_d(x)\right) \wedge \left(\inf_{0<x<1} \eta_d(x)\right).$$

*Proof.* Notice that

$$\delta'(a(x))\,a'(x) = \frac{1}{1-c}\left(1 - cda^{d-1}(x)\,a'(x)\right)$$

for all $x \in \mathbb{I}$. For $\delta$ to be a diagonal function, it must be point-wisely bounded by the identity function, nondecreasing, and Lipschitz with constant $d$. The last two conditions can be fulfill if $0 \leq \delta' \leq d$.

In the case $a(0) > 0$, we may extend $\delta$ linearly by setting

$$\delta(t) = \frac{-ca^{d-1}(0)\,t}{1-c}$$

for all $t < a(0)$. Clearly, $\delta(t) \leq t$ and $0 \leq \delta'(t) \leq 1$ for all $t < a(0)$ as long as $0 \leq \delta(a(0)) = \frac{-ca^d(0)}{1-c} \leq a(0)$ which is a part of the conditions for the original $\delta$ with its domain $[a(0), 1]$. Thus, there is no need to verify the extension.

Denote $\alpha_d(x) = \frac{x}{a^d(x)}$.

*Case* 1. $a(0) = 0$.

Then

$$\frac{1}{1-c} = \delta'(a(0))\,a'(0) \geq 0$$

which implies $c < 1$. Fix $0 < x < 1$.

The fact that $\delta(a(x)) \geq 0$ can be simplified to $x - ca^d(x) \geq 0$ which is equivalent to

$$c \leq \frac{x}{a^d(x)} = \alpha_d(x).$$

The fact that $\delta(a(x)) \leq a(x)$ can be simplified to $x - ca^d(x) \leq (1-c)\,a(x)$ which is equivalent to

$$c \leq \frac{a(x)-x}{a(x)\left(1-a^{d-1}(x)\right)} = \beta_d(x).$$

The fact that $\delta'(a(x)) \geq 0$ is equivalent to $1 - cda^{d-1}(x)\,a'(x) \geq 0$, so we have

$$c \leq \frac{1}{da^{d-1}(x)\,a'(x)} = \gamma_d(x).$$

The fact that $\delta'(a(x)) \leq d$ is equivalent to $1 - dca^{d-1}(x)\,a'(x) \leq d(1-c)\,a'(x)$, so we have

$$c \leq \frac{da'(x) - 1}{da'(x)\left(1 - a^{d-1}(x)\right)} = \eta_d(x).$$

Now, $\lim_{x\to 0^+} \alpha_d(x) = \infty$ and $\alpha_d(1) = 1 > 1 - \frac{1}{a'(0)} = \lim_{x\to 0^+} \beta_d(x)$. Also,

$$\alpha_d'(x) = \frac{a^d(x) - dxa^{d-1}(x)\,a'(x)}{a^{2d}(x)} = 0$$

if and only if $a(x) = dxa'(x)$ which leads to $\alpha_d(x) = \gamma_d(x)$ at that point. Thus,

$$\inf_{0<x<1} \alpha_d(x) \geq \left(\inf_{0<x<1} \beta_d(x)\right) \wedge \left(\inf_{0<x<1} \gamma_d(x)\right)$$

which yields the desired result.

*Case* 2. $a(0) > 0$.

Then $\delta(a(0)) \geq 0$ implies $-\frac{c}{1-c} \geq 0$ which is equivalent to either $c \leq 0$ or $c > 1$. For $c \leq 0$, we may proceed as in the previous case to get

$$c \leq \min\left(\alpha_d(x), \beta_d(x), \gamma_d(x), \eta_d(x)\right)$$

for all $0 < x < 1$. Now, $\min\left(\alpha_d(x), \beta_d(x), \gamma_d(x)\right) \geq 0$ so we are left with

$$c \leq \left(\inf_{0<x<1} \eta_d(x)\right) \wedge 0.$$

For $c > 1$, we may also proceed similarly to get

$$c \geq \max\left(\alpha_d(x), \beta_d(x), \gamma_d(x), \eta_d(x)\right)$$

for all $0 < x < 1$. Now, $\alpha_d(0) = 0$ and $\alpha_d(1) = 1 < \beta(0)$. Again, $\alpha_d'(x) = 0$ if and only if $\alpha_d(x) = \gamma_d(x)$ for any $0 < x < 1$. Thus,

$$\sup_{0<x<1} \alpha_d(x) \leq \left(\sup_{0<x<1} \beta_d(x)\right) \vee \left(\sup_{0<x<1} \gamma_d(x)\right)$$

as desired. □

We will end this work by providing a few construction of idempotent invariant bivariate aggregation functions using this method which in turn relies on the construction of a copula from a diagonal function. For convenience, we will be using the copula $K_\delta$ defined by

$$K_\delta(u, v) = \min\left(u, v, \frac{1}{2}\left(\delta(u) + \delta(v)\right)\right)$$

for all $u, v \in \mathbb{I}$. It is know that $K_\delta$ is a copula whenever $\delta$ is a diagonal functions [22]. For other constructions of copulas given diagonal functions, see for example, [17, 8, 9, 34, 26, 43].

**Example 4.3.** Again, let $a$ be the scaling function of the $Mean : \mathbb{I} \to \mathbb{I}$, that is,

$$a(x) = \frac{1+x}{2}$$

for all $x \in \mathbb{I}$.

In this case, $\beta_2 = \gamma_2 = \frac{1}{a}$ is decreasing while $\eta_2 = 0$. Thus, we obtain the range of possible $c$ as either $c \geq \frac{1}{a(0)} = 2$ or $c \leq 0$. For each $c$, we have the corresponding diagonal function $\delta = \delta_c$ given by

$$\delta_c(t) = \frac{1}{1-c}\left(2t - 1 - ct^2\right)$$

whenever $\frac{1}{2} \leq t \leq 1$.

Next, we need to determine $c$ for which $\partial_i K_\delta \geq c\left(\partial_i K_\delta - \partial_i \pi\right)$ a.e. Since $K_\delta$ is symmetric, it is sufficient to only consider the case $i = 1$. Note that $K_\delta(u,v) = v$ when $(u,v)$ is near $\left(1, \frac{1}{2}\right)$, $K_\delta(u,v) = u$ when $(u,v)$ is near $\left(\frac{1}{2}, 1\right)$, and $K_\delta(u,v) = \frac{1}{2}\left(\delta(u) + \delta(v)\right)$ when $(u,v)$ is near diagonal section. Thus, all three cases are possible.

When $K_\delta(u,v) = u$, we have $\partial_1 K_\delta = 1$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ become $1 \geq c(1-v)$ where $\frac{1}{2} \leq v \leq 1$. This forces either $c = 2$ or $c \leq 0$. When $K_\delta(u,v) = v$, we have $\partial_1 K_\delta = 0$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ becomes $0 \geq -cv$ which holds only for $c \geq 0$. When $K_\delta(u,v) = \frac{1}{2}\left(\delta(u) + \delta(v)\right)$, we have $\partial_1 K_\delta = \frac{1}{2}\delta'(u)$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ becomes $\delta'(u) \geq c\left(\delta'(u) - 2v\right)$ which holds for both $c = 0$ and $c = 2$.

Let

$$A_c(x,y) = (1-c)K_{\delta_c}\left(\frac{x+1}{2}, \frac{y+1}{2}\right) + \frac{c}{4}(x+1)(y+1)$$

for all $x, y \in \mathbb{I}$. Notice that $A_0 = \text{Mean}$. Thus, $A_2$ is another idempotent invariant aggregation function with the same scaling function as the Mean.

**Example 4.4.** Let $a(x) = \left(\frac{2}{3}\right)^{1-x}$ for all $x \in \mathbb{I}$. Then $a(x) \geq x$ for all $x \in \mathbb{I}$ which implie $a$ is a scaling function of an idempotent invariant aggregation function. For this function $a$, $\lim_{x \to 1^-} \eta_2(x) = -\infty$. Thus, only positive

$$c \geq \left(\sup_{0<x<1} \beta_2(x)\right) \vee \left(\sup_{0<x<1} \gamma_2(x)\right) \vee \left(\sup_{0<x<1} \eta_2(x)\right)$$

is possible.

Using graphical approach, we can see that $\eta_2 < 0$ while both $\beta_2$ and $\gamma_2$ are nonincreasing. Thus, $\delta_c$ given by

$$\delta_c\left(\left(\frac{2}{3}\right)^{1-x}\right) = \frac{1}{1-c}\left(x - c\left(\frac{2}{3}\right)^{2-2x}\right)$$

is a diagonal function if and only if $c \geq \beta_2(0) \vee \gamma_2(0) = 3$.

Next, we need to determine $c$ for which $\partial_i K_\delta \geq c\left(\partial_i K_\delta - \partial_i \pi\right)$ a.e. Since $K_\delta$ is symmetric, it is sufficient to only consider the case $i = 1$. Note that $K_\delta(u,v) = v$ when $(u,v)$ is near $\left(1, \frac{2}{3}\right)$, $K_\delta(u,v) = u$ when $(u,v)$ is near $\left(\frac{2}{3}, 1\right)$, and $K_\delta(u,v) = \frac{1}{2}\left(\delta(u) + \delta(v)\right)$ when $(u,v)$ is near the diagonal section. Thus, all three cases are possible.

When $K_\delta(u,v) = v$, we have $\partial_1 K_\delta = 0$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ always holds since the right side is negative. When $K_\delta(u,v) = u$, we have $\partial_1 K_\delta = 1$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ becomes $1 \geq c(1-v)$ for all $u \leq v \leq 1$ which yields $c \leq 3$. When $K_\delta(u,v) = \frac{1}{2}\left(\delta(u) + \delta(v)\right)$, we have $\partial_1 K_\delta = \frac{1}{2}\delta'(u)$ so that $\partial_1 K_\delta \geq c\left(\partial_1 K_\delta - \partial_1 \pi\right)$ becomes $\delta'(u) \geq c\left(\delta'(u) - 2v\right)$ which always holds when $c = 3$.

Therefore,

$$
\begin{aligned}
A(x,y) &= -2K_{\delta_3}\left(\left(\frac{2}{3}\right)^{1-x},\left(\frac{2}{3}\right)^{1-y}\right)+3\left(\frac{2}{3}\right)^{2-x-y}\\
&= 3\left(\frac{2}{3}\right)^{2-x-y}-\min\left(2\left(\frac{2}{3}\right)^{1-x\vee y},\left(\frac{2}{3}\right)^{1-2x}+\left(\frac{2}{3}\right)^{1-2y}-\frac{x+y}{2}\right)
\end{aligned}
$$

is an idempotent invariant aggregation function with the scaling function given by $a(x) = \left(\frac{2}{3}\right)^{1-x}$ for all $x \in \mathbb{I}$.

Last, for the case $a(0) = 0$, the following result holds.

**Corollary 4.3.** *Let $a : \mathbb{I} \to \mathbb{I}$ be a continuously differentiable strictly increasing scaling function and $\delta$ be a function that satisfies (4.7). If $\eta_2 \geq 0$, then the function $A$ defined by*

$$
A(x,y) = (1-c)\,K_\delta(a(x),a(y)) + ca(x)\,a(y) \tag{4.8}
$$

*for all $x, y \in \mathbb{I}$ is an idempotent invariant aggregation function with scaling function $a$ whenever $0 \leq c \leq (\inf_{0<x<1}\beta_2(x)) \wedge (\inf_{0<x<1}\gamma_2(x)) \wedge (\inf_{0<x<1}\eta_d(x))$.*

*Furthermore, all $(1-c)\,K_\delta + c\pi$ are copulas with the same diagonal function $a^{-1}$.*

*Proof.* Clearly, $\beta_2 \geq 0$ and $\gamma_2 \geq 0$. Therefore, the condition $\eta_2 \geq 0$ guarantee the existence of $c$. Since $c \in \mathbb{I}$, we know $(1-c)\,K_\delta + c\pi$ is a copula and that $A$ is an idempotent invariant aggregation function. Therefore, $(1-c)\,K_\delta + c\pi$ must have the same diagonal function which is $a^{-1}$.

To see why we can not extend the range of $c$ to a negative number, notice that $K_\delta(1,0) = 0$ which implies $K_\delta(u,v) = v$ in a neighborhood of $(1,0)$. For this case, $\partial_1 K_\delta = 0$ so that the condition $\partial_1 K_\delta \geq c(\partial_1 K_\delta - \partial_1 \pi)$ becomes $0 \geq -cv$ for small $v > 0$. This forces $c \geq 0$. □

**Example 4.5.** Let $a(x) = \frac{\sqrt{2}x}{\sqrt{1+x^2}}$ for all $x \in \mathbb{I}$ be a sigmoid function. Then $a(x) \geq x$ for all $x \in \mathbb{I}$ which implies $a$ is a scaling function of an idempotent invariant aggregation function.

In this case, $\beta_2 \leq \gamma_2$ and $\beta_2 \leq \eta_2$ pointwisely and $\beta_2$ is nondecreasing. Thus, the range of $c$ becomes $0 \leq c \leq \beta_2(0) = 1 - \frac{1}{a'(0)} = \frac{\sqrt{2}-1}{\sqrt{2}}$.

For such $c$, let $A_c(x,y) = (1-c)\,K_{\delta_c}(a(x),a(y)) + ca(x)\,a(y)$ where

$$
\delta_c(t) = \frac{1}{1-c}\left(a^{-1}(t) + ct^2\right) = \frac{1}{1-c}\left(\frac{t}{\sqrt{2-t^2}} + ct^2\right)
$$

is the diagonal function corresponding to $c$ as in (4.7). Then $A_c$ is a family of idempotent invariant aggregation functions with the same scaling function $a$.

**Example 4.6.** Consider an Ali–Mikhail–Haq copula defined by $C(u,v) = \frac{uv}{u+v-uv}$ for all $u, v \in \mathbb{I}$. We will use our result to construct a family of copulas with the same diagonal function of $C$ which is given by $\delta_C(t) = \frac{t}{2-t}$.

Let $a = \delta_C^{-1}$. Then $a(x) = \frac{2x}{x+1}$. Again, we found that $\beta_2 \leq \gamma_2$ and $\beta_2 \leq \eta_2$ pointwisely while $\beta_2$ is nondecreasing. Thus, we can choose $c$ such that $0 \leq c \leq \beta_2(0) = 1 - \frac{1}{a'(0)} = \frac{1}{2}$. Now, any function $D_c = (1-c)\,K_{\delta_c} + c\pi$ where

$$
\delta_c(t) = \frac{1}{1-c}\left(\frac{t}{2-t} + ct^2\right)
$$

for all $t \in \mathbb{I}$ is a copula with diagonal function $\delta_{D_c} = \delta_C$ whenever $0 \leq c \leq \frac{1}{2}$.

## 5. Conclusions and discussions

In this work, we make a slight change to the Sklar's formula which immensely effect the class of aggregation functions that can be constructed via this two-step method. When proceed with the Sklar's formula as in [18], only $k$-increasing aggregation functions can be constructed. This leaves out many aggregation functions. For example, only half of quadratic aggregation functions can be constructed via the Sklar's formula while the new formula only fails to represent some quadratic aggregation functions lying on the boundary. This fact also holds in general. The new formula is able to represent any aggregation function lying in the interior of this set. This is a vast improvement over the Sklar's formula – simple yet effective. This also means we could use a vast family of copulas already constructed in literature to construct aggregation functions. One interesting example would be using this method to construct aggregation functions with special properties. This should provide many such aggregation functions. We also present an example in the case of idempotent invariant aggregation functions. The construction for other classes should be an interesting research topic in the future.

Another problem related to data representation is data prediction / regression. In the latter case, aggregation functions can be used as a possible form of relationship between a response variable and explanatory variables under the assumption of monotonicity. With our result, this can be done by choosing a parametric family of copulas and scaling functions. Then the best parameters are seek via, for example, the least square method. It would also be interesting to see whether a two-step regression can be done by first work on a regression of each explanatory variable to get the corresponding scaling function and then follow by a regression on copulas. This is similar to how a joint distribution function is estimated in a copula model - first estimate marginal distribution functions and then estimate the copula. Whether this two-step method is possible for aggregation functions will again be the future topic of study.

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