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Strong duality in parametric robust semi-definite linear programming and exact relaxations

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ABSTRACT. This paper addresses the issue of which strong duality holds between parametric robust semi-definite linear optimization problems and their dual programs. In the case of a spectral norm uncertainty set, it yields a corresponding strong duality result with a semi-definite programming as its dual. We also show that the dual can be reformulated as a second-order cone programming problem or a linear programming problem when the constraint uncertainty sets of parametric robust semi-definite linear programs are given in terms of affinely parameterized diagonal matrix.

1. INTRODUCTION

As we know, a semi-definite linear programming model problem (SDP) under data uncertainty, due to modelling or estimation errors that come from the lack of information [2, 4, 12, 23], arises in a wide range of engineering applications, in particular in control theory analysis and design [6, 13, 14, 27]. It has also been extensively studied without taking into account data uncertainty as a valuable modeling tool for many optimization problems because SDP can efficiently be solved; see [5, 10, 15, 16, 18, 22, 26] and other references therein.

In this paper, we will be mainly concerned with the *parametric uncertain* semi-definite linear program that is defined as follows: for each *parameter* $v := (v_1, \dots, v_m)$, $v_l \in \mathbb{R}^n$, $l \in \{1, \dots, m\}$, $m, n \in \mathbb{N} := \{1, 2, \dots\}$, an uncertain semi-definite linear program can be captured by the problem:

$$(\mathrm{UP}_v) \qquad \inf_{x\in\mathbb{R}^n} \left\{ c^T x \;\middle|\; A_0 + \sum_{i=1}^{n} x_i A_i \succeq \overline{0}_p \right\},$$

where $c \in V(v)$ and $A_i \in \mathcal{V}_i$, $i = 1, \dots, n$, are *uncertain*, and the uncertainty set $V(v) \subset \mathbb{R}^n$ is assumed to be *polytope* given by $V(v) := \operatorname{conv}\{v_1, \dots, v_m\}$ while the uncertainty sets $\mathcal{V}_i$, $i = 1, \dots, n$, are assumed to be nonempty closed and convex sets in $\mathbb{S}^p$.

Following the deterministic approach, a computationally powerful approach to dealing with data uncertainty in optimization, the *parametric robust* semi-definite linear program (PRSDLP) is defined as follows: for each *parameter* $v \in \mathbb{R}^{m\times n}$, a robust semi-definite linear program (the robust counterpart of (UP_v)) is given by

$$(\mathrm{RP}_v) \qquad \inf_{x\in\mathbb{R}^n} \left\{ \max_{c\in V(v)} c^T x \;\middle|\; A_0 + \sum_{i=1}^{n} x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \mathcal{V}_i, i \in I \right\},$$

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$I := \{0, 1, \ldots, n\}$, where the uncertain objective and constraints are enforced for every possible value of the data within the corresponding uncertainty sets. For a given parameter $v \in \mathbb{R}^{m\times n}$, model problem of the form (RP$_v$) aims at finding a worst-case solution that is immunized against the data uncertainty. Unfortunately, however, (RP$_v$) may not be easily solvable for certain classes of uncertainties.

The study of classes of robust programs which either possess relaxation/ duality properties with numerically tractable dual programs or permit numerically tractable approximations has been made intensively by exploiting special algebraic features of the uncertainty sets as well as linear or polynomial structures of objective/constraints; see [3, 7, 8, 11, 20, 21, 25]. For instance, in the case that there is no uncertain on the objective function, the author in [19] have given necessary and sufficient conditions for the validity of strong duality, in the sense that the optimal values of (RP$_v$) equals the optimal value of its associated dual and the optimal solution of the dual problem is attained. They also showed that the dual can be reformulated as a simple semi-definite linear program under spectral norm uncertainty [1, 2], and so is computationally tractable.

This work has facilitated a way of identifying the tractable class of parametric robust semi-definite linear programs by examining the tractability of (RP$_v$), and so, it provides us with the motivation for establishing strong duality for parametric uncertain semi-definite linear programs. We invite the reader to consult [17] for various characterizations of strong duality in robust optimization problems.

To this aim, we first show that the closedness and the convexity of the characteristic cone of the uncertain linear matrix inequality constraints $A_0 + \sum_{i=1}^{n} x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \mathcal{V}_i,\ i = 0, 1, \ldots, n$, is a necessary and sufficient condition for (RP$_v$), where $v \in \mathbb{R}^{m\times n}$, to have strong duality, whenever the optimal value of (RP$_v$) is finite. This is done by first transforming the (UP$_v$) into a linear optimization problem with uncertain linear matrix inequality constraints using a special variable transformation and then employing the strong duality in [19]. We then provide the weakest condition that exhibits exact SDP relaxations for a class of parametric robust semi-definite linear programs under spectral norm uncertainty. We also present exact second-order cone programming (SOCP) and linear programming (LP) relaxations when the constraint uncertainty sets of parametric robust semi-definite linear programs are given in terms of affinely parameterized diagonal matrix.

It is worth mentioning here that our results differ from the work in [19] which examined strong duality between the robust counterpart of an uncertain semi-definite linear programming model problem with only uncertain constraints and the optimistic counterpart of its uncertain dual. Note also that the objective function of (RP$_v$) is not a linear function, and so, [19, Theorem 2.2] cannot apply directly.

The rest of the paper is organized as follows. Section 2 presents uniform strong duality results for the parametric robust semi-definite linear program (RP$_v$) in terms of the robust characteristic cone. Section 3 present a characterization in terms of uniform exact SDP relaxation for the parametric robust semi-definite linear program (RP$_v$) under spectral norm uncertainty, where for each parameter $v \in \mathbb{R}^{n\times m}$, the relaxation problem of (RP$_v$) is a semi-definite linear program. Section 4 provides simple classes of uncertain semi-definite linear programs with computationally dual programs when the data uncertainty is affinely parametrized diagonal matrix.

Notations: Before we move to the next section, we introduce some necessary notations. The notation $\mathbb{R}^n$ signifies the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $n \in \mathbb{N}$. The origin of any space is denoted by 0 but we may use 0_n for the origin of $\mathbb{R}^n$ in situations where some confusion might be possible, while the symbol $\overline{0}_n$ stands for the zero $(n \times n)$ matrix. For a nonempty set $\Omega \subset \mathbb{R}^n$, conv$\Omega$ denotes the convex hull of Ω and clΩ stands for the closure of Ω, while coneco$\Omega := \mathbb{R}_+\text{conv}\Omega$ stands for the convex conical hull of $\Omega \cup \{0_n\}$, where $\mathbb{R}_+ := [0, +\infty) \subset \mathbb{R}$. As usual, the symbol I_n refers to the identity $(n \times n)$ matrix. A symmetric $(n \times n)$ matrix A is said to be positive semi-definite, denoted by $A \succeq \overline{0}_n$, whenever $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. Note that $A \succeq \overline{0}_n$ if and only if $\text{Tr}(AB) \geq 0$ for all $B \succeq \overline{0}_n$, where $\text{Tr}(\cdot)$ refers to the trace operation. For $A_i \in \mathbb{S}^p$, $i \in I\backslash\{0\}$, the linear operator $\widehat{A} : \mathbb{R}^n \to \mathbb{S}^p$ is defined by $\widehat{A}(x) := \sum_{i=1}^n x_i A_i$. Then the adjoint operator $\widehat{A}^* : \mathbb{S}^p \to \mathbb{R}^n$ is given by $(\widehat{A}^*(F))_i := (\text{Tr}(A_i F))$.

2. Uniform strong duality for Robust SDPs

In this section, we present a characterization of an uniform strong duality for the parametric robust semi-definite linear program (RP$_v$) in terms of the robust characteristic cone which is defined by

$$C := \bigcup_{A_i \in \mathcal{V}_i,\ i \in I} \{(-\text{Tr}(A_1 F), \ldots, -\text{Tr}(A_n F), \text{Tr}(A_0 F) + r) |\ F \succeq \overline{0}_p,\ r \geq 0\},$$

and the proof is motivated by Remark 2.1 along with Theorem 2.3 in [19] and Theorem 2.1 in [9].

Theorem 2.1. *Assume that the parametric robust semi-definite linear program* (PRSDLP) *is feasible, i.e., $X := \{x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^n x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \mathcal{V}_i,\ i \in I\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) *C is closed and convex.*
(ii) *For each $v \in \mathbb{R}^{n\times m}$ with* inf (RP$_v$) $> -\infty$,

$$\inf(\text{RP}_v) = \max_{(A_i, F, \lambda_l)} \Big\{ -Tr(A_0 F) \mid Tr(A_i F) = \sum_{l=1}^m v_l^i \lambda_l, i \in I\backslash\{0\},$$

$$\sum_{l=1}^m \lambda_l = 1,\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ i \in I,\ l = 1, \ldots, m\Big\}. \tag{2.1}$$

Proof. [(i)⇒(ii)] Let $v \in \mathbb{R}^{n\times m}$ be such that inf (RP$_v$) $> -\infty$. Since $V(v) = \text{conv}\{v_1, \ldots, v_m\}$, the problem (RP$_v$) can be equivalently reformulated as

$$\text{(AR}_v) \qquad \inf_{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}} \Big\{x_{n+1} \mid v_l^T x - x_{n+1} \leq 0,\ l = 1, \ldots, m,\ A_0 + \sum_{i=1}^n x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \mathcal{V}_i,\ i \in I\Big\}.$$

Now, letting

$$\mathcal{V}_0' := \left\{ \begin{pmatrix} \overline{0}_p & & & \\ & \ddots & & \\ & & \overline{0}_p & \\ & & & A_0 \end{pmatrix} \middle|\ A_0 \in \mathcal{V}_0 \right\},$$

$$\mathcal{V}_i' := \left\{ \begin{pmatrix} -v_1^i I_p & & & \\ & \ddots & & \\ & & -v_m^i I_p & \\ & & & A_i \end{pmatrix} \middle|\ A_i \in \mathcal{V}_i \right\},\ i \in I\backslash\{0\},$$

$$\mathcal{V}_{n+1}' := \left\{ \begin{pmatrix} I_p & & & \\ & \ddots & & \\ & & I_p & \\ & & & \overline{0}_p \end{pmatrix} \right\}.$$

It can be directly verified that problem (AR_v) amounts to the following one

$$\inf_{(x,x_{n+1})\in\mathbb{R}^n\times\mathbb{R}}\Big\{x_{n+1}\ \Big|\ A_0'+\sum_{i=1}^{n+1}x_iA_i'\succeq\overline{0}_p,\ \forall A_i'\in\mathcal{V}_i',\ i\in I\cup\{n+1\}\Big\}.$$

(Formulating the characteristic cone of (AR_v)**):** Let us first show that the characteristic cone of problem (AR_v), i.e.,

$$\bigcup_{A_i'\in\mathcal{V}_i',\ i\in I\cup\{n+1\}}\{(-\mathrm{Tr}(A_1'F'),\dots,-\mathrm{Tr}(A_n'F'),-\mathrm{Tr}(A_{n+1}'F'),\mathrm{Tr}(A_0'F')+r')\ \big|\ F'\succeq\overline{0}_{p(m+1)},\ r'\geq 0\},$$

is equivalent to the following one

$$K:=\mathrm{coneco}\{(v_l,-1,0)\mid l=1,\dots,m\}+\widehat{C},\tag{2.2}$$

where

$$\widehat{C}:=\cup_{A_i\in\mathcal{V}_i,\ i\in I}\{(-\mathrm{Tr}(A_1F),\dots,-\mathrm{Tr}(A_nF),0,\mathrm{Tr}(A_0F)+r)\mid F\succeq\overline{0}_p,\ r\geq 0\}.$$

Indeed, let $(x,x_{n+1},\alpha)\in K$ be arbitrary. Then, there exist $\lambda_l\geq 0$, $l=1,\dots,m$, $A_i\in\mathcal{V}_i$, $i\in I$, $F\succeq\overline{0}_p$, and $r\geq 0$ such that $x_i=\sum_{l=1}^m\lambda_lv_l^i-\mathrm{Tr}(A_iF),\ i\in I\backslash\{0\}$, $x_{n+1}=-\sum_{l=1}^m\lambda_l$ and $\alpha=\mathrm{Tr}(A_0F)+r$. By letting $r':=r$, for each $i\in I\backslash\{0\}$,

$$A_0':=\begin{pmatrix}\overline{0}_p&&&\\&\ddots&&\\&&\overline{0}_p&\\&&&A_0\end{pmatrix},\quad A_i':=\begin{pmatrix}-v_1^iI_p&&&\\&\ddots&&\\&&-v_m^iI_p&\\&&&A_i\end{pmatrix},$$

$$A_{n+1}':=\begin{pmatrix}I_p&&&\\&\ddots&&\\&&I_p&\\&&&\overline{0}_p\end{pmatrix},\quad F':=\begin{pmatrix}\frac{\lambda_1}{p}I_p&&&\\&\ddots&&\\&&\frac{\lambda_m}{p}I_p&\\&&&F\end{pmatrix},$$

we obtain that

$$F'\succeq\overline{0}_{p(m+1)},\ \mathrm{Tr}(A_0'F')+r'=\mathrm{Tr}(A_0F)+r=\alpha$$

$$\mathrm{Tr}(A_i'F_i')=\sum_{l=1}^m\mathrm{Tr}\left(\frac{-\lambda_lv_l^i}{p}I_p\right)+\mathrm{Tr}(A_iF)=-x_i\text{ for all }i\in I\backslash\{0\}\text{ and}$$

$$\mathrm{Tr}(A_{n+1}'F_{n+1}')=\sum_{l=1}^m\mathrm{Tr}\left(\frac{\lambda_l}{p}I_p\right)=-x_{n+1},$$

showing that

$$(x,x_{n+1},\alpha)\in\bigcup_{\substack{A_i'\in\mathcal{V}_i',\\ i\in I\cup\{n+1\}}}\{(-\mathrm{Tr}(A_1'F'),\dots,-\mathrm{Tr}(A_n'F'),-\mathrm{Tr}(A_{n+1}'F'),\mathrm{Tr}(A_0'F')+r')\ \big|\ F'\succeq\overline{0}_{p(m+1)},\ r'\geq 0\}.$$

Conversely, let (x,x_{n+1},α) be an arbitrary element of the set

$$\bigcup_{\substack{A_i'\in\mathcal{V}_i',\\ i\in I\cup\{n+1\}}}\{(-\mathrm{Tr}(A_1'F'),\dots,-\mathrm{Tr}(A_n'F'),-\mathrm{Tr}(A_{n+1}'F'),\mathrm{Tr}(A_0'F')+r')\ \big|\ F'\succeq\overline{0}_{p(m+1)},\ r'\geq 0\},$$

which in turn implies to the assertion that there exist $A_i'\in\mathcal{V}_i$, $i\in I\cup\{n+1\}$, $r'\geq 0$ and $F'=\begin{pmatrix}Z_1&&*&*\\&\ddots&&*\\ *&&Z_m&\\ *&*&&Z\end{pmatrix}\succeq\overline{0}_{p(m+1)}$, for some $Z,Z_l\in\mathbb{S}^p$, $l=1,\dots,m$, such that $\alpha=\mathrm{Tr}(A_0'F')+r'=\mathrm{Tr}(A_0Z)+r'$, $x_i=-\mathrm{Tr}(A_i'F_i')=-\sum_{l=1}^m\mathrm{Tr}\left(-v_l^iZ_l\right)-\mathrm{Tr}(A_iZ)$ for all $i\in I\backslash\{0\}$ and $x_{n+1}=-\mathrm{Tr}(A_{n+1}'F_{n+1}')=-\sum_{l=1}^m\mathrm{Tr}\left(Z_l\right)$. For any $B\succeq\overline{0}_p$, letting

$\widetilde{B} := \begin{pmatrix} \overline{0}_p & & & \\ & \ddots & & \\ & & \overline{0}_p & \\ & & & B \end{pmatrix}$, and $\widetilde{B}^l$, $l = 1, \ldots, m$, is the $((m+1) \times (m+1))$ diagonal block matrix with B in the (l,l)th entry and zeros elsewhere. Then, it follows that $\widetilde{B} \succeq \overline{0}_{p(m+1)}$ and $\widetilde{B}^l \succeq \overline{0}_{p(m+1)}$ for all $l = 1, \ldots, m$. As $F' \succeq \overline{0}_{p(m+1)}$, we have that for each $l = 1, \ldots, m$, $\mathrm{Tr}(ZB) = \mathrm{Tr}(F'\widetilde{B}) \geq 0$ and $\mathrm{Tr}(Z_l B) = \mathrm{Tr}(F'\widetilde{B}^l) \geq 0$ which imply that $Z \succeq \overline{0}_p$ and $Z_l \succeq \overline{0}_p$. Letting $\lambda_l := \mathrm{Tr}(Z_l) \geq 0$ for each $l = 1, \ldots, m$, we arrive at

$$(x, x_{n+1}, \alpha) = \sum_{l=1}^{m} \lambda_l (v_l, -1, 0) + (-\widehat{A}^*(Z), 0, \mathrm{Tr}(A_0 Z) + r') \in K.$$

(C is closed $\Leftrightarrow$ K is closed): Assume that K is closed and consider any sequence $(x^k, x^k_{n+1}) \to (x, x_{n+1})$ as $k \to +\infty$, where $(x^k, x^k_{n+1}) \in C$, $\forall k \in \mathbb{N}$. Then, there exist $A^k_i \in \mathcal{V}_i$, $i \in I$, $F^k \succeq \overline{0}_p$ and $r^k \geq 0$ such that

$$(x^k, x^k_{n+1}) = (-\widehat{A^k}^*(F^k), \mathrm{Tr}(A^k_0 F^k) + r^k).$$

We first show that the sequence $\{r^k\}$ is bounded. Otherwise, by taking a subsequence if necessary we may assume that $r^k \to +\infty$ as $k \to +\infty$. By letting,

$$\widetilde{z}_k := \frac{1}{r^k}(-\widehat{A^k}^*(F^k), 0, \mathrm{Tr}(A^k_0 F^k)) \in K$$

for all $k \in \mathbb{N}$ and $\widetilde{z}_k = \frac{1}{r^k}(x^k, 0, x^k_{n+1}) - (0_n, 0, 1) \to -(0_n, 0, 1)$ as $k \to +\infty$. Since K is closed, it follows that $-(0_n, 0, 1) \in K$. Then, there exist $\widetilde{\lambda}_l \geq 0$, $l = 1, \ldots, m$, $\widetilde{A}_i \in \mathcal{V}_i$, $i \in I$, $\widetilde{F} \succeq \overline{0}_p$, and $\widetilde{r} \geq 0$ such that

$$0_n = \sum_{l=1}^{m} \widetilde{\lambda}_l v_l - \widehat{\widetilde{A}}^*(\widetilde{F}),\ 0 = -\sum_{l=1}^{m} \widetilde{\lambda}_l,\ -1 = \mathrm{Tr}(\widetilde{A}_0 \widetilde{F}) + \widetilde{r}.$$

On the other hand, as $X \neq \emptyset$, there is $x^0 \in \mathbb{R}^n$ such that $\widetilde{A}_0 + \sum_i^n x^0_i \widetilde{A}_i \succeq \overline{0}_p$. So,

$$0 \leq \mathrm{Tr}\left(\left(\widetilde{A}_0 + \sum_{i=1}^{n} x^0_i \widetilde{A}_i\right)\widetilde{F}\right) = \mathrm{Tr}(\widetilde{A}_0 \widetilde{F}) + \sum_{i=1}^{n} x^0_i \mathrm{Tr}(\widetilde{A}_i \widetilde{F}) = \mathrm{Tr}(\widetilde{A}_0 \widetilde{F}) = -1 - \widetilde{r},$$

which contradicts the fact that $\widetilde{r} \geq 0$ and hence the sequence $\{r^k\}$ must be bounded. Thus, by passing to subsequence if necessary, we can assume that $r^k \to r_0 \geq 0$ as $k \to +\infty$. Then, by letting $z_k := (-\widehat{A^k}^*(F^k), 0, \mathrm{Tr}(A^k_0 F^k))$, we obtain that $z_k \in K$ for all $k \in \mathbb{N}$ and $z_k \to (x, 0, x_{n+1} - r_0) \in K$ as $k \to +\infty$. Therefore, we find $\lambda_l \geq 0$, $l = 1, \ldots, m$, $A_i \in \mathcal{V}_i$, $i \in I$, $F \succeq \overline{0}_p$, and $r \geq 0$ such that

$$x = \sum_{l=1}^{m} \lambda_l v_l - \widehat{A}^*(F),\ 0 = -\sum_{l=1}^{m} \lambda_l,\ x_{n+1} - r_0 = \mathrm{Tr}(A_0 F) + r,$$

and consequently, $(x, x_{n+1}) = (-\widehat{A}^*(F), \mathrm{Tr}(A_0 F) + r + r_0) \in C$. This shows that C is closed.

Conversely, let C be closed. It can be verified that the closedness of C implies the closedness of $\widehat{C}$ and thus, we can employ [24, Corollary 9.1.3] to assert that

$$\begin{aligned} \mathrm{cl}K &= \mathrm{clconeco}\{(v_l, -1, 0) \mid l = 1, \ldots, m\} + \mathrm{cl}\widehat{C} \\ &= \mathrm{coneco}\{(v_l, -1, 0) \mid l = 1, \ldots, m\} + \widehat{C} = K, \end{aligned}$$

which entails that K is closed.

(Guaranteeing robust strong duality): Let us note that, by assumption, $\widehat{C}$ is convex. Therefore, K is convex as well by the expression (2.2). Now, invoking [19, Theorem 2.1] with $c := (0_n^T, 1)^T$, we see that the closedness together with the convexity of K guarantees that the robust strong duality holds for the problem (AR$_v$) with the attainment of the dual problem. Namely,

$$\inf(\mathrm{AR}_v) = \max_{(A_i', F')} \Big\{ -\mathrm{Tr}(A_0'F') \mid \mathrm{Tr}(A_i'F') = 0,\ i \in I\backslash\{0\},$$
$$\mathrm{Tr}(A_{n+1}'F') = 1,\ A_i' \in \mathcal{V}_i',\ F' \succeq \overline{0}_{p(m+1)},\ i \in I \cup \{n+1\}\Big\}. \tag{2.3}$$

We claim that the problem in the right hand-side of (2.3) is equivalent to the following one

$$\max_{(A_i,F,\lambda_l)} \Big\{ -\mathrm{Tr}(A_0F) \mid \mathrm{Tr}(A_iF) = \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},\ \sum_{l=1}^m \lambda_l = 1,\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ i \in I,$$
$$l = 1, \ldots, m\Big\}.$$

To see this, consider any $A_i' \in \mathcal{V}_i'$, $F' \succeq \overline{0}_{p(m+1)}$, $i \in I \cup \{n+1\}$ with $\mathrm{Tr}(A_i'F') = 0$, $\forall i \in I\backslash\{0\}$ and $\mathrm{Tr}(A_{n+1}'F') = 1$. Arguing as before, we arrive at $\mathrm{Tr}(A_0'F') = \mathrm{Tr}(A_0Z)$, $0 = \mathrm{Tr}(A_i'F_i') = \sum_{l=1}^m \mathrm{Tr}\left(-v_l^iZ_l\right) + \mathrm{Tr}(A_iZ),\ i \in I\backslash\{0\}$ and $1 = \mathrm{Tr}(A_{n+1}'F_{n+1}') = \sum_{l=1}^m \mathrm{Tr}\left(Z_l\right)$, for some $Z, Z_l \succeq \overline{0}_p$, $l = 1, \ldots, m$. Letting $\overline{F} := Z$, $\overline{\lambda}_l := \mathrm{Tr}\left(Z_l\right) \geq 0$ for all $l = 1, \ldots, m$, we get

$$\max_{(A_i',F')} \Big\{ -\mathrm{Tr}(A_0'F') \mid \mathrm{Tr}(A_i'F') = 0,\ i \in I\backslash\{0\}, \mathrm{Tr}(A_{n+1}'F') = 1,\ A_i' \in \mathcal{V}_i',\ F' \succeq \overline{0}_{p(m+1)},\ i \in I \cup \{n+1\}\Big\}$$
$$\leq \max_{(A_i,F,\lambda_l)} \Big\{ -\mathrm{Tr}(A_0F) \mid \mathrm{Tr}(A_iF) = \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},\ \sum_{l=1}^m \lambda_l = 1,\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ i \in I,$$
$$l = 1, \ldots, m\Big\}.$$

On the other hand, for any $A_i \in \mathcal{V}_i$, $F \succeq \overline{0}_p$, $\lambda_l \geq 0$, $i \in I$, $l = 1, \ldots, m$ with $\mathrm{Tr}(A_iF) = \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\}$ and $\sum_{l=1}^m \lambda_l = 1$, letting

$$F' := \begin{pmatrix} \frac{\lambda_1}{p} I_p & & & \\ & \ddots & & \\ & & \frac{\lambda_m}{p} I_p & \\ & & & F \end{pmatrix} \succeq \overline{0}_{p(m+1)},$$

we have, $\mathrm{Tr}(A_0'F') = \mathrm{Tr}(A_0F)$, $\mathrm{Tr}(A_i'F_i') = -\sum_{l=1}^m v_l^i\lambda_l + \mathrm{Tr}(A_iF) = 0,\ i \in I\backslash\{0\}$ and $\mathrm{Tr}(A_{n+1}'F_{n+1}') = \sum_{l=1}^m \lambda_l = 1$. This gives us that

$$\max_{(A_i,F,\lambda_l)} \Big\{ -\mathrm{Tr}(A_0F) \mid \mathrm{Tr}(A_iF) = \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},$$
$$\sum_{l=1}^m \lambda_l = 1,\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ i \in I,\ l = 1, \ldots, m\Big\}$$
$$\leq \max_{(A_i',F')} \Big\{ -\mathrm{Tr}(A_0'F') \mid \mathrm{Tr}(A_i'F') = 0,\ i \in I\backslash\{0\},$$
$$\mathrm{Tr}(A_{n+1}'F') = 1,\ A_i' \in \mathcal{V}_i',\ F' \succeq \overline{0}_{p(m+1)},\ i \in I \cup \{n+1\}\Big\}.$$

Hence, the equality holds, and completes the proof the implication (i) $\Rightarrow$ (ii).

[(ii) $\Rightarrow$ (i)] Assume that (ii) holds. Let $(z, z_{n+1}) \in \mathrm{clconv}C$. So, for each $x \in X$, we have $-z^Tx \geq -z_{n+1} > -\infty$ (see e.g., in the proof of [19, Theorem 2.2]). Now, applying (ii) with $m := 1$ and $v := -z \in \mathbb{R}^n$, we obtain that

$$\max_{(A_i,F)} \Big\{ -\mathrm{Tr}(A_0F) \mid \mathrm{Tr}(A_iF) = -z_i,\ i \in I\backslash\{0\},\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_p,\ i \in I\Big\} \geq -z_{n+1}.$$

It follows that there exist $A_i \in \mathcal{V}_i$, $i \in I$, and $F \succeq \overline{0}_p$ with $\mathrm{Tr}(A_i F) = -z_i$, such that $z_{n+1} \geq \mathrm{Tr}(A_0 F)$. Consequently, $(z, z_{n+1}) \in C$, showing that clconv$C \subseteq C$ and so, clconv$C = C$. Therefore, C is closed and convex, thereby establishing the desired result. □

Remark 2.1. It should be noted that Theorem 2.1 differ from the work in [19] which examined strong duality between the robust counterpart of an uncertain semi-definite linear programming model problem with only uncertain constraints and the optimistic counterpart of its uncertain dual. Note also that the objective function of (RP$_v$), in general, is not a linear function, and so, [19, Theorem 2.2] cannot apply directly. Let us illustrate this remark by the following simple example, which is motivated by [19, Example 2.1].

Example 2.1. Consider a parametric robust semi-definite linear programming problem defined as follows: for each $v = (v_1, v_2) \in \mathbb{R}^{2\times 2}$, a robust semi-definite linear program is given by

$$(\mathrm{EP}_v) \qquad \inf_{x\in\mathbb{R}^2} \Big\{ \max_{c\in V(v)} \{c_1x_1 + c_2x_2\} \;\Big|\; A_0 + x_1A_1 + x_2A_2 \succeq \overline{0}_3,\ \forall A_i \in \mathcal{V}_i, i = 0,1,2\Big\},$$

where $x = (x_1, x_2)^T$, $c = (c_1, c_2)^T$, the uncertainty set $V(v)$ is a polytope given by $V(v) = \mathrm{conv}\{v_1, v_2\} \subseteq \mathbb{R}^2$ and the uncertainty sets $\mathcal{V}_i$, $i \in I = \{0,1,2\}$, are given by

$$\mathcal{V}_0 = \mathcal{V}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \text{ and } \mathcal{V}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -a & 0 \end{pmatrix} \;\middle|\; a \in [0,1] \right\}.$$

In this case,

$$\begin{aligned} C &= \bigcup_{A_i\in\mathcal{V}_i,\ i\in I} \{(-\mathrm{Tr}(A_1F), -\mathrm{Tr}(A_2F), \mathrm{Tr}(A_0F) + r) |\ F \succeq \overline{0}_p,\ r \geq 0\} \\ &= \bigcup_{a\in[0,1]} \left\{ (-2af_5, -f_1, f_1 + r) \;\middle|\; F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_2 & f_4 & f_5 \\ f_3 & f_5 & f_6 \end{pmatrix} \succeq \overline{0}_3,\ r \geq 0 \right\} \\ &= \mathbb{R} \times (-\mathbb{R}_+) \times \mathbb{R}_+, \end{aligned}$$

which is closed and convex. So, we assert by Theorem 2.1 that strong duality holds for all robust semi-definite linear programs (EP$_v$), $v \in \mathbb{R}^{2\times 2}$, whenever $\inf$ (EP$_v$) $> -\infty$. Let us now verify the optimal value of problem (EP$_v$) with (for instance) $\bar{v} := (\bar{v}_1, \bar{v}_2)$, $\bar{v}_1 := (0,-1)^T$, $\bar{v}_2 := (0,1)^T$, which amounts to the following one

$$(\mathrm{EP}_{\bar{v}}) \qquad \inf_{x\in\mathbb{R}^2} \{|x_2| \mid x_1 = 0,\ x_2 \geq -1\}.$$

It can be verified that $\bar{x} := (0,0)^T$ is an optimal solution of problem (EP$_{\bar{v}}$) with the optimal value $\inf$ (EP$_{\bar{v}}$) $= 0$. Consider the following the optimistic counterpart of (EP$_{\bar{v}}$):

$$(\mathrm{OP}_{\bar{v}}) \qquad \begin{aligned} &\max_{(A_i,F,\lambda_1,\lambda_2)} \Big\{ -\mathrm{Tr}(A_0F) \mid \mathrm{Tr}(A_1F) = \bar{v}_1^1\lambda_1 + \bar{v}_2^1\lambda_2,\ \mathrm{Tr}(A_2F) = \bar{v}_1^2\lambda_1 + \bar{v}_2^2\lambda_2, \\ &\qquad \lambda_1 + \lambda_2 = 1,\ A_i \in \mathcal{V}_i,\ F \succeq \overline{0}_3,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0\ i \in I\Big\} \\ &= \max_{a\in[0,1]} \max_{F\in\mathbb{S}^3} \max_{(\lambda_1,\lambda_2)} \Big\{ -f_1 \;\Big|\; -2af_5 = 0,\ f_1 = -\lambda_1 + \lambda_2, \\ &\qquad \lambda_1 + \lambda_2 = 1,\ F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_2 & f_4 & f_5 \\ f_3 & f_5 & f_6 \end{pmatrix} \succeq \overline{0}_3,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0\Big\}. \end{aligned}$$

As $F \succeq \overline{0}_3$, we get $f_1 \geq 0$ and so, $\max$ (OP$_{\bar{v}}$) $= 0$. This shows that the strong duality holds for (EP$_{\bar{v}}$), that is, $\max$ (OP$_{\bar{v}}$) $= 0 = \min$ (EP$_{\bar{v}}$).

We close this section with a remark that in the spacial case where there is no uncertainty in the objective function, i.e. $m := 1$ and $v := c$ with a given $c \in \mathbb{R}^n$, the preceding theorem collapses to the strong duality result for semi-definite linear programming problem under constraint data uncertainly which was established in [19].

3. Uniform exact SDP relaxation

This section is devoted to presenting an application of our characterization of uniform strong duality to classes of parametric robust semi-definite linear programming problems with spectral norm uncertainty. In this circumstance, we show that the relaxation problem of each robust semi-definite linear program is a single semi-definite linear program.

In what follows, let us consider the uncertain semi-definite programming problem with spectral norm uncertainty:

$$\widetilde{\mathcal{V}}_i := \{\overline{A}_i + \rho_i \Delta_i \mid \Delta_i \in \mathbb{S}^p,\ \|\Delta_i\|_{\text{spec}} \leq 1\}, \tag{3.4}$$

where $\overline{A}_i \in \mathbb{S}^p$, $\rho_i \geq 0$ and $\|\Delta_i\|_{\text{spec}}$ denotes the square root of the largest eigenvalue of the matrix $\Delta_i^T \Delta_i$. In this case, the uncertainty set $\widetilde{\mathcal{V}}_i$ is just a closed ball with center $\overline{A}_i$ and radius ρ_i in the matrix space $\mathbb{S}^p$.

In this way, we need the following proposition.

Proposition 3.1. [19, Proposition 3.1] *Let $\widetilde{\mathcal{V}}_i$ as in* (3.4), $\overline{A}_i \in \mathbb{S}^p$. *Then, the following statements hold:*

(i) *The robust characteristic cone*

$$C := \bigcup_{A_i \in \widetilde{\mathcal{V}}_i,\ i \in I} \{(-Tr(A_1 F), \ldots, -Tr(A_n F), Tr(A_0 F) + r) \mid F \succeq \overline{0}_p,\ r \geq 0\}$$

is convex.

(ii) *For each fixed $F \succeq \overline{0}_p$, one has*

$$[Tr((\overline{A}_i - \rho_i I_p)F), Tr((\overline{A}_i + \rho_i I_p)F)] = \bigcup_{\|\Delta_i\|_{spec} \leq 1} \{Tr((\overline{A}_i + \rho_i \Delta_i)F)\},\ i \in I,$$

and so,

$$\inf_{\|\Delta_i\|_{spec} \leq 1} \{Tr((\overline{A}_i + \rho_i \Delta_i)F)\} = Tr((\overline{A}_i - \rho_i I_p)F). \tag{3.5}$$

Theorem 3.2. *Assume that the parametric robust semi-definite linear program* (PRSDLP) *is feasible, i.e., $X := \{x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^n x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \widetilde{\mathcal{V}}_i,\ i \in I\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) *The robust characteristic cone*

$$C := \bigcup_{\|\Delta_i\|_{spec} \leq 1} \{(-Tr((\overline{A}_1 + \rho_1 \Delta_1)F), \ldots, -Tr((\overline{A}_n + \rho_n \Delta_n)F),\ Tr((\overline{A}_0 + \rho_0 \Delta_0)F) + r) \Big| F \succeq \overline{0}_p,\ r \geq 0\}$$

is closed.

(ii) *For each $v \in \mathbb{R}^{n \times m}$ with* $\inf(\mathrm{RP}_v) > -\infty$, *the uniform exact SDP relaxation holds for* (PRSDLP) *in the sense that*

$$\begin{aligned} \inf(\mathrm{RP}_v) = \max_{(F,\lambda_l)} \Big\{ -Tr((\overline{A}_0 - \rho_0 I_p)F) \mid & Tr((\overline{A}_i - \rho_i I_p)F) \leq \sum_{l=1}^m v_l^i \lambda_l,\ i \in I \backslash \{0\}, \\ & Tr((\overline{A}_i + \rho_i I_p)F) \geq \sum_{l=1}^m v_l^i \lambda_l,\ i \in I \backslash \{0\},\ \sum_{l=1}^m \lambda_l = 1, F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ l = 1, \ldots, m\Big\}. \end{aligned} \tag{3.6}$$

Proof. We first note that

$$\begin{aligned}C :=& \bigcup_{A_i\in\widetilde{\mathcal{V}}_i,\ i\in I} \{(-\mathrm{Tr}(A_1F),\ldots,-\mathrm{Tr}(A_nF),\mathrm{Tr}(A_0F)+r)|\ F\succeq \overline{0}_p,\ r\geq 0\}\\ =& \bigcup_{\|\Delta_i\|_{\text{spec}}\leq 1,\ i\in I} \{(-\mathrm{Tr}((\overline{A}_1+\rho_1\Delta_1)F),\ldots,-\mathrm{Tr}((\overline{A}_n+\rho_n\Delta_n)F),\ \mathrm{Tr}((\overline{A}_0+\rho_0\Delta_0)F)+r)\mid F\succeq \overline{0}_p,\ r\geq 0\}.\end{aligned}$$

Invoking Proposition 3.1(i), we conclude that the cone C is convex. Note also that

$$\begin{aligned}&\max_{(\Delta_i,F,\lambda_l)}\{-\mathrm{Tr}((\overline{A}_0+\rho_0\Delta_0)F)\mid \mathrm{Tr}((\overline{A}_i+\rho_i\Delta_i)F)=\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\},\\ &\sum_{l=1}^m\lambda_l=1,\ \|\Delta_i\|_{\text{spec}}\leq 1,\ F\succeq\overline{0}_p,\ \lambda_l\geq 0,\ i\in I,\ l=1,\ldots,m\}\\ =&\max_{(\Delta_i,F,\lambda_l)}\{-\mathrm{Tr}((\overline{A}_0-\rho_0I_p)F)\mid \mathrm{Tr}((\overline{A}_i+\rho_i\Delta_i)F)=\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\},\\ &\sum_{l=1}^m\lambda_l=1,\ \|\Delta_i\|_{\text{spec}}\leq 1,\ F\succeq\overline{0}_p,\ \lambda_l\geq 0,\ i\in I\backslash\{0\},\ l=1,\ldots,m\},\end{aligned}\tag{3.7}$$

where the last equality holds due to (3.5). Here, for each $F\succeq\overline{0}_p$ fulfilling $\mathrm{Tr}((\overline{A}_i+\rho_i\Delta_i)F)=\sum_{l=1}^m v_l^i\lambda_l$, we get $\mathrm{Tr}((\overline{A}_i-\rho_iI_p)F)\leq\sum_{l=1}^m v_l^i\lambda_l$ and $\mathrm{Tr}((\overline{A}_i+\rho_iI_p)F)\geq\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\}$. Thus,

$$\begin{aligned}&\max_{(\Delta_i,F,\lambda_l)}\{-\mathrm{Tr}((\overline{A}_0-\rho_0I_p)F)\mid \mathrm{Tr}((\overline{A}_i+\rho_i\Delta_i)F)=\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\},\\ &\sum_{l=1}^m\lambda_l=1,\ \|\Delta_i\|_{\text{spec}}\leq 1,\ F\succeq\overline{0}_p,\ \lambda_l\geq 0,\ i\in I\backslash\{0\},\ l=1,\ldots,m\}\\ \leq&\max_{(F,\lambda_l)}\Big\{-\mathrm{Tr}((\overline{A}_0-\rho_0I_p)F)\mid \mathrm{Tr}((\overline{A}_i-\rho_iI_p)F)\leq\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\},\\ &\mathrm{Tr}((\overline{A}_i+\rho_iI_p)F)\geq\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\},\ \sum_{l=1}^m\lambda_l=1,\\ &F\succeq\overline{0}_p,\ \lambda_l\geq 0,\ l=1,\ldots,m\Big\}.\end{aligned}\tag{3.8}$$

On the one hand, for any $F\succeq\overline{0}_p$ with $\mathrm{Tr}((\overline{A}_i-\rho_iI_p)F)\leq\sum_{l=1}^m v_l^i\lambda_l$ and $\mathrm{Tr}((\overline{A}_i+\rho_iI_p)F)\geq\sum_{l=1}^m v_l^i\lambda_l,\ i\in I\backslash\{0\}$, Proposition 3.1(ii) gives us that there exist $\Delta_i\in\mathbb{S}^p$ satisfying

$\|\Delta_i\|_{\text{spec}} \leq 1$ such that $\text{Tr}((\overline{A}_i + \rho_i\Delta_i)F) = \sum_{l=1}^m v_l^i \lambda_l$. This entails that

$$\max_{(\Delta_i,F,\lambda_l)} \{-\text{Tr}((\overline{A}_0 - \rho_0 I_p)F) \mid \text{Tr}((\overline{A}_i + \rho_i\Delta_i)F) = \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\}$$

$$\sum_{l=1}^m \lambda_l = 1,\ \|\Delta_i\|_{\text{spec}} \leq 1,\ F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ i \in I\backslash\{0\},\ l = 1,\ldots,m\}$$

$$\geq \max_{(F,\lambda_l)} \Big\{ -\text{Tr}((\overline{A}_0 - \rho_0 I_p)F) \mid \text{Tr}((\overline{A}_i - \rho_i I_p)F) \leq \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},$$

$$\text{Tr}((\overline{A}_i + \rho_i I_p)F) \geq \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},\ \sum_{l=1}^m \lambda_l = 1,$$

$$F \succeq \overline{0}_p,\ \lambda_l \geq 0,\ l = 1,\ldots,m\Big\}. \tag{3.9}$$

Therefore, in view of (3.7), (3.8) and (3.9), the conclusion will follow by applying Theorem 2.1. □

As a corollary, we now derive a characterization of solution in terms of linear inequalities for the robust semi-definite linear program (RP_v) with each fixed $v \in \mathbb{R}^{m\times n}$.

Corollary 3.1 (Characterization of solution for (RP_v))**.** *Let $v \in \mathbb{R}^{n\times m}$ be such that the problem (RP_v) has an optimal solution, and let $\bar{x} \in \{x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^n x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \widetilde{\mathcal{V}}_i,\ i \in I\}$. Assume that the characteristic cone*

$$C := \bigcup_{\|\Delta_i\|_{spec}\leq 1} \{(-Tr((\overline{A}_1 + \rho_1\Delta_1)F), \ldots, -Tr((\overline{A}_n + \rho_n\Delta_n)F),\ Tr((\overline{A}_0 + \rho_0\Delta_0)F) + r) \Big| F \succeq \overline{0}_p,\ r \geq 0\}$$

is closed. Then, $\bar{x}$ is an optimal solution of problem (RP_v) if and only if there exist $F \succeq \overline{0}_p$, $\lambda_l \geq 0$, $l = 1,\ldots,m$, such that

$$Tr((\overline{A}_i - \rho_i I_p)F) \leq \sum_{l=1}^m v_l^i\lambda_l,$$

$$Tr((\overline{A}_i + \rho_i I_p)F) \geq \sum_{l=1}^m v_l^i\lambda_l,\ i \in I\backslash\{0\},$$

$$\sum_{l=1}^m \lambda_l = 1 \textit{ and } \max_{l=1,\ldots,m} v_l^T\bar{x} + Tr((\overline{A}_0 - \rho_0 I_p)F) = 0.$$

Proof. The proof is completed immediately with the aid of Theorem 3.2. □

4. Affine parameterizations and exact relaxations

In this section, we provide two classes of parametric robust semi-definite linear programming problems where the corresponding dual problems can be reformulated as simple linear programming problems or second order programming problems, and so, are computationally tractable.

In the sequel, for a given parameter $v \in \mathbb{R}^{m\times n}$, we consider the parametric robust SDP under affinely paramterized diagonal matrix data uncertainty:

$$(\text{RP}_v) \qquad \inf_{x\in\mathbb{R}^n} \left\{ \max_{c\in V(v)} c^T x \;\Big|\; A_0 + \sum_{i=1}^n x_i A_i \succeq \overline{0}_p,\ \forall A_i \in \overline{\mathcal{V}}_i,\ i \in I \right\},$$

where for each $i \in I$,

$$\overline{\mathcal{V}}_i := \Big\{\mathrm{diag}\left(a_{i1}^{(0)} + \sum_{j=1}^{k} u_{i1}^{(j)} a_{i1}^{(j)}, \ldots, a_{ip}^{(0)} + \sum_{j=1}^{k} u_{ip}^{(j)} a_{ip}^{(j)}\right) \Big| \, u_{is} := (u_{is}^1, \ldots, u_{is}^k) \in \overline{\mathcal{U}}_{is}, \ s = 1, \ldots, p\Big\},$$

$a_{is}^{(j)} \in \mathbb{R}$ for each $s = 1, \ldots, p$ and $j = 1, \ldots, k$ and $\overline{\mathcal{U}}_{is}, \ s = 1, \ldots, p$, are convex compact sets in $\mathbb{R}^k$. Here, $\mathrm{diag}(d_1, \ldots, d_p)$ denotes a diagonal matrix with diagonal elements $d_1, \ldots, d_p$. In this case, the characteristic cone C collapses to

$$C = \bigcup_{u_{is} \in \overline{\mathcal{U}}_{is}} \Big\{\Big(-\sum_{s=1}^{p} \theta_s\Big(a_{1s}^{(0)} + \sum_{j=1}^{k} u_{1s}^{(j)} a_{1s}^{(j)}\Big), \ldots, -\sum_{s=1}^{p} \theta_s\Big(a_{ns}^{(0)} + \sum_{j=1}^{k} u_{ns}^{(j)} a_{ns}^{(j)}\Big),$$
$$r + \sum_{s=1}^{p} \theta_s\Big(a_{0s}^{(0)} + \sum_{j=1}^{k} u_{0s}^{(j)} a_{0s}^{(j)}\Big)\Big) : \theta_s \geq 0, \ r \geq 0\Big\},$$

which is convex [19, Proposition 4.1], and the dual problem of (RP_v) can be simplified as

$$\max_{(A_i, F, \lambda_l)} \Big\{ -\mathrm{Tr}(A_0 F) \mid \mathrm{Tr}(A_i F) = \sum_{l=1}^{m} v_l^i \lambda_l, \ i \in I \backslash \{0\},$$
$$\sum_{l=1}^{m} \lambda_l = 1, \ A_i \in \overline{\mathcal{V}}_i, \ F \succeq \overline{0}_p, \ \lambda_l \geq 0, \ i \in I, \ l = 1, \ldots, m\Big\}$$
$$= \max_{(u_{is}, \theta_s, \lambda_l)} \Big\{ -\sum_{s=1}^{p} \theta_s\Big(a_{0s}^{(0)} + \sum_{j=1}^{k} u_{0s}^{(j)} a_{0s}^{(j)}\Big) \Big| \sum_{s=1}^{p} \theta_s\Big(a_{is}^{(0)} + \sum_{j=1}^{k} u_{is}^{(j)} a_{is}^{(j)}\Big) = \sum_{l=1}^{m} v_l^i \lambda_l, \ i \in I \backslash \{0\},$$
$$\sum_{l=1}^{m} \lambda_l = 1, \ u_{is} \in \overline{\mathcal{U}}_{is}, \ \theta_s \geq 0, \ \lambda_l \geq 0, \ i \in I, \ s = 1, \ldots, p, \ l = 1, \ldots, m\Big\}. \tag{4.10}$$

4.1. **Scenario Uncertainty.** In the case that for each $i \in I$, $s = 1, \ldots, p$,

$$\overline{\mathcal{U}}_{is} := \mathrm{conv}\{z_{is}^{(1)}, \ldots, z_{is}^{(q)}\}$$

which is the convex hull of a given finitely generated scenarios $z_{is}^{(t)} := (z_{is}^{(1t)}, \ldots, z_{is}^{(kt)}) \in \mathbb{R}^k$, $t = 1, \ldots, q$, and $q \in \mathbb{N}$, it can be verified that the characteristic cone C is closed and so, strong duality between the robust counterpart and the corresponding dual always holds.

For the robust (RP_v) with scenario uncertainty sets $\overline{\mathcal{U}}_{is}$, named ($\mathrm{RPS}_v$), and the corresponding relaxation problem (DPS_v) can be equivalently rewritten as the following simple linear programming problem:

$$\max_{(\theta_s, \theta_{is}^{(jt)}, \lambda_l)} -\sum_{s=1}^{p} \Big(\theta_s a_{0s}^{(0)} + \sum_{j=1}^{k} \sum_{t=1}^{q} \theta_{0s}^{(jt)} z_{0s}^{(jt)} a_{0s}^{(j)}\Big)$$
$$\text{s.t.} \ \sum_{s=1}^{p} \Big(\theta_s a_{is}^{(0)} + \sum_{j=1}^{k} \sum_{t=1}^{q} \theta_{is}^{(jt)} z_{is}^{(jt)} a_{is}^{(j)}\Big) = \sum_{l=1}^{m} v_l^i \lambda_l, \ i \in I \backslash \{0\}$$
$$\sum_{t=1}^{q} \theta_{is}^{(jt)} = \theta_s, \ \sum_{l=1}^{m} \lambda_l = 1, \ \theta_s \geq 0, \ \theta_{is}^{(jt)} \geq 0, \ \lambda_l \geq 0,$$
$$i \in I, \ s = 1, \ldots, p, \ j = 1, \ldots, k, \ t = 1, \ldots, q,$$
$$l = 1, \ldots, m, . \tag{DPS_v}$$

Theorem 4.3. *Consider the parametric robust semi-definite linear program* (RPS_v) *under scenario data uncertainty with the parameter* $v \in \mathbb{R}^{n \times m}$ *such that* $\inf(\mathrm{RPS}_v) > -\infty$ *and its relaxation problem* (DPS_v). *Suppose that* $X := \{x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^{n} x_i A_i \succeq \overline{0}_p, \ \forall A_i \in \overline{\mathcal{V}}_i, \ i \in I\} \neq \emptyset$. *Then,* $\inf(\mathrm{RPS}_v) = \max(\mathrm{DPS}_v)$.

Proof. Applying the uniform strong duality given in Theorem 2.1, we obtain that

$$\begin{aligned}\inf(\mathrm{RPS}_v) = \max_{(\theta_s,\mu_{is}^{(jt)},\lambda_l)} \Big\{ & -\sum_{s=1}^{p}\theta_s\Big(a_{0s}^{(0)}+\sum_{j=1}^{k}\sum_{t=1}^{q}\mu_{0s}^{(jt)}z_{0s}^{(jt)}a_{0s}^{(j)}\Big) \;\Big| \\ & \sum_{s=1}^{p}\theta_s\Big(a_{is}^{(0)}+\sum_{j=1}^{k}\sum_{t=1}^{q}\mu_{is}^{(jt)}z_{is}^{(jt)}a_{is}^{(j)}\Big)=\sum_{l=1}^{m}v_l^i\lambda_l,\ i\in I\backslash\{0\}, \\ & \sum_{t=1}^{q}\mu_{is}^{(jt)}=1,\ \sum_{l=1}^{m}\lambda_l=1,\ \theta_s\geq 0,\ \mu_{is}^{(jt)}\geq 0,\ \lambda_l\geq 0, \\ & i\in I,\ s=1,\ldots,p,\ j=1,\ldots,k,\ t=1,\ldots,q,\ l=1,\ldots,m\Big\}.\end{aligned}$$

Then, by letting $\theta_{is}^{(jt)} := \theta_s\mu_{0s}^{(jt)}$, we assert that, for each $i\in I$, $s=1,\ldots,p$, $j=1,\ldots,k$, $t=1,\ldots,q$,

$$\theta_s\geq 0,\ \mu_{is}^{(jt)}\geq 0,\ \sum_{t=1}^{q}\mu_{is}^{(jt)}=1$$

is equivalent to $\theta_s\geq 0$, $\theta_{is}^{(jt)}\geq 0$ and $\sum_{t=1}^{q}\theta_{is}^{(jt)}=\theta_s$. So, the maximization problem in (4.10) collapses to (DPS$_v$). □

4.2. **Ellipsoidal Uncertainty.** If for each $i\in I$, $s=1,\ldots,p$, $\overline{\mathcal{U}}_{is} := \mathbb{B}\subset\mathbb{R}^k$ and the *robust Slater condition* holds, i.e.

$$\Big\{x\in\mathbb{R}^n \mid A_0+\sum_{i=1}^{n}x_iA_i\succ\overline{0}_p,\ \forall A_i\in\overline{\mathcal{V}}_i,\ i\in I\Big\}\neq\emptyset,$$

then the characteristic cone C is closed [19, Theorem 2.3], and so, strong duality between the robust counterpart and the corresponding dual always holds by Theorem 2.1.

For the robust (RP$_v$) with ellipsoidal uncertainty sets $\overline{\mathcal{U}}_{is}$, named (RPE$_v$), the corresponding relaxation problem (DPE$_v$) can be stated as

$$\begin{aligned}\max_{(\theta_s,\theta_{is}^{(j)},\lambda_l)}\ & -\sum_{s=1}^{p}\theta_s a_{0s}^{(0)}-\sum_{s=1}^{p}\sum_{j=1}^{k}\theta_{0s}^{(j)}a_{0s}^{(j)} \\ \text{s.t. } & \sum_{s=1}^{p}\theta_s a_{is}^{(0)}+\sum_{s=1}^{p}\sum_{j=1}^{k}\theta_{is}^{(j)}a_{is}^{(j)}=\sum_{l=1}^{m}v_l^i\lambda_l,\ i\in I\backslash\{0\} \\ & \|(\theta_{is}^{(1)},\ldots,\theta_{is}^{(k)})\|\leq\theta_s,\ \sum_{l=1}^{m}\lambda_l=1,\ \theta_s\in\mathbb{R},\ \theta_{is}^{(j)}\in\mathbb{R}, \\ (\mathrm{DPE}_v)\qquad & \lambda_l\geq 0,\ i\in I,\ s=1,\ldots,p,\ j=1,\ldots,k,\ l=1,\ldots,m,\end{aligned}$$

which is a second-order cone linear programming problem.

Theorem 4.4. *Consider the parametric robust semi-definite linear program* (RPE$_v$) *under ellipsoidal data uncertainty with the parameter* $v\in\mathbb{R}^{n\times m}$ *such that* $\inf(\mathrm{RPE}_v)>-\infty$ *and its relaxation problem* (DPE$_v$)*. Suppose that the robust Slater condition holds. Then,* $\inf(\mathrm{RPE}_v)=\max(\mathrm{DPE}_v)$.

Proof. By employing Theorem 2.1, we get that

$$\inf(\mathrm{RPE}_v) = \max_{(\theta_s, u_{is}^{(j)}, \lambda_l)} \Big\{ -\sum_{s=1}^{p} \theta_s \Big(a_{0s}^{(0)} + \sum_{j=1}^{k} u_{0s}^{(j)} a_{0s}^{(j)} \Big) \;\Big|\; \sum_{s=1}^{p} \theta_s \Big(a_{is}^{(0)} + \sum_{j=1}^{k} u_{is}^{(j)} a_{is}^{(j)} \Big) = \sum_{l=1}^{m} v_l^i \lambda_l,\ i \in I \backslash \{0\},$$
$$\|(u_{is}^{(1)}, \ldots, u_{is}^{(k)})\| \le 1,\ \sum_{l=1}^{m} \lambda_l = 1,\ \theta_s \ge 0,\ \mu_{is}^{(j)} \in \mathbb{R},\ \lambda_l \ge 0,$$
$$i \in I,\ s = 1, \ldots, p,\ j = 1, \ldots, k,\ l = 1, \ldots, m \Big\}.$$

To see the conclusion, defining $\theta_{is}^{(j)} := \theta_s u_{is}^{(j)}$. So, it can be verified that, for each $i \in I$, $s = 1, \ldots, p$, $j = 1, \ldots, k$,

$$\theta_s \ge 0,\ \mu_{is}^{(j)} \ge 0,\ \|(u_{is}^{(1)}, \ldots, u_{is}^{(k)})\| \le 1$$

is equivalent to $\theta_s \ge 0$, $\theta_{is}^{(j)} \ge 0$ and $\|(\theta_{is}^{(1)}, \ldots, \theta_{is}^{(k)})\| \le \theta_s$. Therefore, the maximization problem in (4.10) gives way to (DPE_v). □

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References

[1] Beck, A.; Ben-Tal, A. *Duality in robust optimization: Primal worst equals dual best*. Operations Research Letters **37** (2009), no. 1, 1–6.

[2] Ben-Tal, A.; Ghaoui, L. E.; Nemirovski, A. *Robust Optimization*. (2009) Princeton Series in Applied Mathematics, Princeton University Press, New Jersey.

[3] Bertsimas, D.; Brown, D. Constructing uncertainty sets for robust linear optimization. *Oper. Res.* **57** (2009), no. 6, 1483–1495.

[4] Bertsimas, D.; Pachamanova, D.; Sim, M. Robust linear optimization under general norms. *Operations Research Letters* **32** (2004), no. 6, 510–516.

[5] Blanco, V.; Puerto, J.; Ali, S. E. H. B. A semidefinite programming approach for solving multiobjective linear programming. *J. Glob Optim* **58** (2014), no. 3, 465–480.

[6] Boyd, S.; Vandenberghe, L. *Convex Optimization*. Combridge University Press, Cambridge, (2004).

[7] Chieu, N. H.; Feng, J. W.; Gao, W. Li, G.; Wu, D. SOS-Convex Semialgebraic Programs and its Applications to Robust Optimization: A Tractable Class of Nonsmooth Convex Optimization. *Set-Valued Var. Anal.* **26** (2018), no. 2, 305–326.

[8] Chuong, T. D. Linear matrix inequality conditions and duality for a class of robust multiobjective convex polynomial programs. *SIAM Journal on Optimization* **28** (2018), no. 3, 2466–2488.

[9] Chuong, T. D. Exact relaxations for parametric robust linear optimization problems. *Oper. Res. Lett.* **47** (2019), no. 2, 105–109.

[10] Chuong, T. D.; Jeyakumar, V. Convergent hierarchy of SDP relaxations for a class of semi-infinite convex polynomial programs and applications. *Appl. Math. Comput.* **315** (2017), 381–399.

[11] Chuong, T. D.; Jeyakumar, V. Tight SDP relaxations for a class of robust SOS-convex polynomial programs without the slater condition. *J. Convex Anal.* **25** (2018), no. 4, 1159–1182.

[12] Ghaoui, L. E.; Oustry, F.; Lebret, H. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization* **9** (1999), no. 1, 33–52.

[13] Goberna, M. A. Linear semi-infinite optimization: Recent advances. In: Jeyakumar, V., Rubinov, A. M. (Ed.), *Continuous Optimization*. Springer, New York, (2005), pp. 3–22.

[14] Goberna, M. A.; López, M. A. *Linear Semi-infinite Optimization*. Wiley, Chichester, (1998).

[15] Helmberg, C. Semidefinite programming. *Eur. J. Oper. Res.* **137** (2002), no. 3, 461–482.

[16] Jeyakumar, V. A note on strong duality in convex semidefinite optimization: necessary and sufficient conditions. *Optimization Letters* **2** (2008), no. 1, 15–25.

[17] Jeyakumar, V.; Li, G. Strong duality in robust convex programming: Complete characterizations. *SIAM Journal on Optimization* **20** (2010), no. 6, 3384–3407.

[18] Jeyakumar, V.; Li, G. Exact SDP relaxations for classes of nonlinear semidefinite programming problems. *Operations Research Letters* **40** (2012), no. 6, 529–536.

[19] Jeyakumar, V.; Li, G. Y. Strong duality in robust semi-definite linear programming under data uncertainty. *Optimization* **63** (2014), no. 5, 713–733.

[20] Jeyakumar, V., Li, G., Vicente-Pérez, J. Robust SOS-convex polynomial optimization problems: exact SDP relaxations. *Optimization Letters* **9** (2015), no. 1 1–18.

[21] Jiao, L.; Lee, J. H. Finding efficient solutions in robust multiple objective optimization with SOS-convex polynomial data. *Ann. Oper. Res.* **296** (2021), no.(1-2), 803–820.

[22] Lee, J. H.; Sisarat, N.; Jiao, L. Multi-objective convex polynomial optimization and semidefinite programming relaxations. *J. Global Optim.* **80** (2021), no. 1, 117–138.

[23] Pinar, M. C.; Tutuncu, R. H. Robust profit opportunities in risky financial portfolios. *Oper. Res. Lett.* **33** (2005), no. 4 331–340.

[24] Rockafellar, R. T. Convex Analysis. Princeton University Press, Princeton, New Jersey, (1970).

[25] Scherer, C. W. Relaxations for robust linear matrix inequality problemswith verifications for exactness. *SIAM Journal on Matrix Analysis and Applications* **27** (2005), no. 2, 365–395.

[26] Vandenberghe, L.; Boyd, S. Semidefinite programming. *SIAM Review* **38** (1996), no. 1, 49–95.

[27] Wolkowicz, H.; Saigal, R.: Vandenberghe, L. Handbook of Semi-definite Programming: Theory, Algorithms, and Applications. Kluwer Academic, Boston, MA., (2000).

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