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Convergence theorem for an intermixed iteration in p-uniformly convex metric space

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ABSTRACT. In this paper, we first introduce the intermixed algorithm in p-uniformly convex metric spaces, and then we prove Δ-convergence of the proposed iterative method for finding a common element of the sets of fixed points of finite families of nonexpansive mappings in the framework of complete p-uniformly convex metric spaces. Furthermore, we apply our main theorem to prove Δ-convergence to solve the minimization problems in the framework of complete p-uniformly convex metric spaces. Finally, we give two examples in L^p spaces and numerical examples to support our main results.

1. INTRODUCTION

The iteration construction for approximating fixed points problem of convergence theorems is usually divided into two categories. One is weak convergence, such as the Mann iteration algorithm [19] and the Ishikawa iteration algorithm [12]. On the other hand are the algorithms with strong convergence, such as the Halpern iteration algorithm [10] and the viscosity algorithm [21].

In 1967, Halpern [10] proposed *the Halpern iteration* for nonexpansive mapping $S : C \to C$ and the sequence $\{x_n\}$ generated by $x_0, u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n, \qquad \text{for all } n \geq 0,$$

where C is a closed convex subset of a real Hilbert space H and he proved the strong convergence of $\{x_n\}$ to $P_{F(S)}(x_0)$ provided that $\alpha_n = n^{-\theta}$ with $\theta \in (0, 1)$.

By extended the Halpern iteration. Moudafi [21] introduced *the viscosity algorithm* for a contraction $f : C \to C$ and a nonexpansive mapping $T : C \to C$. The sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \qquad \text{for all } n \geq 0,$$

where C is a closed convex subset of a real Hilbert space H and $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$. Then he proved the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}f(z)$ under some suitable condition α_n. After that, many researchers have modified the viscosity algorithm in which the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is also involved in the sequence $\{x_n\}$, see, for instance [31, 27, 28].

In 2015, Yao et al. [31] proposed the intermixed algorithm for two strict pseudo-contractions S and T. The sequences $\{x_n\}$ and $\{y_n\}$ generated by $x_0, y_0 \in C$ and

(1.1)
$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C\left(\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n\right), & \text{for all } n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C\left(\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n\right), & \text{for all } n \geq 0, \end{cases}$$

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where $T : C \to C$ is a λ-strictly pseudo-contraction, $f, g : C \to H$ are a ρ_1 and ρ_2-contraction, respectively, $k \in (0, 1-\lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0, 1)$. Furthermore, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.1) converge independently to $P_{F(T)}f(y^*)$ and $P_{F(S)}g(x^*)$, respectively, where $x^* \in F(T)$ and $y^* \in F(S)$. Obviously, $\lim\limits_{n\to\infty} \|x_n - y_n\|$ is an essential tool for proving the theorem in Hilbert space. So, proving convergence of the intermixed theorem in the p-uniformly convex metric space requires creating an apparatus similar to $\lim\limits_{n\to\infty} \|x_n - y_n\|$.

A nonempty metric space (X, d) is said to be a *geodesic space* if every two points $x, y \in X$ are joined by a geodesic path $c : [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. In this case, c is called an *isometry*, and the image of c is called a *geodesic segment* joining x to y. When this image is unique, it is denoted by $[x, y]$. The metric space X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic segment. The foundation examples of geodesic spaces are normed vector spaces, complete Riemannian manifolds, and polyhedral complexes of piecewise constant curvature, etc.

Let $x, y \in X$ and $t \in [0, 1]$. We write $tx \oplus (1-t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y) \qquad \text{and} \qquad d(y, z) = (1-t)d(x, y).$$

A function $f : X \to (-\infty, \infty]$ is called *convex* if for any geodesic $[x, y] := \{tx \oplus (1-t)y : 0 \le t \le 1\}$ joining $x, y \in X$, we have

$$f(tx \oplus (1-t)y) \le tf(x) + (1-t)f(y),$$

and is called *uniformly convex* [6] if there exists a strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(\frac{1}{2}x \oplus \frac{1}{2}y) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) - \phi\left(d(x, y)\right).$$

In 1994, Ball, Carlen and Lieb [3] introduced the notion of p-uniform convexity which plays an essential role in Banach space theory. Recall that a normed space $(X, \|\cdot\|)$ is said to be p-uniformly convex for $2 \le p < \infty$ if and only if there exists a constant $c \ge 1$ such that for any $x, y \in X$,

$$\left\|\frac{x+y}{2}\right\|^p \le \frac{1}{2}\|x\|^p + \frac{1}{2}\|y\|^p - \frac{1}{c^p}\left\|\frac{x-y}{2}\right\|^p.$$

For any fixed $2 \le p < \infty$, a geodesic space (X, d) is called to be p-uniformly convex with parameter c [17, 22, 25] if there exists a constant $0 < c \le 1$ such that for any $x, y, z \in X$ and any geodesic $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$,

$$d^p(z, \gamma(t)) \le (1-t)d^p(z, x) + td^p(z, y) - ct(1-t)d^p(x, y), \qquad t \in [0, 1].$$

Over the past decade, Naor and Silberman [22] introduced p-uniformly convex metric space for $1 < p < \infty$ as following: A metric space (X, d) is called p-uniformly convex with parameter $c > 0$ if and only if (X, d) is a geodesic space and

$$d^p\left(z, (1-t)\,x \oplus ty\right) \le (1-t)d^p(z, x) + td^p(z, y) - \frac{c}{2}t(1-t)d^p(x, y), \tag{1.2}$$

for all $x, y, z \in X$, $t \in [0, 1]$. Furthermore, every closed and convex subset of a p-uniformly convex normed space is a p-uniformly convex metric space with the same parameter [1]. Moreover, when $p = 2 = c$ in (1.2), we obtain the $CAT(0)$ property [22, 4]. In addition, numerous problems in Finster geometry and metric geometry, the nonlinearization of the geometry of Banach space and other related fields reduce to find an element of (1.2), see more detail in [17, 23, 24, 25, 26].

Many mathematicians proposed their algorithms for solving various problems in the framework of complete p-uniformly convex metric spaces, see, for instance [29, 6].

Recently, Godwin et al. [29] introduced a modified Mann type proximal point algorithm involving nonexpansive mapping. Moreover, they proved that the sequence generated by the algorithm converges to a common solution of finite families of minimization problems and a common element of the set of solutions of the fixed point of a nonexpansive mapping in the framework of complete p-uniformly convex metric spaces as follows:

Theorem 1.1. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$, and let $f_i : X \to (-\infty, \infty]$, for all $i = 1, 2, ..., N$, be finite families of proper, convex and lower semi-continuous functions. Let the p-resolvent $J_{\lambda^{(i)}}$ of f be Δ-demiclosed at 0 for all $i = 1, 2, ..., N$, and let $T : X \to X$ be a nonexpansive mapping. Suppose that $\Gamma := F(T) \cap (\bigcap_{i=1}^{N} \arg\min_{y\in X} f_i(y)) \neq \emptyset$, and for arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ ... \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T y_n, \qquad \text{for all } n \geq 1, \end{cases}$$

where $\{\lambda_n^{(i)}\}$ are a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for all $n \geq 1$, $i = 1, 2, ..., N$, and $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ Δ-converges to some $x^* \in \Gamma$.

Inspired and motivated by K-mapping in [15], we define K-mapping in p-uniformly convex metric space as follows.

Definition 1.1. Let $p > 0$ and (X, d) be a complete p-uniformly convex metric space with $c > 0$. Let $\{T_i\}_{i=1}^{N}$ be finite families of nonlinear mappings of X into itself and let $\lambda_i \in [0, 1]$ for all $i = 1, 2, ..., N$. Define a mapping $K : X \to X$ by

$$\begin{aligned} U_1 =& \lambda_1 T_1 \oplus (1 - \lambda_1) I, \\ U_2 =& \lambda_2 T_2 U_1 \oplus (1 - \lambda_2) U_1, \\ U_3 =& \lambda_3 T_3 U_2 \oplus (1 - \lambda_3) U_2, \\ &\vdots \\ U_{N-1} =& \lambda_{N-1} T_{N-1} U_{N-2} \oplus (1 - \lambda_{N-1}) U_{N-2}, \\ K = U_N =& \lambda_N T_N U_{N-1} \oplus (1 - \lambda_N) U_{N-1}. \end{aligned}$$

This mapping is called the K-mapping generate by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$.

Based on the result mentioned above, we first introduce the intermixed algorithm in p-uniformly convex metric spaces to prove Δ-convergence of the proposed iterative method for finding a common element of the sets of fixed points of finite families of nonexpansive mappings by using the concept of the K-mapping in the framework of complete p-uniformly convex metric spaces. Moreover, we apply our main theorem to prove Δ-convergence to solve the minimization problems in the framework of complete p-uniformly convex metric spaces. Finally, we give two examples in L^p spaces and numerical examples to support our main results.

2. Preliminaries

In this section, we recall some definitions and lemmas that will be needed to prove our main results. Let $\{x_n\}$ be a bounded sequence in a metric space X, and let $r(\cdot, \{x_n\}) : X \to [0, \infty)$ be a continuous functional defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\},$$

while the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [8, 16]). The notion of Δ-convergence in metric spaces was introduced by Lim [18].

Remark 2.1. Let X be a complete p-uniformly convex metric space. Then

(i) every bounded sequence in X has a unique asymptotic center (see [9]),

(ii) every bounded sequence in X has a Δ-convergent subsequence (see [29]).

Definition 2.2. A mapping $T : X \to X$ is called *nonexpansive mapping* if

$$d(Tx, Ty) \leq d(x, y), \qquad \text{for all } x, y \in X.$$

Definition 2.3. A mapping $T : X \to X$ is called *the fixed point problem* is to find $x \in X$ such that

$$Tx = x,$$

the set of fixed points of T is denoted by $F(T)$.

Definition 2.4. Let X be a complete convex metric space, and let $T : X \to X$ be any nonlinear mapping. The mapping T is said to be Δ-demiclosed at 0 if, for any bounded sequence $\{x_n\}$ in X such that $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $z = Tz$.

Lemma 2.1. (See [7]) *Let X be a complete $CAT(0)$ space, and let $T : X \to X$ be a nonexpansive mapping. Then T is Δ-demiclosed at 0.*

Remark 2.2. (see [29]) Following the same argument as in the proof of Lemma 2.1, one can easily show that Lemma 2.1 holds if X is a complete p-uniformly convex metric space.

The following lemma is crucial for proving our main theorem.

Lemma 2.2. *Let $p > 0$ and (X, d) be a complete p-uniformly convex metric space with $c > 0$. Let $\{T_i\}_{i=1}^N$ be finite families of nonexpansive mappings of X into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\lambda_i \in (0, 1)$ for all $i = 1, 2, ..., N-1$ and $\lambda_N \in (0, 1]$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$.*

Proof. It easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(K)$. Let $x_0 \in F(K)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. By the definition of K, we have

$$\begin{aligned}
d^p(x_0, x^*) =& d^p(Kx_0, x^*)\\
=& d^p(\lambda_N T_N U_{N-1}x_0 \oplus (1-\lambda_N)U_{N-1}x_0, x^*)\\
\leq& \lambda_N d^p(x^*, T_N U_{N-1}x_0) + (1-\lambda_N) d^p(x^*, U_{N-1}x_0)\\
&- \frac{c}{2}\lambda_N(1-\lambda_N) d^p(T_N U_{N-1}x_0, U_{N-1}x_0) \qquad (2.3)\\
\leq& d^p(x^*, U_{N-1}x_0)\\
\leq& \lambda_{N-1} d^p(x^*, T_{N-1}U_{N-2}x_0) + (1-\lambda_{N-1}) d^p(x^*, U_{N-2}x_0)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{c}{2}\lambda_{N-1}(1-\lambda_{N-1})d^p(T_{N-1}U_{N-2}x_0, U_{N-2}x_0)\\
&\le d^p(x^*, U_{N-2}x_0)\\
&\ \vdots\\
&\le d^p(x^*, U_2x_0)\\
&\le \lambda_2 d^p(x^*, T_2U_1x_0) + (1-\lambda_2)d^p(x^*, U_1x_0)\\
&\quad - \frac{c}{2}\lambda_2(1-\lambda_2)d^p(T_2U_1x_0, U_1x_0)\\
&\le \lambda_2 d^p(x^*, U_1x_0) + (1-\lambda_2)d^p(x^*, U_1x_0)\\
&\quad - \frac{c}{2}\lambda_2(1-\lambda_2)d^p(T_2U_1x_0, U_1x_0) \qquad (2.4)\\
&\le d^p(x^*, U_1x_0)\\
&\le \lambda_1 d^p(x^*, T_1x_0) + (1-\lambda_1)d^p(x^*, x_0) - \frac{c}{2}\lambda_1(1-\lambda_1)d^p(T_1x_0, x_0)\\
&\le \lambda_1 d^p(x^*, x_0) + (1-\lambda_1)d^p(x^*, x_0) - \frac{c}{2}\lambda_1(1-\lambda_1)d^p(T_1x_0, x_0)\\
&= d^p(x^*, x_0) - \frac{c}{2}\lambda_1(1-\lambda_1)d^p(T_1x_0, x_0). \qquad (2.5)
\end{aligned}
$$

This implies that

$$\frac{c}{2}\lambda_1(1-\lambda_1)d^p(T_1x_0, x_0) \le 0.$$

We obtain that

$$d^p(T_1x_0, x_0) = 0,$$

it follow that $T_1x_0 = x_0$, that is $x_0 \in F(T_1)$.
From the definition of U_1 and $x_0 \in F(T_1)$, we obtain

$$
\begin{aligned}
d^p(U_1x_0, x_0) &= d^p(\lambda_1 T_1x_0 \oplus (1-\lambda_1)x_0, x_0)\\
&\le \lambda_1 d^p(T_1x_0, x_0) + (1-\lambda_1)d^p(x_0, x_0) - \frac{c}{2}\lambda_1(1-\lambda_1)d^p(T_1x_0, x_0)\\
&= 0,
\end{aligned}
$$

then,

$$U_1x_0 = x_0. \qquad (2.6)$$

By (2.4) and (2.6), we have

$$\frac{c}{2}\lambda_2(1-\lambda_2)d^p(T_2x_0, x_0) \le 0.$$

We obtain that

$$d^p(T_2x_0, x_0) = 0,$$

it follow that $T_2x_0 = x_0$, that is $x_0 \in F(T_2)$.
From the definition of U_2, (2.6) and $x_0 \in F(T_2)$, we obtain

$$
\begin{aligned}
d^p(U_2x_0, x_0) &= d^p(\lambda_2 T_2U_1x_0 \oplus (1-\lambda_2)U_1x_0, x_0)\\
&\le \lambda_2 d^p(T_2U_1x_0, x_0) + (1-\lambda_2)d^p(U_1x_0, x_0) - \frac{c}{2}\lambda_2(1-\lambda_2)d^p(T_2U_1x_0, x_0)\\
&= \lambda_2 d^p(T_2x_0, x_0) + (1-\lambda_2)d^p(x_0, x_0) - \frac{c}{2}\lambda_2(1-\lambda_2)d^p(T_2x_0, x_0)\\
&= 0,
\end{aligned}
$$

then,

$$U_2x_0 = x_0. \qquad (2.7)$$

By using the same argument, we can conclude that $T_i x_0 = x_0$ and $U_i x_0 = x_0$ for all $i = 1, 2, ..., N-1$.
By (2.3), we have

$$\frac{c}{2}\lambda_N(1-\lambda_N)d^p(T_N x_0, x_0) \leq 0.$$

We obtain that

$$d^p(T_N x_0, x_0) = 0,$$

it follow that $T_N x_0 = x_0$, that is $x_0 \in F(T_N)$.

Therefore $x_0 \in \bigcap_{i=1}^{N} F(T_i)$.

Hence $F(K) \subseteq \bigcap_{i=1}^{N} F(T_i)$. $\square$

Now, we give the following example in $\mathbb{R}$ to support Lemma 2.2.

Example 2.1. Let $\mathbb{R}$ be the set of real numbers, and let $X = \mathbb{R}^2$ be endowed with a metric $d : \mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

for all $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a complete p-uniformly convex metric space with $p = 2$ and parameter $c = 2$, and with the geodesic joining x to y given by

$$(1-t)\mathbf{x} \oplus t\mathbf{y} = ((1-t)x_1 + ty_1, (1-t)x_2 + ty_2),$$

for all $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and $t \in [0, 1]$.

For every $i = 1, 2, ..., N$, let the mappings $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_i(\mathbf{x}) = (\frac{x_1}{2i}, \frac{x_2}{3i}),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Suppose that K is the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ where $\lambda_i = \frac{1}{i+1}$, for all $i = 1, 2, ..., N$. Then $(0, 0) \in F(K) = \cap_{i=1}^{N} F(T_i)$.

3. Main results

In this section, we prove Δ-convergence of the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.8) for finding a common element of the sets of fixed points of finite families of nonexpansive mappings in the framework of complete p-uniformly convex metric spaces.

Theorem 3.2. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$, and let $\{T_i\}_{i=1}^{N}$ and $\{S_i\}_{i=1}^{N}$ be finite families of nonexpansive mappings from X into itself with $\xi = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. For every $N \in \mathbb{N}$, let $K_T : X \to X$ be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$, let $K_S : X \to X$ be the K-mapping generated by $S_1, S_2, ..., S_N$ and $\eta_1, \eta_2, ..., \eta_N$, where $\{\lambda_i\}_{i=1}^{N}$ and $\{\eta_i\}_{i=1}^{N}$ are the sequences in $[a, b]$ and $[c, d]$ with $0 < a \leq b < 1$ and $0 < c \leq d < 1$, respectively. For given $x_1, y_1 \in X$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}K_T x_n \oplus \frac{\gamma_n}{1-\alpha_n}x_n\right), \\ y_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}K_S y_n \oplus \frac{\gamma_n}{1-\alpha_n}y_n\right), \end{cases} \tag{3.8}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < s \leq \alpha_n, \beta_n, \gamma_n \leq q < 1$, for all $n \in \mathbb{N}$ and for some $s, q > 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ Δ-converge to some $x^* \in \xi$.

Proof. Let $z \in \xi$.
From the definition of K_T and (1.2), we have

$$\begin{aligned}
d^p(K_Tx_n, z) =& d^p(\lambda_N T_N U_{N-1}x_n \oplus (1-\lambda_N)U_{N-1}x_n, z)\\
\leq& \lambda_N d^p(T_NU_{N-1}x_n, z) + (1-\lambda_N)d^p(U_{N-1}x_n, z)\\
&- \frac{c}{2}(\lambda_N)(1-\lambda_N)d^p(T_NU_{N-1}x_n, U_{N-1}x_n)\\
\leq& \lambda_N d^p(U_{N-1}x_n, z) + (1-\lambda_N)d^p(U_{N-1}x_n, z)\\
=& d^p(U_{N-1}x_n, z)\\
=& \lambda_{N-1}d^p(T_{N-1}U_{N-2}x_n, z) + (1-\lambda_{N-1})d^p(U_{N-2}x_n, z)\\
&- \frac{c}{2}(\lambda_{N-1})(1-\lambda_{N-1})d^p(T_{N-1}U_{N-2}x_n, U_{N-2}x_n)\\
\leq& \lambda_{N-1}d^p(U_{N-2}x_n, z) + (1-\lambda_{N-1})d^p(U_{N-2}x_n, z)\\
=& d^p(U_{N-2}x_n, z)\\
&\vdots\\
\leq& d^p(U_2x_n, z)\\
=& \lambda_2 d^p(T_2U_1x_n, z) + (1-\lambda_2)d^p(U_1x_n, z)\\
&- \frac{c}{2}(\lambda_2)(1-\lambda_2)d^p(T_2U_1x_n, U_1x_n)\\
\leq& \lambda_2 d^p(U_1x_n, z) + (1-\lambda_2)d^p(U_1x_n, z)\\
=& d^p(U_1x_n, z)\\
=& \lambda_1 d^p(T_1x_n, z) + (1-\lambda_1)d^p(x_n, z)\\
&- \frac{c}{2}(\lambda_1)(1-\lambda_1)d^p(T_1x_n, x_n)\\
\leq& \lambda_1 d^p(x_n, z) + (1-\lambda_1)d^p(x_n, z)\\
=& d^p(x_n, z). \qquad (3.9)
\end{aligned}$$

Using the same method as derived in (3.9), we have

$$d^p(K_Sy_n, z) \leq d^p(y_n, z). \qquad (3.10)$$

From the definition of x_n, (1.2) and (3.9), we have

$$\begin{aligned}
d^p(x_{n+1}, z) =& d^p\left(\alpha_n y_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}K_Tx_n \oplus \frac{\gamma_n}{1-\alpha_n}x_n\right), z\right)\\
\leq& \alpha_n d^p(y_n, z) + (1-\alpha_n)d^p\left(\frac{\beta_n}{1-\alpha_n}K_Tx_n \oplus \frac{\gamma_n}{1-\alpha_n}x_n, z\right)\\
&- \frac{c}{2}(\alpha_n)(1-\alpha_n)d^p\left(y_n, \frac{\beta_n}{1-\alpha_n}K_Tx_n \oplus \frac{\gamma_n}{1-\alpha_n}x_n\right) \qquad (3.11)\\
\leq& \alpha_n d^p(y_n, z) + (1-\alpha_n)\Bigg(\frac{\beta_n}{1-\alpha_n}d^p(K_Tx_n, z) + \frac{\gamma_n}{1-\alpha_n}d^p(x_n, z)\\
&- \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right)d^p(K_Tx_n, x_n)\Bigg)\\
\leq& \alpha_n d^p(y_n, z) + (1-\alpha_n)\Bigg(\frac{\beta_n}{1-\alpha_n}d^p(x_n, z) + \frac{\gamma_n}{1-\alpha_n}d^p(x_n, z)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_T x_n, x_n)\Bigg)\\
&= \alpha_n d^p(y_n, z) + (1-\alpha_n)\Bigg(d^p(x_n, z) - \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_T x_n, x_n)\Bigg) \qquad (3.12)\\
&\leq \alpha_n d^p(y_n, z) + (1-\alpha_n) d^p(x_n, z). \qquad (3.13)
\end{aligned}
$$

From the definition of y_n, (1.2) and (3.10), we have

$$
\begin{aligned}
d^p(y_{n+1}, z) &= d^p\left(\alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} K_S y_n \oplus \frac{\gamma_n}{1-\alpha_n} y_n\right), z\right)\\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p\left(\frac{\beta_n}{1-\alpha_n} K_S y_n \oplus \frac{\gamma_n}{1-\alpha_n} y_n, z\right)\\
&\quad - \frac{c}{2}(\alpha_n)(1-\alpha_n) d^p\left(x_n, \frac{\beta_n}{1-\alpha_n} K_S y_n \oplus \frac{\gamma_n}{1-\alpha_n} y_n\right) \qquad (3.14)\\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n)\Bigg(\frac{\beta_n}{1-\alpha_n} d^p(K_S y_n, z) + \frac{\gamma_n}{1-\alpha_n} d^p(y_n, z)\\
&\quad - \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_S y_n, y_n)\Bigg)\\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n)\Bigg(\frac{\beta_n}{1-\alpha_n} d^p(y_n, z) + \frac{\gamma_n}{1-\alpha_n} d^p(y_n, z)\\
&\quad - \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_S y_n, y_n)\Bigg)\\
&= \alpha_n d^p(x_n, z) + (1-\alpha_n)\Bigg(d^p(y_n, z) - \frac{c}{2}\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_S y_n, y_n)\Bigg) \qquad (3.15)\\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(y_n, z). \qquad (3.16)
\end{aligned}
$$

From (3.13) and (3.16), we get

$$
\begin{aligned}
d^p(x_{n+1}, z) &+ d^p(y_{n+1}, z)\\
&\leq \alpha_n d^p(y_n, z) + (1-\alpha_n) d^p(x_n, z) + \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(y_n, z)\\
&= d^p(y_n, z) + d^p(x_n, z),
\end{aligned}
$$

which implies that $\lim_{n\to\infty} (d^p(x_n, z) + d^p(y_n, z))$ exists for all $z \in \xi$.
Thus $\{x_n\}$ and $\{y_n\}$ are bounded.
From (3.12), we have
(3.17)

$$
(1-\alpha_n)\left(\frac{c}{2}\right)\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_T x_n, x_n) \leq \alpha_n d^p(y_n, z) + (1-\alpha_n) d^p(x_n, z) - d^p(x_{n+1}, z).
$$

From (3.15), we have
(3.18)

$$
(1-\alpha_n)\left(\frac{c}{2}\right)\left(\frac{\beta_n}{1-\alpha_n}\right)\left(\frac{\gamma_n}{1-\alpha_n}\right) d^p(K_S y_n, y_n) \leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(y_n, z) - d^p(y_{n+1}, z).
$$

Combining (3.17) and (3.18), we have

$$\begin{aligned}\left(\frac{c}{2}\right)\left(\frac{\beta_n\gamma_n}{1-\alpha_n}\right)&(d^p(K_Tx_n,x_n)+d^p(K_Sy_n,y_n))\\ &\le\alpha_n d^p(y_n,z)+(1-\alpha_n)d^p(x_n,z)-d^p(x_{n+1},z)\\ &\quad+\alpha_n d^p(x_n,z)+(1-\alpha_n)d^p(y_n,z)-d^p(y_{n+1},z)\\ &=(d^p(x_n,z)+d^p(y_n,z))-(d^p(x_{n+1},z)+d^p(y_{n+1},z)).\end{aligned}\tag{3.19}$$

From (3.19) and the conditions of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, we obtain that

$$\lim_{n\to\infty}(d^p(K_Tx_n,x_n)+d^p(K_Sy_n,y_n))=0,$$

it implies that

$$d^p(K_Tx_n,x_n)\to 0 \text{ and } d^p(K_Sy_n,y_n)\to 0 \text{ as } n\to\infty.\tag{3.20}$$

From (3.11) and (3.9), we have

$$\begin{aligned}&d^p(x_{n+1},z)\\ \le&\alpha_n d^p(y_n,z)+(1-\alpha_n)d^p\left(\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n,z\right)\\ &-\frac{c}{2}(\alpha_n)(1-\alpha_n)d^p\left(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n\right)\\ \le&\alpha_n d^p(y_n,z)+(1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}d^p(x_n,z)+\frac{\gamma_n}{1-\alpha_n}d^p(x_n,z)\right)\\ &-\frac{c}{2}(\alpha_n)(1-\alpha_n)d^p\left(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n\right)\\ =&\alpha_n d^p(y_n,z)+(1-\alpha_n)d^p(x_n,z)\\ &-\frac{c}{2}(\alpha_n)(1-\alpha_n)d^p\left(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n\right),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{c}{2}(\alpha_n)(1-\alpha_n)&d^p\left(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n\right)\\ &\le\alpha_n d^p(y_n,z)+(1-\alpha_n)d^p(x_n,z)-d^p(x_{n+1},z).\end{aligned}\tag{3.21}$$

From (3.14), (3.10), and by using the same process above, we have

$$\begin{aligned}d^p(y_{n+1},z)=&\alpha_n d^p(x_n,z)+(1-\alpha_n)d^p(y_n,z)\\ &-\frac{c}{2}(\alpha_n)(1-\alpha_n)d^p\left(x_n,\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n\right),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{c}{2}(\alpha_n)(1-\alpha_n)&d^p\left(x_n,\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n\right)\\ &\le\alpha_n d^p(x_n,z)+(1-\alpha_n)d^p(y_n,z)-d^p(y_{n+1},z).\end{aligned}\tag{3.22}$$

Combining (3.21) and (3.22), we get

$$\begin{aligned}\frac{c}{2}(\alpha_n)(1-\alpha_n)&\left(d^p(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n)+d^p(x_n,\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n)\right)\\ \le&\alpha_n(d^p(x_n,z)+d^p(y_n,z))+(1-\alpha_n)(d^p(y_n,z)+d^p(x_n,z))\\ &-(d^p(y_{n+1},z)+d^p(x_{n+1},z))\end{aligned}$$

(3.23)

$$=d^p(y_n,z)+d^p(x_n,z)-(d^p(y_{n+1},z)+d^p(x_{n+1},z)).$$

From (3.23) and the condition of $\{\alpha_n\}$, we obtain that

$$\lim_{n\to\infty}\left(d^p(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n)+d^p(x_n,\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n)\right)=0,$$

then

(3.24)

$d^p(y_n,\frac{\beta_n}{1-\alpha_n}K_Tx_n\oplus\frac{\gamma_n}{1-\alpha_n}x_n)\to 0$ and $d^p(x_n,\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n)\to 0$ as $n\to\infty$.

From the definition of y_n, we get

$$d^p(y_{n+1},x_n)=d^p\left(\alpha_nx_n\oplus(1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n\right),x_n\right)$$

$$\leq\alpha_nd^p(x_n,x_n)+(1-\alpha_n)d^p\left(\frac{\beta_n}{1-\alpha_n}K_Sy_n\oplus\frac{\gamma_n}{1-\alpha_n}y_n,x_n\right). \tag{3.25}$$

From (3.24) and (3.25), we have

$$d^p(y_{n+1},x_n)\to 0 \text{ as } n\to\infty. \tag{3.26}$$

Similarly way (3.26), we have

$$d^p(x_{n+1},y_n)\to 0 \text{ as } n\to\infty.$$

From (3.26), we have

$$\begin{aligned}\limsup_{n\to\infty}d(x,x_n)&\leq\limsup_{n\to\infty}d(x,y_{n+1})+\limsup_{n\to\infty}d(y_{n+1},x_n)\\&=\limsup_{n\to\infty}d(x,y_{n+1}),\qquad\text{for all } x\in X,\end{aligned}$$

then

$$r(x,\{x_n\})\leq r(x,\{y_{n+1}\}). \tag{3.27}$$

By using the same method as (3.27), we have

$$\begin{aligned}\limsup_{n\to\infty}d(x,y_{n+1})&\leq\limsup_{n\to\infty}d(x,x_n)+\limsup_{n\to\infty}d(x_n,y_{n+1})\\&=\limsup_{n\to\infty}d(x,x_n),\qquad\text{for all } x\in X,\end{aligned}$$

then

$$r(x,\{y_{n+1}\})\leq r(x,\{x_n\}). \tag{3.28}$$

From (3.27) and (3.28), we have

$$r(x,\{x_n\})=r(x,\{y_{n+1}\}),\qquad\text{for all } x\in X. \tag{3.29}$$

So, we get

$$r(\{x_n\})=r(\{y_{n+1}\}). \tag{3.30}$$

Since $\{x_n\}$ is bounded and X is a complete p-uniformly convex metric space, then, by Remark 2.1 (i), $\{x_n\}$ have a unique asymptotic center. That is, $A(\{x_n\})=\{x^*\}$.
From (3.29) and $A(\{x_n\})=\{x^*\}$, we get

$$r(x^*,\{x_n\})=r(\{x_n\})=r(x^*,\{y_{n+1}\}). \tag{3.31}$$

From (3.29) and (3.30), we have

$$A(\{x_n\})=\{x\in X;r(x,\{x_n\})=r(\{x_n\})\}$$

$$=\{x \in X; r(x, \{y_{n+1}\}) = r(\{y_{n+1}\})\}$$
$$=A(\{y_{n+1}\}).$$

It follows that $A(\{x_n\}) = A(\{y_{n+1}\}) = \{x^*\}$.
Let $\{x_{n_k}\}$ and $\{y_{n_k}\}$ be any subsequences of $\{x_n\}$ and $\{y_n\}$, respectively, such that $A(\{x_{n_k}\}) = A(\{y_{n_k+1}\}) = \{u\}$.
From (3.20), we have

$$\lim_{k\to\infty} d^p(K_T x_{n_k}, x_{n_k}) = 0 \text{ and } \lim_{k\to\infty} d^p(K_S y_{n_k+1}, y_{n_k+1}) = 0.$$

By Remark 2.1 (ii), 2.2 and by the Δ-demicloseness of K_T and K_S at 0, we obtain that $u \in F(K_T)$ and $u \in F(K_S)$. From Lemma 2.2, then $u \in \bigcap_{i=1}^{N} F(T_i)$ and $u \in \bigcap_{i=1}^{N} F(S_i)$.
Hence $u \in \xi$.
From (3.26), we have

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, u) &\leq \limsup_{k\to\infty} d(x_{n_k}, y_{n_k+1}) + \limsup_{k\to\infty} d(y_{n_k+1}, u)\\
&= \limsup_{k\to\infty} d(y_{n_k+1}, u)\\
&\leq \limsup_{k\to\infty} d(y_{n_k+1}, x^*)\\
&\leq \limsup_{k\to\infty} d(y_{n_k+1}, x_{n_k}) + \limsup_{k\to\infty} d(x_{n_k}, x^*)\\
&= \limsup_{k\to\infty} d(x_{n_k}, x^*)\\
&= r(x^*, \{x_{n_k}\})\\
&= r(\{x_{n_k}\})\\
&= \inf\{r(z^*, \{x_{n_k}\}) : z^* \in X\}\\
&\leq r(u, \{x_{n_k}\})\\
&= \limsup_{k\to\infty} d(x_{n_k}, u),
\end{aligned}$$

which implies that $x^* = u$.
Therefore, $\{x_n\}$ Δ-converges to $x^* \in \xi$.
Similarly, as derived above and since $A(\{x_n\}) = A(\{y_{n+1}\}) = \{x^*\}$, we also have $\{y_n\}$ Δ-converges to $x^* \in \xi$. □

Corollary 3.1. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$, and let $\{T_i\}_{i=1}^N$ be finite families of nonexpansive mappings from X into itself with $\xi = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. For every $N \in \mathbb{N}$, let $K_T : X \to X$ be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$, where $\{\lambda_i\}_{i=1}^N$ are the sequences in $[a, b]$ with $0 < a \leq b < 1$. For given $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} K_T x_n \oplus \frac{\gamma_n}{1-\alpha_n} x_n\right), \tag{3.32}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < s \leq \alpha_n, \beta_n, \gamma_n \leq q < 1$, for all $n \in \mathbb{N}$ and for some $s, q > 0$. Then, the sequence $\{x_n\}$ Δ-converges to some $x^* \in \xi$.

Proof. If we put $\{T_i\}_{i=1}^N \equiv \{S_i\}_{i=1}^N$, $\{\lambda_i\}_{i=1}^N = \{\eta_i\}_{i=1}^N$ and $x_n = y_n$, in Theorem 3.2, we obtain the desired conclusion. □

4. APPLICATION

In this section, we apply our main theorem to prove Δ-convergence to solve the minimization problems in the framework of complete p-uniformly convex metric spaces.

Let f be a real-valued function defined on metric space X. The *minimization problem (MP)* is to find a point $x \in X$ such that

$$f(x) = \inf_{y\in X} f(y),$$

which is denoted by $x := \arg\min_{y\in X} f(y)$. The MP is very important problems in optimization theory, convex analysis, nonlinear analysis and geometry, see more detail in [30, 20, 14].

In 2016, Choi and Ji [6] introduced the notion of p-resolvent map of a proper, convex and lower semi-continuous function f in p-uniformly convex metric space X as follows: For $x \in X$ and $\lambda > 0$,

$$J_\lambda^f(x) = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda} d^p(y,x)\right].$$

Clearly, the p-resolvent mapping generalizes the Moreau-Yosida resolvent mapping defined in $CAT(0)$ spaces. Moreover, Choi and Ji [6] proved the convergence of the proximal point algorithm by the p-resolvent map in p-uniformly convex metric spaces.

Before proving Theorem 4.3, we need the following lemma.

Lemma 4.3. (See [29]) *For $p > 1$, let (X, d) be a p-uniformly convex metric space with parameter $c \geq 2$, and let $f : X \to (-\infty, +\infty]$ be a convex, lower semi-continuous function not identically ∞. Let J_λ^f be the p-resolvent mapping of f such that $F(J_\lambda^f) \neq \emptyset$. Then, for all $\lambda > 0$, we have the following:*

(i) $x^* \in F(J_\lambda^f)$ *if and only if* x^* *is a minimizer of* f*;*

(ii) $d^p(x^*, J_\lambda^f x) + d^p(J_\lambda^f x, x) \leq d^p(x^*, x)$ *for all* $x \in X$ *and* $x^* \in F(J_\lambda^f)$*;*

(iii) J_λ^f *is a generalized quasi-nonexpansive mapping, i.e.,*

$$d^p(J_\lambda^f x, x^*) \leq d^p(x, x^*) \qquad \text{for all } x \in X, x^* \in F(J_\lambda^f);$$

(iv) $d^p(J_\lambda x, x) \leq d^p(J_\mu x, x)$ *for all* $\lambda < \mu$ *and* $x \in X$.

Theorem 4.3. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$. For all $i = 1, 2, ..., N$, let $\{f_i\}, \{g_i\}, X \to (-\infty, \infty)$, be finite families of proper, convex and lower semi-continuous functions and let $J^i_{\lambda_i^f}$ and $J^i_{\lambda_i^g}$ be the p-resolvent mappings of f_i and g_i, respectively, with $\xi = \left(\bigcap_{i=1}^N \arg\min_{y\in X} f_i(y)\right) \cap \left(\bigcap_{i=1}^N \arg\min_{y\in X} g_i(y)\right) \neq \emptyset$. For every $N \in \mathbb{N}$, let $K_f : X \to X$ be the K-mapping generated by $J^1_{\lambda_1^f}, J^2_{\lambda_2^f}, ..., J^{N-1}_{\lambda_{N-1}^f}, J^N_{\lambda_N^f}$ and $\lambda_1, \lambda_2, ..., \lambda_N$, let $K_g : X \to X$ be the K-mapping generated by $J^1_{\lambda_1^g}, J^2_{\lambda_2^g}, ..., J^{N-1}_{\lambda_{N-1}^g}, J^N_{\lambda_N^g}$ and $\eta_1, \eta_2, ..., \eta_N$, where $\{\lambda_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ are the sequences in $[a, b]$ and $[c, d]$ with $0 < a \leq b < 1$ and $0 < c \leq d < 1$, respectively, for all $\lambda_i^f, \lambda_i^g > 0$, for all $i = 1, 2, ..., N$. For given $x_1, y_1 \in X$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} K_f x_n \oplus \frac{\gamma_n}{1-\alpha_n} x_n\right), \\ y_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} K_g y_n \oplus \frac{\gamma_n}{1-\alpha_n} y_n\right), \end{cases} \tag{4.33}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < s \leq \alpha_n, \beta_n, \gamma_n \leq q < 1$, for all $n \in \mathbb{N}$ and for some $s, q > 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ Δ-converge to some $x^* \in \xi$.

Proof. For all $i = 1, 2, ..., N$, we replace $T_i \equiv J^i_{\lambda^f_i}$ and $S_i \equiv J^i_{\lambda^g_i}$ in Theorem 3.2.
From Lemma 4.3 (iii) and the definitions of K_f and K_g, we have

$$d^p(K_f x_n, z) \leq d^p(x_n, z) \qquad \text{and} \qquad d^p(K_g y_n, z) \leq d^p(y_n, z), \quad \text{for all } z \in \xi.$$

From Lemma 4.3 (i), we have $\bigcap_{i=1}^{N} F(J^i_{\lambda^f_i}) = \bigcap_{i=1}^{N} \arg\min_{y\in X} f_i(y)$ and $\bigcap_{i=1}^{N} F(J^i_{\lambda^g_i}) = \bigcap_{i=1}^{N} \arg\min_{y\in X} g_i(y)$, for all $N \in \mathbb{N}$.
By using the same method in Theorem 3.2, we can conclude Theorem 4.3. □

Corollary 4.2. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$. For all $i = 1, 2, ..., N$, let $\{f_i\} : X \to (-\infty, \infty)$ be finite families of proper, convex and lower semi-continuous functions and let $J^i_{\lambda^f_i}$ be the p-resolvent mappings of f_i with $\xi = \bigcap_{i=1}^{N} \arg\min_{y\in X} f_i(y) \neq \emptyset$. For every $N \in \mathbb{N}$, let $K_f : X \to X$ be the K-mapping generated by $J^1_{\lambda^f_1}, J^2_{\lambda^f_2}, ..., J^{N-1}_{\lambda^f_{N-1}}, J^N_{\lambda^f_N}$ and $\lambda_1, \lambda_2, ..., \lambda_N$, where $\{\lambda_i\}_{i=1}^N$ are the sequences in $[a, b]$ with $0 < a \leq b < 1$, for all $\lambda^f_i > 0$, for all $i = 1, 2, ..., N$. For given $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} K_f x_n \oplus \frac{\gamma_n}{1-\alpha_n} x_n\right), \tag{4.34}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < s \leq \alpha_n, \beta_n, \gamma_n \leq q < 1$, for all $n \in \mathbb{N}$ and for some $s, q > 0$. Then, the sequence $\{x_n\}$ Δ-converges to some $x^* \in \xi$.

Proof. For all $i = 1, 2, ..., N$. If we put $f_i \equiv g_i$, $J^i_{\lambda^f_i} \equiv J^i_{\lambda^g_i}$, $\{\lambda_i\}_{i=1}^N = \{\eta_i\}_{i=1}^N$ and $x_n = y_n$, in Theorem 4.3, we obtain the desired conclusion. □

Remark 4.3. Theorem 4.2 can be reduced as follows:

(i) If we put $N = 1$, in Theorem 4.3, then we obtain

$$\begin{cases} x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} J^1_{\varphi_1} x_n \oplus \frac{\gamma_n}{1-\alpha_n} x_n\right), \\ y_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} J^1_{\psi_1} y_n \oplus \frac{\gamma_n}{1-\alpha_n} y_n\right), \end{cases}$$

for all $n \in \mathbb{N}$, by using the same mappings and parameters as in Theorem 4.3. Then, the sequences $\{x_n\}$ and $\{y_n\}$ Δ-converge to some $x^* \in (\arg\min_{y\in X} f(y)) \cap (\arg\min_{y\in X} g(y))$.

(ii) If we put $f \equiv g$, $J^1_{\varphi_1} \equiv J^1_{\psi_1}$ and $x_n = y_n$ in Remark 4.3 (i), then we obtain

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} J^1_{\varphi_1} x_n \oplus \frac{\gamma_n}{1-\alpha_n} x_n\right), \tag{4.35}$$

for all $n \in \mathbb{N}$, by using the same mappings and parameters as in Remark 4.3 (i). Then, the sequence $\{x_n\}$ Δ-converges to some $x^* \in \arg\min_{y\in X} f(y)$.

5. Numerical examples

In this section, we give the following examples to support Theorem 3.2 and Theorem 4.3.

Example 5.2. Let $X = L^p(W, F, \mu)$ be a measure space, where $W = [0, 1]$, F is σ-algebra on W, and $\mu : F \to [0, \infty)$. Let a metric $d : L^p \times L^p \to \mathbb{R}$ be defined by

$$d(f, g) = (\int_W |f - g|^p d\mu)^{\frac{1}{p}}, \qquad \text{for all } p \geq 1,\ f, g \in L^p(W, F, \mu),$$

and with the geodesic joining x to y given by

$$(1 - t)x \oplus ty = (1 - t)x + ty, \qquad \text{for all } t \in [0, 1].$$

For every $i = 1, 2, ..., N$, let the mappings $T_i : X \to X$ be defined by

$$T_i(f(x)) = \frac{f(x)}{2i}, \qquad \text{for all } f \in X \text{ and } x \in W,$$

and let the mappings $S_i : X \to X$ be defined by

$$S_i(g(x)) = \frac{g(x)}{3i}, \qquad \text{for all } g \in X \text{ and } x \in W.$$

Let K_T be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ where $\lambda_i = \frac{1}{i+1}$, for all $i = 1, 2, ..., N$, and K_S be the K-mapping generated by $S_1, S_2, ..., S_N$ and $\eta_1, \eta_2, ..., \eta_N$ where $\eta_i = \frac{1}{i+2}$, for all $i = 1, 2, ..., N$.

Let $x_1, y_1 \in X$ and the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.8), where $\alpha_n = \frac{n+2}{6n}$, $\beta_n = \frac{2n-1}{6n}$, and $\gamma_n = \frac{3n-1}{6n}$, for all $n \in \mathbb{N}$. By the definitions of K_T and K_S, we have $0 \in (\cap_{i=1}^N F(T_i)) \cap (\cap_{i=1}^N F(S_i)) \equiv \xi$. For every $n \in \mathbb{N}$, we can rewrite (3.8) as follows:

$$\begin{aligned} x_{n+1} =& (\frac{n+2}{6n})y_n \oplus (\frac{5n-2}{6n})\left((\frac{2n-1}{5n-2})K_T x_n \oplus (\frac{3n-1}{5n-2})x_n\right), \\ y_{n+1} =& (\frac{n+2}{6n})x_n \oplus (\frac{5n-2}{6n})\left((\frac{2n-1}{5n-2})K_S y_n \oplus (\frac{3n-1}{5n-2})y_n\right). \end{aligned}$$

From Theorem 3.2, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ Δ-converge to 0.

We have given a numerical example to guarantee the convergence of Theorem 3.2, we give $f(x) = x^2$ and $g(x) = 2x^2 - x$, for all $x \in W$. The table 1 and figure 1 show the values of $\{x_n\}$ and $\{y_n\}$, where $x_1 = -1$, $y_1 = 1$ and $n = N = 25$.

TABLE 1. The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = -1$, $y_1 = 1$ and $n = N = 25$

n	x_n	y_n
1	-1.0000	1.0000
2	0.1250	-0.0370
3	0.0508	-0.0224
⋮	⋮	⋮
13	0.0018	0.0019
⋮	⋮	⋮
23	0.0001	0.0001
24	0.0000	0.0000
25	0.0000	0.0000

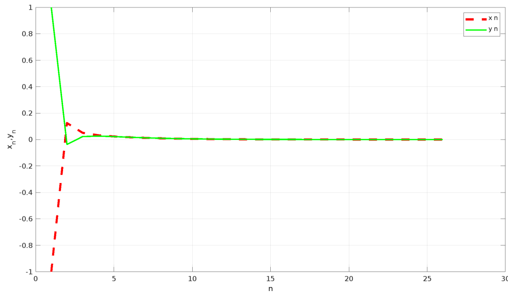

FIGURE 1. The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = -1$, $y_1 = 1$ and $n = N = 25$

Example 5.3. In this example, we use the same mappings and parameters as in Example 5.2 unless the following mappings f_i and g_i, for all $i = 1, 2, ..., N$. Define $f_i(h(x)) = i(h(x))^2$ and $g_i(\hat{h}(x)) = i|\hat{h}(x)|$, for all $h, \hat{h} \in X$ and $x \in W$.

For every $i = 1, 2, ..., N$, let the mappings $J^i_{\lambda^f_i} : X \to X$ be defined by

$$J^i_{\lambda^f_i}(r(x)) = \arg\min_{\hat{r}\in X}[f_i(\hat{r}(x)) + \frac{1}{2\lambda^f_i} d^p(\hat{r}(x), r(x))],$$

where $\lambda^f_i = \frac{1}{i^2+1}$ and for all $r(x) \in X$, and $x \in W$,
and let the mappings $J^i_{\lambda^g_i} : X \to X$ be defined by

$$J^i_{\lambda^g_i}(q(x)) = \arg\min_{\hat{q}\in X}[g_i(\hat{q}(x)) + \frac{1}{2\lambda^g_i} d^p(\hat{q}(x), q(x))],$$

where $\lambda^g_i = \frac{1}{2i+1}$ and for all $q(x) \in X$, and $x \in W$.

Let K_f be the K-mapping generated by $J^1_{\lambda^f_1}, J^2_{\lambda^f_2}, ..., J^N_{\lambda^f_N}$ and $\lambda_1, \lambda_2, ..., \lambda_N$ where $\lambda_i = \frac{1}{i+1}$, for all $i = 1, 2, ..., N$, and K_g be the K-mapping generated by $J^1_{\lambda^g_1}, J^2_{\lambda^g_2}, ..., J^N_{\lambda^g_N}$ and $\eta_1, \eta_2, ..., \eta_N$ where $\eta_i = \frac{1}{i+2}$, for all $i = 1, 2, ..., N$.

Let $x_1, y_1 \in X$ and the sequences $\{x_n\}$ and $\{y_n\}$ generated by (4.33), where the parameters $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ define as the same in Example 5.2. By the definitions of K_f and K_g, we have $0 \in (\cap^N_{i=1} \arg\min_{y\in X} f_i(y)) \cap (\cap^N_{i=1} \arg\min_{y\in C} g_i(y)) \equiv \xi$. For every $n \in \mathbb{N}$, we can rewrite (4.33) as follows:

$$x_{n+1} =(\frac{n+2}{6n})y_n \oplus (\frac{5n-2}{6n})\Big((\frac{2n-1}{5n-2})K_f x_n \oplus (\frac{3n-1}{5n-2})x_n\Big),$$
$$y_{n+1} =(\frac{n+2}{6n})x_n \oplus (\frac{5n-2}{6n})\Big((\frac{2n-1}{5n-2})K_g y_n \oplus (\frac{3n-1}{5n-2})y_n\Big).$$

From Theorem 4.3, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ Δ-converge to 0.

We have given a numerical example to guarantee the convergence of Theorem 4.3, we give $h(x) = \hat{h}(x) = r(x) = \hat{r}(x) = q(x) = \hat{q}(x) = x$, for all $x \in W$. The table 2 and figure 2 show the values of $\{x_n\}$ and $\{y_n\}$, where $x_1 = -1$, $y_1 = 1$, $p = 2$ and $n = N = 30$.

TABLE 2. The values of $\{x_n\}$ and $\{y_n\}$ with $x_1 = -1$, $y_1 = 1$, $p = 2$ and $n = N = 30$

n	x_n	y_n
1	-1.0000	1.0000
2	0.1250	0.0278
3	0.0772	0.0601
⋮	⋮	⋮
15	0.0036	0.0041
⋮	⋮	⋮
28	0.0001	0.0001
29	0.0000	0.0000
30	0.0000	0.0000

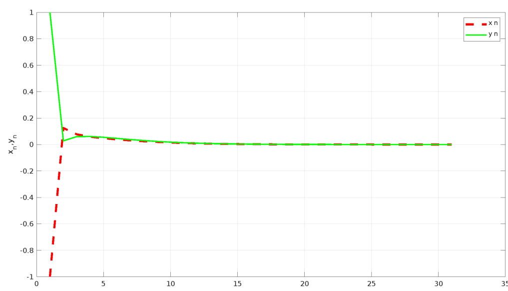

FIGURE 2. The convergences of $\{x_n\}$ and $\{y_n\}$ with $x_1 = -1$, $y_1 = 1$, $p = 2$ and $n = N = 30$

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