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On the transfer of convergence between two sequences in Banach spaces

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ABSTRACT. Let $(X, \|\cdot\|)$ be a Banach space and $T : A \to X$ a contraction mapping, where $A \subset X$ is a closed set. Consider a sequence $\{x_n\} \subset A$ and define the sequence $\{y_n\} \subset X$, by $y_n = x_n + T\left(x_{\sigma(n)}\right)$, where $\{\sigma(n)\}$ is a sequence of natural numbers. We highlight some general conditions so that the two sequences $\{x_n\}$ and $\{y_n\}$ are simultaneously convergent. Both cases: 1) $\sigma(n) < n$, for all n, and 2) $\sigma(n) \geq n$, for all n, are discussed. In the first case, a general Picard iteration procedure is inferred. The results are then extended to sequences of mappings and some appropriate applications are also proposed.

1. INTRODUCTION

Our study focuses on a particular problem of convergence in Banach spaces. A comprehensive treatment of Banach space theory can be found, for example, in [6] and [10]. The problem we are studying is related to the *Banach's contraction mapping principle*. The theory of fixed points is intensively studied in the literature. Rich information on this topic can be found in the monographs [2] and [3].

Given a closed subset A of a Banach space $(X, \|\cdot\|)$, we consider a contraction mapping $T : A \to X$ and we study the simultaneous convergence of the sequences $\{x_n\} \subset A$ and $\{y_n\} \subset X$ linked by a relation of the type

$$y_n = x_n + T(x_{\sigma(n)}),\ n = 1, 2, \ldots.$$

Here, $\{\sigma(n)\}$ is a sequence of non-negative integers. We study two situations. In the first case, we suppose that $\sigma(n) < n$, for all n. Note that the proposed convergence theorem highlights a general Picard iteration procedure. The second case studied refers to the dual condition $\sigma(n) \geq n$, for all n. If T is a non self mapping, i.e., the closed set A is not invariant with respect to T ($T(A) \not\subset A$), then the Picard-Banach fixed point theorem cannot be used in the proofs. However, if $T(A) \subset A$, which happens for example when $A = X$, then the results could be obtained by applying the Picard-Banach fixed point theorem. The results are then extended to sequences of mappings.

Although such kind of problems is common in literature, we do not know a systematic study of this topic in the general framework of Banach spaces. For $X = \mathbb{R}$, a particular related study can be found in [13].

Our results could be used to solve a wide class of problems. We illustrate some such applications in Section 3.

In the following, we will denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_i = \{i, i+1, i+2, \ldots\}$, where i is a positive integer.

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2. MAIN RESULTS

Firstly we study the case of a sequence $\{\sigma(n)\}$ of natural numbers with the property $\sigma(n) < n$, for all n. The following theorem provides a general convergence criterion in a Banach space.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $T : A \to X$ be a contraction mapping defined on a closed subset A of X. Let us consider a positive integer i and a function $\sigma : \mathbb{N}_i \to \mathbb{N}$, with the properties $\lim_{n\to\infty} \sigma(n) = \infty$ and $\sigma(n) < n$, for all $n \in \mathbb{N}_i$. Let $\{x_n\}_{n=0}^{\infty} \subset A$ and define the sequence $\{y_n\}_{n=i}^{\infty}$ by $y_n = x_n + T(x_{\sigma(n)})$, for all $n \in \mathbb{N}_i$. Then $\{x_n\}_{n=0}^{\infty}$ is convergent if and only if $\{y_n\}_{n=i}^{\infty}$ is convergent.*

Proof. Assume that $\{x_n\}$ converges to u. Since A is closed, we have $u \in A$. From the assumption $\lim_{n\to\infty} \sigma(n) = \infty$ and taking into account the continuity on the set A of the contraction mapping T, we conclude that $\{y_n\}$ converges to $u + T(u)$.

Let us prove the converse implication. From the hypothesis, there is $a \in [0, 1)$ such that $\|T(x)-T(y)\| \le a\|x-y\|$, for all $x, y \in A$. Since $\{y_n\}_{n=i}^{\infty}$ is assumed to be convergent, there is $C > 0$ such that $\|y_n\| \le C$, for all $n \in \mathbb{N}_i$. First of all, we will prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is also bounded. Let us denote $M_n = \max\{\|x_0\|, \|x_1\|, \ldots, \|x_n\|\},\ n \in \mathbb{N}$, and $K = C + M_i + \|T\left(x_{\sigma(i)}\right)\|$. We will prove by induction the inequalities

$$\|x_n\| \le M_i + K\left(1 + a + \ldots + a^{n-i}\right),\ \text{for all } n \in \mathbb{N}_i. \tag{2.1}$$

For $n = i$, we have $\|x_i\| \le M_i < M_i + K$. Suppose now that, for a given $n \in \mathbb{N}_i$, the following inequalities are true:

$$\|x_k\| \le M_i + K\left(1 + a + \ldots + a^{k-i}\right),\ \text{for } k = i, \ldots, n. \tag{2.2}$$

From the assumption, we have $\sigma(n+1) \in \{0, 1, \ldots, n\}$. Then we obtain

$$\begin{aligned}\|x_{n+1}\| = \left\|y_{n+1} - T\left(x_{\sigma(n+1)}\right)\right\| &= \left\|y_{n+1} - T\left(x_{\sigma(i)}\right) + \left[T\left(x_{\sigma(i)}\right) - T\left(x_{\sigma(n+1)}\right)\right]\right\| \\ &\le \|y_{n+1}\| + \left\|T\left(x_{\sigma(i)}\right)\right\| + a\left\|x_{\sigma(i)} - x_{\sigma(n+1)}\right\| \\ &\le C + \left\|T\left(x_{\sigma(i)}\right)\right\| + a\left\|x_{\sigma(i)}\right\| + a\left\|x_{\sigma(n+1)}\right\| \le K + aM_n.\end{aligned}$$

We have $\|x_k\| \le M_i < M_i + K\left(1 + a + \ldots + a^{n-i}\right)$, for $k = 0, \ldots, i-1$. Then, from (2.2), we deduce $M_n \le M_i + K\left(1 + a + \ldots + a^{n-i}\right)$. Therefore

$$\|x_{n+1}\| \le K + a\left[M_i + K\left(1 + a + \ldots + a^{n-i}\right)\right] \le M_i + K\left(1 + a + \ldots + a^{n-i} + a^{n+1-i}\right).$$

Thus, inequalities (2.1) are proved. Let us denote $M = M_i + \dfrac{K}{1-a}$. From (2.1) we find $\|x_n\| \le M$, for all $n \in \mathbb{N}_i$. Clearly, $\|x_n\| \le M_i < M$, for $n = 0, \ldots, i-1$. As follows,

$$\|x_n\| \le M,\ \text{for all } n \in \mathbb{N},$$

that is, the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Let us define

$$\Delta_n = \sup\{\|x_p - x_q\| :\ p, q \in \mathbb{N},\ p, q \ge n\},\ n \in \mathbb{N}.$$

We have $\Delta_n \le 2M$ and $\Delta_n \ge \Delta_{n+1}$, for all $n \in \mathbb{N}$.

Suppose $\varepsilon > 0$ and denote $\varepsilon_1 = \dfrac{\varepsilon(1-a)}{2}$. Since $\{y_n\}_{n=i}^{\infty}$ is convergent, it is a Cauchy sequence. Then, there is $n_1 \in \mathbb{N}_i$ such that $\|y_p - y_q\| < \varepsilon_1$, for all $p, q \in \mathbb{N}_i$, with $p, q \ge n_1$. Based on the assumption $\lim_{n\to\infty} \sigma(n) = \infty$, we define the sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ by the recurrence relation

$$n_{k+1} = \min\{m \in \mathbb{N}_i :\ \sigma(p) \ge n_k,\ \text{for all } p \ge m\},\ \text{for } k = 1, 2, \ldots$$

Note that $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers. The following relations

$$\Delta_{n_k} \leq \varepsilon_1 \left(1 + a + \ldots + a^{k-1}\right) + 2Ma^k, \ k \geq 1, \tag{2.3}$$

will be proved by induction. For $p, q \in \mathbb{N}_i$, with $p, q \geq n_1$, we have

$$\begin{aligned}\|x_p - x_q\| &= \left\|\left[y_p - T\left(x_{\sigma(p)}\right)\right] - \left[y_q - T\left(x_{\sigma(q)}\right)\right]\right\| \leq \|y_p - y_q\| + \left\|T\left(x_{\sigma(p)}\right) - T\left(x_{\sigma(q)}\right)\right\| \\ &< \varepsilon_1 + a\left\|x_{\sigma(p)} - x_{\sigma(q)}\right\| \leq \varepsilon_1 + 2aM.\end{aligned}$$

It turns out that $\Delta_{n_1} \leq \varepsilon_1 + 2aM$.

Suppose now that (2.3) holds for a positive integer k. For $p, q \in \mathbb{N}_i$, with $p, q \geq n_{k+1}$, we have $\sigma(p), \sigma(q) \geq n_k \geq n_1$. As follows, we obtain

$$\|x_p - x_q\| < \varepsilon_1 + a\|x_{\sigma(p)} - x_{\sigma(q)}\| \leq \varepsilon_1 + a\Delta_{n_k} \leq \varepsilon_1\left(1 + a + \ldots + a^{k-1} + a^k\right) + 2Ma^{k+1}.$$

Then $\Delta_{n_{k+1}} \leq \varepsilon_1(1 + a + \ldots + a^{k-1} + a^k) + 2Ma^{k+1}$. Thus, (2.3) is proved by induction. In relation (2.3), by choosing a positive integer k_1 such that $a^{k_1} < \dfrac{\varepsilon}{4M}$, we obtain

$$\Delta_{n_{k_1}} \leq \varepsilon_1\left(1 + a + \ldots + a^{k_1-1}\right) + 2Ma^{k_1} < \varepsilon_1 \cdot \frac{1}{1-a} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon.$$

Hence $\|x_p - x_q\| < \varepsilon$, for all $p, q \in \mathbb{N}, \ p, q \geq n_{k_1}$. Since $\varepsilon > 0$ is arbitrarily chosen, we conclude that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. So the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent. □

Remark 2.1. In the particular case $A = X$, $i = 1$ and $\sigma(n) = n - 1$, for all $n \geq 1$, the result of Theorem 2.1 can be derived from Banach's contraction mapping principle.

Thus, denote $\lim\limits_{n\to\infty} y_n = \ell$ and consider the a-contraction mapping $U = \ell - T : X \to X$. Let $u \in X$ be the unique fixed point of U, that is $u + T(u) = \ell$. We will prove $\lim\limits_{n\to\infty} x_n = u$. For all $n \in \mathbb{N}_1$, we have

$$\|x_n - u\| = \|y_n - T(x_{n-1}) - \ell + T(u)\| \leq \|y_n - \ell\| + \|T(u) - T(x_{n-1})\| \leq \|y_n - \ell\| + a\|x_{n-1} - u\|.$$

We easily obtain by induction $\|x_n - u\| \leq \sum\limits_{k=1}^{n} a^{n-k}\|y_k - \ell\| + a^n\|x_0 - u\|$, for all $n \in \mathbb{N}_1$.

Then, from Silverman-Toeplitz theorem (see, for example, [9]) we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} a^{n-k}\|y_k - \ell\| = \frac{1}{1-a} \lim_{n\to\infty} \|y_n - \ell\| = 0.$$

Therefore, $\lim\limits_{n\to\infty} \|x_n - u\| = 0$, that is, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to u.

Remark 2.2. The condition $\lim\limits_{n\to\infty} \sigma(n) = \infty$ cannot be removed from the hypothesis of Theorem 2.1.

The following elementary example supports the above remark. Consider two distinct elements of A, let us say x_0 and x_1, such that $T(x_0) \neq T(x_1)$, and define $\sigma(n) = 0$, for odd positive integers n, and $\sigma(n) = 1$, for even positive integers $n > 1$. Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ cannot be simultaneously convergent.

The following general Picard iteration procedure is inferred.

Corollary 2.1. *Let $T : X \to X$ be a contraction mapping defined on a Banach space X. Let us consider a positive integer i and a function $\sigma : \mathbb{N}_i \to \mathbb{N}$, with the properties $\lim\limits_{n\to\infty} \sigma(n) = \infty$ and $\sigma(n) < n$, for all $n \in \mathbb{N}_i$. Assume that a sequence $\{x_n\}_{n=0}^{\infty}$ of X satisfies the recurrent relation $x_n = T\left(x_{\sigma(n)}\right) + y_n$, for all $n \in \mathbb{N}_i$, where $\{y_n\}_{n=i}^{\infty}$ is a sequence of X with the property $\lim\limits_{n\to\infty} y_n = 0$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.*

Proof. We have $y_n = x_n + (-T)(x_{\sigma(n)}),\ n \geq i$. Since $y_n \to 0$ and $-T$ is a contraction mapping, Theorem 2.1 ensures the convergence of $\{x_n\}_{n=0}^{\infty}$. Let $u \in X$ be the limit of $\{x_n\}_{n=0}^{\infty}$. Since T is continuous and $y_n \to 0$, we get $u = T(u)$. □

We now propose some extensions of Theorem 2.1 for sequences of functions. In this context, we recall the following classical result due to Bonsall (see the monograph [5]): *if a sequence $\{T_n\}_{n=1}^{\infty}$ of contraction mappings of a complete metric space, with the same Lipschitz constant, is pointwise convergent to a contraction mapping T, then the sequence of the fixed points of $T_n,\ n \geq 1$, converges to the fixed point of T.* This result has been extended by many researchers. We mention the papers [1], [7], [11], [12], [15] and [16].

Our first result deals with the uniform convergence of a sequence of functions to a contraction mapping.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space and let $T : A \to X$ be a contraction mapping defined on a closed subset A of X. Let us consider a positive integer i and a function $\sigma : \mathbb{N}_i \to \mathbb{N}$, with the properties $\lim\limits_{n\to\infty} \sigma(n) = \infty$ and $\sigma(n) < n$, for all $n \in \mathbb{N}_i$. Suppose a sequence of mappings $T_n : A \to X,\ n \geq i$, which uniformly converges to T on A. Consider the sequences $\{x_n\}_{n=0}^{\infty}$, with the terms in A, and $\{y_n\}_{n=i}^{\infty}$, defined by $y_n = x_n + T_n\left(x_{\sigma(n)}\right)$, for all $n \in \mathbb{N}_i$. Then $\{x_n\}_{n=0}^{\infty}$ is convergent if and only if $\{y_n\}_{n=i}^{\infty}$ is convergent.*

Proof. Assume that $\{x_n\}_{n=0}^{\infty}$ converges to $u \in \mathrm{cl}_X(A) = A$. To prove the convergence of sequence $\{y_n\}_{n=i}^{\infty}$ it is enough to show $\lim\limits_{n\to\infty} T_n\left(x_{\sigma(n)}\right) = T(u)$. Assume $\varepsilon > 0$. Since the contraction mapping T is a continuous function on A and $\lim\limits_{n\to\infty} \sigma(n) = \infty$, there is $n_1 \in \mathbb{N}_i$ such that $\left\|T\left(x_{\sigma(n)}\right) - T(u)\right\| < \dfrac{\varepsilon}{2}$, for all $n \in \mathbb{N}_i,\ n \geq n_1$. On the other hand, there is $n_2 \in \mathbb{N}_i$ such that $\|T_n(x) - T(x)\| < \dfrac{\varepsilon}{2}$, for all $x \in A$ and for all $n \in \mathbb{N}_i,\ n \geq n_2$. Therefore, $\left\|T_n\left(x_{\sigma(n)}\right) - T(u)\right\| \leq \left\|T_n\left(x_{\sigma(n)}\right) - T\left(x_{\sigma(n)}\right)\right\| + \left\|T\left(x_{\sigma(n)}\right) - T(u)\right\| < \varepsilon$, for all $n \in \mathbb{N}_i,\ n \geq \max\{n_1, n_2\}$. Hence $\lim\limits_{n\to\infty} T_n(x_{\sigma(n)}) = T(u)$. So, $\lim\limits_{n\to\infty} y_n = u + T(u)$.

Assume now that $\{y_n\}_{n=i}^{\infty}$ is convergent. Let us consider the sequence $\{z_n\}_{n=i}^{\infty}$, defined by $z_n = x_n + T\left(x_{\sigma(n)}\right),\ n \in \mathbb{N}_i$. We will show that $\{z_n\}_{n=i}^{\infty}$ is a Cauchy sequence.

Let ε be an arbitrary positive number. The convergence of $\{y_n\}_{n=i}^{\infty}$ involves the existence of $n_1 \in \mathbb{N}_i$ with the property $\|y_p - y_q\| \leq \dfrac{\varepsilon}{3},\ \forall p, q \in \mathbb{N}_i,\ p, q \geq n_1$. Since $\{T_n\}_{n=i}^{\infty}$ is uniformly convergent to T on A, there is $n_2 \in \mathbb{N}_i$ such that $\|T_n(x) - T(x)\| < \dfrac{\varepsilon}{3}$, for all $x \in A$ and for all $n \in \mathbb{N}_i$, with $n \geq n_2$. Thus, for $p, q \in \mathbb{N}_i$, with $p, q \geq \max\{n_1, n_2\}$, we obtain

$$\|z_p - z_q\| = \left\|(y_p - y_q) + \left[T\left(x_{\sigma(p)}\right) - T_p\left(x_{\sigma(p)}\right)\right] + \left[T_q\left(x_{\sigma(q)}\right) - T\left(x_{\sigma(q)}\right)\right]\right\|$$

$$\leq \|y_p - y_q\| + \left\|T\left(x_{\sigma(p)}\right) - T_p\left(x_{\sigma(p)}\right)\right\| + \left\|T_q\left(x_{\sigma(q)}\right) - T\left(x_{\sigma(q)}\right))\right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence we conclude that $\{z_n\}_{n=i}^{\infty}$ is a Cauchy sequence. As follows, $\{z_n\}_{n=i}^{\infty}$ is convergent. From Theorem 2.1, we conclude that $\{x_n\}_{n=0}^{\infty}$ is also convergent. □

The pointwise convergence of a sequence of mappings $\{T_n\}$ to a contraction mapping T do not ensure the result of Theorem 2.2. However, we provide below a version of this theorem that regards this kind of convergence.

Theorem 2.3. *Let $T_n : X \to X,\ n \geq 1$ be sequence of a-contractions mappings defined on a Banach space X, such that $\{T_n\}_{n=1}^{\infty}$ is pointwise convergent to $T : X \to X$. Assume a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $\{y_n\}_{n=1}^{\infty}$, defined by $y_n = x_n + T_n(x_{n-1})$, for all $n \in \mathbb{N}_1$, is convergent. Then $\{x_n\}_{n=0}^{\infty}$ is convergent.*

Proof. From the pointwise convergence of $\{T_n\}$ to T, we easily deduce that the limit mapping T is also an a-contraction on X. Let $\ell \in X$ be the limit of $\{y_n\}_{n=1}^{\infty}$. Denote by $u \in X$ the fixed point of the a-contraction $U = \ell - T$, that is, $u = \ell - T(u)$. For $n \geq 1$, we have

$$\|x_n - u\| = \|[y_n - T_n(x_{n-1})] - [\ell - T(u)]\|$$

$$\leq \|y_n - \ell\| + \|T_n(u) - T_n(x_{n-1})\| + \|T(u) - T_n(u)\| \leq a\|x_{n-1} - u\| + t_n,$$

where $t_n := \|y_n - \ell\| + \|T_n(u) - T(u)\|$, with $\lim_{n\to\infty} t_n = 0$. We obtain by induction:

$$\|x_n - u\| \leq a^n\|x_0 - u\| + \sum_{k=1}^{n} a^{n-k}t_k,\ n = 1, 2, \ldots.$$

From Silverman-Toeplitz theorem, we get $\lim_{n\to\infty} \sum_{k=1}^{n} a^{n-k}t_k = \frac{1}{1-a}\lim_{n\to\infty} t_n = 0$. Then $\lim_{n\to\infty} \|x_n - u\| = 0$, that is, $\{x_n\}_{n=0}^{\infty}$ converges to u. □

We are now studying the complementary case $\sigma(n) \geq n,\ n \in \mathbb{N}$.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a Banach space and consider a contraction mapping $T : A \to X$, where $A \subset X$ is a closed set. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers, with the property $\sigma(n) \geq n$, for all $n \in \mathbb{N}$. For a bounded sequence $\{x_n\}_{n=0}^{\infty}$ with the terms in A, we define the sequence $y_n = x_n + T\left(x_{\sigma(n)}\right)$, for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ is convergent if and only if $\{y_n\}_{n=0}^{\infty}$ is convergent.*

Proof. Since $\sigma(n) \geq n$, for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \sigma(n) = \infty$. Then the convergence of $\{x_n\}_{n=0}^{\infty}$ involves the convergence of $\{y_n\}_{n=0}^{\infty}$ (see the first part of the proof of Theorem 2.1).

Suppose now that $\{y_n\}_{n=0}^{\infty}$ is convergent. Since $\{x_n\}_{n=0}^{\infty}$ is bounded, there is a positive constant M such that $\|x_n\| \leq M$, for all $n \in \mathbb{N}$. Following the line of proving Theorem 2.1, we define $\{\Delta_n\}_{n=0}^{\infty}$ by

$$\Delta_n = \sup\{\|x_p - x_q\| :\ p, q \in \mathbb{N},\ p, q \geq n\},\ n \in \mathbb{N}.$$

We have $\Delta_n \leq 2M$ and $\Delta_n \geq \Delta_{n+1}$, for all $n \in \mathbb{N}$. Since T is a contraction, there is $a \in [0, 1)$ such that $\|T(x) - T(y)\| \leq a\|x - y\|$, for all $x, y \in A$.

Suppose $\varepsilon > 0$ and denote $\varepsilon_1 = \frac{\varepsilon(1-a)}{2}$. The convergent sequence $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Then there is $n_1 \in \mathbb{N}$ such that $\|y_p - y_q\| < \varepsilon_1$, for all $p, q \in \mathbb{N}$, with $p, q \geq n_1$. For $p, q \in \mathbb{N}$, with $p, q \geq n_1$, we have $\sigma(p), \sigma(q) \geq n_1$ and

$$\|x_p - x_q\| = \left\|\left[y_p - T\left(x_{\sigma(p)}\right)\right] - \left[y_q - T\left(x_{\sigma(q)}\right)\right]\right\| \leq \|y_p - y_q\| + \left\|T\left(x_{\sigma(p)}\right) - T\left(x_{\sigma(q)}\right)\right\|$$

$$< \varepsilon_1 + a\|x_{\sigma(p)} - x_{\sigma(q)}\| \leq \varepsilon_1 + a\Delta_{n_1}.$$

Therefore $\Delta_{n_1} \leq \varepsilon_1 + a\Delta_{n_1}$. So, $\Delta_{n_1} \leq \frac{\varepsilon_1}{1-a} = \frac{\varepsilon}{2} < \varepsilon$. It follows

$$\|x_p - x_q\| \leq \Delta_{n_1} < \varepsilon,\ \text{for all } p, q \in \mathbb{N},\ p, q \geq n_1.$$

We conclude that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. So, $\{x_n\}_{n=0}^{\infty}$ is convergent. □

Remark 2.3. Theorem 2.4 does not hold without the assumption on $\{x_n\}$ to be bounded.

Counterexample. Define $A = \{\lambda v,\ \lambda \in \mathbb{R}\} \subset X$, where $v \in X\backslash\{0\}$. Consider the contraction mapping $T : A \to X,\ T(x) = -2^{-1}x,\ x \in A$, and sequences $x_n = 2^n v$ and $\sigma(n) = n + 1$, for all $n \in \mathbb{N}$. Then $y_n = x_n + T(x_{n+1}) = 0$, for all $n \in \mathbb{N}$, but $\|x_n\| = 2^n\|v\| \to \infty$.

Remark 2.4. Theorem 2.4 can be extended to sequences of functions with the limit T, in the similar frame as in Theorem 2.2 and Theorem 2.3.

Remark 2.5. If $A = X$ and $\lim_{n\to\infty} y_n = \ell$, then $\{x_n\}$ converges to the unique fixed point of the contraction $U = \ell - T$. In particular, for $X = \mathbb{R}$, the function $g(x) = x + T(x),\ x \in \mathbb{R}$, is invertible and $u = g^{-1}(\ell)$ (see [13]).

3. Applications

We illustrate the above theoretical results with some interesting applications. The first two examples show that the proved transfer of convergence between sequences in Banach spaces allows a deeper understanding of some known results for real sequences. Statement (i) of Example 3.1 extends Problem 11e, p. 97, in [4], while Theorem 1, p. 102, in [4] is extended by Example 3.2. The last example provides two criteria for uniform convergence in the Banach space of real continuous functions, defined on a compact topological space. A version (for $\lambda = 1/2$) of the statement (ii) of Example 3.3 is presented in [8].

Example 3.1. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in the complex Banach space X. Assume a positive integer i and $\lambda \in \mathbb{C}$, with $|\lambda| < 1$. The following two statements hold.

(i) $\{x_n + \lambda x_{n-i}\}_{n=i}^{\infty}$ is convergent if and only if $\{x_n\}_{n=0}^{\infty}$ is convergent. In addition,
$$\lim_{n\to\infty} x_n = \ell \Leftrightarrow \lim_{n\to\infty} (x_n + \lambda x_{n-i}) = (1+\lambda)\ell.$$

(ii) If $\{x_n\}_{n=0}^{\infty}$ is bounded, then $\{x_n + \lambda x_{n+i}\}_{n=0}^{\infty}$ is convergent if and only if $\{x_n\}_{n=0}^{\infty}$ is convergent. In addition,
$$\lim_{n\to\infty} x_n = \ell \Leftrightarrow \lim_{n\to\infty} (x_n + \lambda x_{n+i}) = (1+\lambda)\ell.$$

Proof. We apply Theorem 2.1 and Theorem 2.4, respectively, for the contraction mapping $T(x) = \lambda x,\ x \in X$. In both cases (i) and (ii), the connection between the limits of the two sequences is obvious. □

Example 3.2. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in the complex Banach space X. Assume that equation $z^p + a_{p-1}z^{p-1} + \ldots + a_0 = 0$ with complex coefficients has the roots in the unit open disc $U(0,1) = \{z \in \mathbb{C},\ |z| < 1\}$. Then $\{y_n\}_{n=0}^{\infty}$, defined by $y_n = x_{n+p} + \sum_{k=0}^{p-1} a_k x_{n+k},\ n \in \mathbb{N}$, is convergent if and only if $\{x_n\}_{n=0}^{\infty}$ is convergent. In addition,
$$\lim_{n\to\infty} x_n = \ell \Leftrightarrow \lim_{n\to\infty} y_n = \ell\left(1 + a_0 + a_1 + \ldots + a_{p-1}\right).$$

Proof. Obviously, if $\{x_n\}_{n=0}^{\infty}$ is convergent, then $\{y_n\}_{n=0}^{\infty}$ is convergent. The converse implication will be proved by induction. From Example 3.1, (i), the statement is true for $p = 1$. Assume now that the property is true for a positive integer p. Let $a_0, a_1, \ldots, a_p \in \mathbb{C}$ such that equation $z^{p+1} + a_p z^p + \ldots + a_1 z + a_0 = 0$ has all the roots in $U(0,1)$. Let $\lambda \in U(0,1)$ be a root of the above equation. Then
$$z^{p+1} + a_p z^p + \ldots + a_1 z + a_0 = (z-\lambda)(z^p + b_{p-1}z^{p-1} + \ldots + b_0),$$
where equation $z^p + b_{p-1}z^{p-1} + \ldots + b_0 = 0$ has the roots in $U(0,1)$. Suppose that the sequence $y_n = x_{n+p+1} + a_p x_{n+p} + \ldots + a_1 x_{n+1} + a_0 x_n,\ n \in \mathbb{N}$, is convergent. We have
$$y_n = x_{n+p+1} + (b_{p-1} - \lambda)x_{n+p} + \ldots + (b_0 - \lambda b_1)x_{n+1} - \lambda b_0 x_n = z_{n+1} - \lambda z_n,$$
where $z_n = x_{n+p} + b_{p-1}x_{n+p-1} + \ldots + b_0 x_n$, for all $n \in \mathbb{N}$. From Example 3.1, (i), we deduce that $\{z_n\}_{n=0}^{\infty}$ is convergent. Hence $\{x_n\}_{n=0}^{\infty}$. Thus, the property is true for $p+1$. The equivalence between the two limits is obvious. □

Example 3.3. Let $(X, \mathcal{T})$ be a compact topological space and let $(C(X), \|\cdot\|_\infty)$ be the Banach space of the continuous functions $\varphi : X \to \mathbb{R}$, where $\|\varphi\|_\infty = \sup_{t \in X} |\varphi(t)|$.

(i) If $T : C(X) \to C(X)$ is a contraction mapping and $\{\varphi_n\}_{n=0}^\infty$ is a sequence in $C(X)$ such that
  (a) the sequence of functions $\{\psi_n\}_{n=1}^\infty$ defined by $\psi_n = \varphi_n + T(\varphi_{n-1}),\ n \geq 1$, is pointwise convergent to a function $\psi \in C(X)$,
  (b) $\psi_{n+1}(t) \geq \psi_n(t)$, for all $t \in X$ and $n \in \mathbb{N}_1$,

  then $\{\varphi_n\}_{n=0}^\infty$ is uniformly convergent to a function $\varphi \in C(X)$, i.e. $\{\varphi_n\}_{n=0}^\infty$ is convergent to φ in the Banach space $(C(X), \|\cdot\|_\infty)$.

(ii) Let $\{\varphi_n\}_{n=0}^\infty$ be a sequence in $C(X)$. If
  (a) $\{\varphi_n\}_{n=1}^\infty$ is pointwise convergent to a function $\varphi \in C(X)$,
  (b) there is $\lambda \in (-1, 1)$ such that $\varphi_{n+2}(t) \geq (1 - \lambda)\varphi_{n+1}(t) + \lambda\varphi_n(t)$, for all $t \in X$ and $n \in \mathbb{N}$,

  then $\{\varphi_n\}_{n=0}^\infty$ is uniformly convergent to φ.

Proof. (i) From Dini's theorem (see [14]) it results that the convergence of $\{\psi_n\}_{n=1}^\infty$ to ψ is uniform, that is, the convergence holds in the Banach space $(C(X), \|\cdot\|_\infty)$. Therefore, from Theorem 2.1, $\{\varphi_n\}_{n=0}^\infty$ is convergent in the Banach space $(C(X), \|\cdot\|_\infty)$.
(ii) Consider the sequence $\{\psi_n\}_{n=0}^\infty$, defined by $\psi_n = \varphi_{n+1} + \lambda\varphi_n$, for all $n \in \mathbb{N}$. From (a) and (b), we deduce that $\{\psi_n\}_{n=0}^\infty$ is pointwise convergent to $(1+\lambda)\varphi \in C(X)$ and $\psi_{n+1}(t) \geq \psi_n(t)$, for all $t \in X$ and $n \in \mathbb{N}$. Dini's theorem ensures the convergence of $\{\psi_n\}_{n=0}^\infty$ to $(1+\lambda)\varphi$ in the Banach space $(C(X), \|\cdot\|_\infty)$. So, by applying Example 3.1 (i), we get the conclusion. □

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