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The Non-Existence of Convex Configuration for a Given Set of Vertex-Norm in Two-Dimensional Space

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ABSTRACT. Given a set of vertex-norm, or distances from the origin, in two-dimensional space, there does not always exist a convex configuration, or convex polygon, whose vertices satisfy the vertex-norm. In this research, we provide the necessary and sufficient conditions, based on the angles spanned by the polygon around the origin, for the existence of such convex configuration. General strategies for constructing a convex configuration satisfying a given vertex-norm set as well as examples of vertex-norm sets for which no such convex configuration exists are also illustrated.

1. INTRODUCTION

Convex polygon is one of the fundamental objects in discrete and computational geometry. Many results in computational geometry are based on convex objects, such as convex hull and Voronoi diagrams [8, 11]. Some applications in computer-aided design (CAD) are also related to convex representation, for instance, the drawing of a given graph in such a way that all faces are convex [9, 2], and the drawing of a graph in convex position [6].

Suppose that a set of points $P = \{A_1, ..., A_n\}$ is given. One typically wants to find the convex hull, or the smallest convex set containing all points in P, which can be uniquely determined. However, some points may lie on the interior of the convex hull. Another common problem setup is whether one can place n points on a plane to satisfy certain constraints while keeping such placement convex. In other words, the question is whether a *convex configuration* exists for a given set of constraints.

Many previous studies focused on the construction of convex polygon whose side lengths, also called linkages, are specified. The *polygon reconfiguration problem* asks whether a polygon can be reconfigured into another polygon [7]. The *polygon convexification problem* asks how a non-convex polygon, such as a star-shaped figure [5], can be transformed into a convex polygon. The *carpenter's rule problem* asks whether a polygon can be moved continuously in such a way that its vertices are in the convex position. These problems were solved by Aichholzer et al. [1], Connelly et al. [4], as compiled in [10].

Different from previous works, Chaidee and Sugihara [3] studied the existence of a convex polygon (in the case of two-dimensional space) or a convex polytope (in the case of three-dimensional space) whose vertices are at specified distances, also called vertex-norms, from the origin. An illustration of the problem is shown in figure 1. The problem can be defined as follow [3].

Given a set of scalars $R = \{r_1, ..., r_n\}$ and the origin O, find a configuration of n points $V = \{v_1, ..., v_n\}$ under the vertex-norm constraints $r_i := |v_i|$ such that none of the points lies strictly inside their convex hull.

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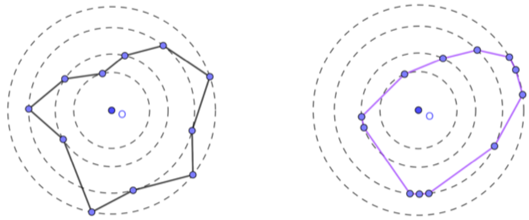

FIGURE 1. (Left) A placement of points on the plane satisfying a given vertex-norm ; (right) a convex configuration of those points

Chaidee and Sugihara [3] proved that the answer is positive for the case of three-dimensional space (i.e., there always exists a convex polytope satisfying a given vertex-norm). For the case of two-dimensional space, the problem seems to be trivial when the number of points is small and the desired convex configuration always exists. However, the result was inconclusive when the number of points is larger. Chaidee and Sugihara conjectured that the convex configuration does not always exist for the case of two-dimensional space.

In this study, we proved the conjecture that the convex polygon satisfying a given vertex-norm does not always exist for the case of two-dimensional space. Some necessary and sufficient conditions on the vertex-norm, $\mathcal{R}$, for such a convex configuration to exist were provided. Based on these conditions, general strategies for constructing a convex configuration satisfying a given vertex-norm set as well as examples of vertex-norm sets for which no such convex configuration exists can be completely illustrated.

This paper is organized as follows. Section 2 provides the basic definitions and lemmas. Section 3 contains preliminary results for proving the main theorems. Important terminology, including *initial sequences* and *core indexes*, together with the conditions on the angle spanned by the vertices of the configuration around the origin, which are key ideas behind our proofs, are also described. Section 4 then shows the necessary and sufficient conditions on the vertex-norm set for which a convex configuration exists. Finally, Section 5 shows examples of vertex-norm sets that do not have a convex configuration with proofs. The summary of this study and possible future directions are provided.

2. PRELIMINARIES

Throughout this paper, we assume that all points lie on the same plane Π and let O be the origin of the plane Π. Denote by $L^{\perp}$ the line perpendicular to a line L and passing through O. Denote by $\measuredangle ABC$ the radian measure of rotation of the ray BA to the ray BC in the counterclockwise direction. Denote by $Int(S)$ the interior of a set S. Let $\mathcal{R}$ be a finite multiset of positive real numbers. Without loss of generality, we assume that $\mathcal{R}$ contains at least two distinct elements and has at least three elements.

For convenience, when we mention the convex polygon $\mathcal{P} = A_1A_2\cdots A_n$, we always assume that $Int(\mathcal{P}) \neq \emptyset$ and its vertices $A_1, A_2, \ldots, A_n$ are arranged in counterclockwise order with respect to its interior point. Furthermore, we define $A_{n+i} = A_i$ for all integer i.

Definition 2.1. Let n be the number of elements in the multiset $\mathcal{R}$. If there exists a (strictly) convex n-gon $\mathcal{P} = A_1A_2\cdots A_n$ such that $\{|OA_i| : i = 1,\ldots,n\} = \mathcal{R}$, then we say $\mathcal{R}$ has a (strictly) convex configuration P. We call $\mathcal{R}$ the vertex-norm set of $\mathcal{P}$ with the origin O and call $r \in \mathcal{R}$ a vertex-norm.

Throughout this paper, we mainly focus on the Euclidean norm in two-dimensional space. For some families of given vertex-norm set $\mathcal{R}$, there are simple strategies to construct the convex configurations. Chaidee and Sugihara [3] have illustrated strategies when all of the vertex-norm are distinct (Figure 2a) and when none of the values of vertex-norm is repeated more than four times (Figure 2b).

Here, we additionally observe that when there are no more than four distinct values of vertex-norm, i.e., $\mathcal{R} = \{r_1, ..., r_1, r_2, ..., r_2, ..., r_k, ..., r_k\}$ for $k \leq 4$ and $r_1 > r_2 > ... > r_k$, one can place points along the arc of circle with radius r_i followed by drawing a tangent line to the circle with radius $r_{i+1} < r_i$, as shown in Figure 2c, to obtain a convex configuration. This is always possible because the total angle around the origin covered by $k \leq 4$ tangent lines is less than $\frac{k\pi}{2} \leq 2\pi$ and the angle covered by each arc can be arbitrarily small.

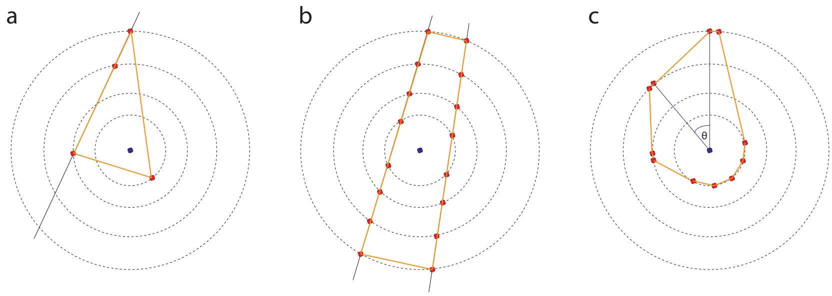

FIGURE 2. Strategies for constructing convex configurations for some $\mathcal{R}$

The following lemma is basic fact in geometry.

Lemma 2.1. *Let L be a line on the plane Π. Let O' be the projection of O onto L. Let A and B be two distinct points on L. Suppose that $|OA| = |OB|$. We have that the point O' is the middle point of A and B, and for any point C that lies on the line L, if $|OC| \leq |OA|$, then C lies on the line segment $\overline{AB}$. Furthermore, we have that $|OC| = |OA|$ if and only if $C = A$ or $C = B$.*

Similar to the hyperplane separation theorem, one can prove the following lemma.

Lemma 2.2. *Let $\mathcal{P} = A_1A_2 \cdots A_n$ be a convex polygon with $O \neq A_i$ $(i = 1, 2, \ldots, n)$. If $O \notin Int(\mathcal{P})$, then there is a line L passing through O such that $L \cap Int(\mathcal{P}) \neq \emptyset$ and $L^{\perp} \cap Int(\mathcal{P}) = \emptyset$.*

3. Some Results for Proving the Main Results

In this section, we compile the basic results which relate to the proof of the main theorem.

3.1. **The existence of convex configuration including the origin as an interior point.** When we consider a convex configuration of a multiset $\mathcal{R}$ as mentioned in Definition 2.1, it may not include the origin O as shown in Figure 3. We will show that we can always find an alternative convex configuration that contains the point O as an interior point. To prove this, we need the following lemma, which considers the number of points that led to the rearrangement of the convex polygon.

Lemma 3.3. *Let L be a line passing through O. Given two points $A, B \notin L \cup L^{\perp}$. Suppose that A and B lie on the same side of $L^{\perp}$, but on the opposite sides of L. Let L_A and L_B be the lines parallel to L and passing through A and B, respectively. Let L_A and L_B intersect $L^{\perp}$ at A' and B', respectively. Let $P_1, P_2, \ldots, P_m$ be any pairwise distinct points in the trapezoid $AA'B'B$ such*

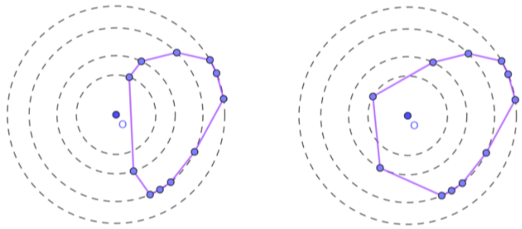

FIGURE 3. (Left) a (strictly) convex configuration which does not contain O ; (right) a (strictly) convex configuration of the same set of points including the origin O

that the polygon $\mathcal{P} = AP_1 \cdots P_m B$ forms a convex polygon. Let $r_A = |OA|$ and $r_B = |OB|$. Suppose that $r_A \geq r_B$. Then we have that $|OP_i| < r_A$ for all $i = 1, 2, \ldots, m$ and

1. *for any positive real number $r \leq r_B$, there are at most two vertices Q of the polygon $\mathcal{P}$ such that $|OQ| = r$;*
2. *for any positive real number $r_B < r \leq r_A$, there is at most one vertex Q of the polygon $\mathcal{P}$ such that $|OQ| = r$.*

Proof. If there is a point $P_i \in \{P_1, P_2, \ldots, P_n\}$ such that P_i lies on the line AB, then by the convexity of $\mathcal{P}$, we have that $P_1, P_2, \ldots, P_n$ all lie on the line AB. Using Lemma 2.1, one can obtain the desired results. Now we assume that $P_1, P_2, \ldots, P_n$ do not lie on the line AB.

Denote by E the intersection of the line L and the line segment $\overline{AB}$. For any point P in the trapezoid $AA'B'B$ with $P \notin \overline{AB}$. If P lies in the trapezoid $AA'OE$, then $|OP| < |OA| = r_A$. If P lies in the trapezoid $EOB'B$, then $|OP| < \max\{|OE|, |OB|\} \leq r_A$. Therefore, $|OP_i| < r_A$ for all $i = 1, 2, \ldots, m$.

For any positive real number $r \leq r_A$. Suppose that there exist three pairwise distinct vertices Q_1, Q_2, Q_3 of $\mathcal{P}$ such that $|OQ_1| = |OQ_2| = |OQ_3| = r$. For $i = 1, 2, 3$, let L_i be the line parallel to L and passing through Q_i, and let Q'_i be the intersection of the line L_i and the line segment $\overline{AB}$ (see Figure 4). Since L_1, L_2, L_3 are perpendicular to $L^\perp$ and Q_1, Q_2, Q_3 all lie on the same semicircle $\mathcal{C}$ with diameter parallel to $L^\perp$, we have that Q'_1, Q'_2, Q'_3 are pairwise distinct. Without loss of generality, we may assume that Q'_2 lies between Q'_1 and Q'_3. By the convexity of the polygon $\mathcal{P}$, we know that Q'_1 and Q'_3 lie in the polygon $\mathcal{P}$, and hence the trapezoid $Q_1Q'_1Q'_3Q_3$ is contained in $\mathcal{P}$. Observe that Q_2 lies on the arc Q_1Q_3 of the semicircle $\mathcal{C}$. It can be deduced that Q_2 is an interior point of the trapezoid $Q_1Q'_1Q'_3Q_3$ which is contained in $\mathcal{P}$. This is impossible since Q_2 is a vertex of $\mathcal{P}$. Therefore, we have that for any positive real number $r \leq r_A$; there are at most two vertices Q of the polygon $\mathcal{P}$ such that $|OQ| = r$

Now consider the case $r_B < r \leq r_A$. Suppose that there are two distinct vertices Q_1, Q_2 of $\mathcal{P}$ such that $|OQ_1| = |OQ_2| = r$. We have already proved that $|OP_i| < r_A$ for all $i = 1, 2, \ldots, m$. Hence we may assume that $r \neq r_A$. For $i = 1, 2$, let L_i be the line parallel to L and passing through Q_i, and let Q'_i be the intersection of the line L_i and the line segment $\overline{AB}$. Since $r_B < r < r_A$, there must be a point Q'_3 on the line segment $\overline{AB}$ such that $|OQ'_3| = r$. One can see that Q'_1, Q'_2 and Q'_3 are pairwise distinct, and the points Q'_1 and Q'_2 must lie between A and Q'_3. We may assume that Q'_2 lie between Q'_1 and Q'_3. Then Q_2 lie inside the triangle $Q_1Q'_1Q'_3$ which is contained in $\mathcal{P}$. This is a contradiction, since Q_2 is a vertex of $\mathcal{P}$. Therefore, for $r_B < r \leq r_A$, there is at most one vertex Q of the polygon $\mathcal{P}$ such that $|OQ| = r$.

□

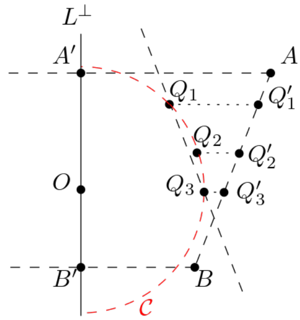

FIGURE 4. Q_2 lies inside the trapezoid $Q_1Q'_1Q'_3Q_3$

This lemma shows the existence of an alternative convex configuration containing O as an interior point.

Lemma 3.4. *If $\mathcal{R}$ has a (strictly) convex configuration, then $\mathcal{R}$ has a (strictly) convex configuration that contains O as an interior point.*

Proof. First, we will prove the result for the non-strictly case. For a positive real number r, we denote by $\mathcal{C}(r)$ the circle with center O and radius r.

Suppose that $\mathcal{R}$ has a convex configuration $\mathcal{P} = A_1A_2\cdots A_n$, where $A_1, A_2, \ldots, A_n$ are arranged in counterclockwise order with respect to an interior point of $\mathcal{P}$. For $i = 1, 2, \ldots, n$, we let $r_i = |OA_i|$.

If $A_1A_2\cdots A_n$ does not contain O as an interior point, then it follows from Lemma 2.2 that there is a line L passing through O such that $L \cap Int(\mathcal{P}) \neq \emptyset$ and $L^{\perp} \cap Int(\mathcal{P}) = \emptyset$.

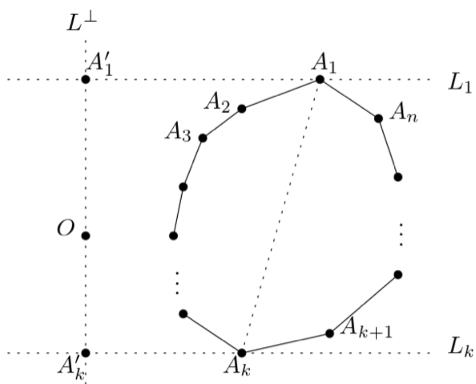

FIGURE 5. the points $A_1, A_2, \ldots, A_k$ lie in the trapezoid $A_1A'_1A'_kA_k$

For $i = 1, 2, \ldots, n$, let L_i be the line parallel to L and passing through A_i, and let A'_i be the projection of A_i onto $L^{\perp}$. Without loss of generality, we may assume that L_1 and L_k are two distinct supporting lines of the convex polygon $\mathcal{P}$, and the points $A_1, A_2, \ldots, A_k$ all lie in the trapezoid $A_1A'_1A'_kA_k$ (see Figure 5). Furthermore, we also assume that for $i = 2, \ldots, k-1$, the line L_i is not a supporting line of $\mathcal{P}$, and assume that $r_k \leq r_1$. We consider the following two cases:

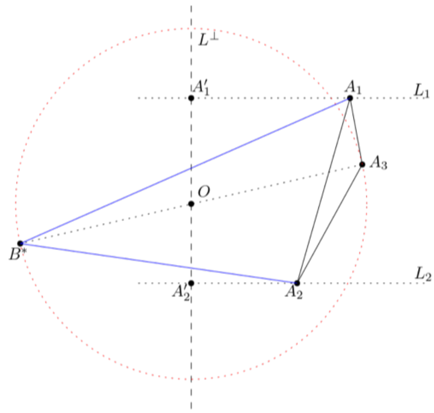

FIGURE 6. the points $A_1, A_2, \ldots, A_k$ lie in the trapezoid $A_1A'_1A'_kA_k$

Case 1: $k = 2$. If $n = 3$, then let B^* be the intersection of the line OA_3 and the circle $\mathcal{C}(r_3)$, where B^* is different than A_3. Then one can see that $A_1A_2B^*$ is a convex configuration of $\mathcal{R}$, and $A_1A_2B^*$ contains O as an interior point (see Figure 6). If $n \geq 4$, then we can divide into two cases:

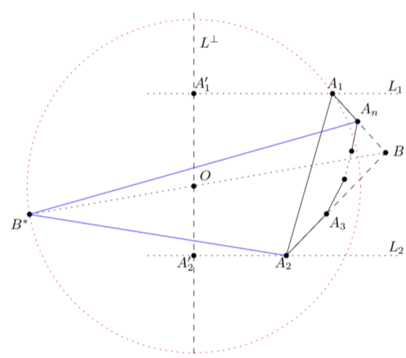

FIGURE 7. the case: A_1A_n and A_2A_3 are not parallel

Case 1.1: the lines A_1A_n and A_2A_3 are not parallel (see Figure 7). Let B be the intersection of the line A_1A_n and the line A_2A_3. If $B \neq A_n$, then let B^* be the intersection of the line OB and the circle $\mathcal{C}(r_1)$, where O lies between B^* and B. One can see that $B^*A_2 \ldots A_n$ is a convex configuration of $\mathcal{R}$ and contains O as an interior point. If $B = A_n$, then $B \neq A_3$. Let B^* be the intersection of the line OB and the circle $\mathcal{C}(r_2)$. We have that $A_1B^*A_3 \ldots A_n$ is a convex configuration of $\mathcal{R}$ and contains O as an interior point.

Case 1.2: the lines A_1A_n and A_2A_3 are parallel to L (see Figure 8). Then Let B^* be the intersection of the line L and the circle $\mathcal{C}(r_1)$, where B^* and A_1 lie on the opposite sides of $L^\perp$. Then $B^*A_2 \ldots A_n$ is a convex configuration of $\mathcal{R}$ and contains O as an interior point.

Case 2: $k \geq 3$. We convert the multiset $\{r_2, r_3, \ldots, r_{k-1}\}$ to the set (not multiset) $\mathcal{R}'$, i.e., $\mathcal{R}'$ and $\{r_2, r_3, \ldots, r_{k-1}\}$ have the same elements, but each element in $\mathcal{R}'$ has multiplicity

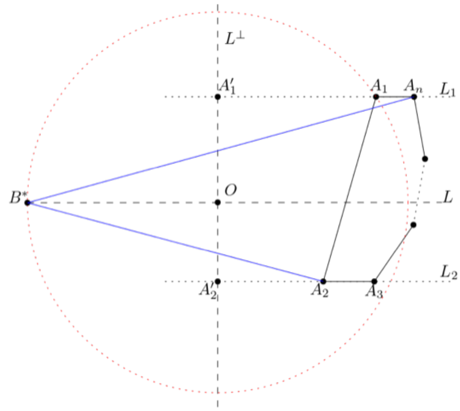

FIGURE 8. the case: A_1A_n and A_2A_3 are parallel to L

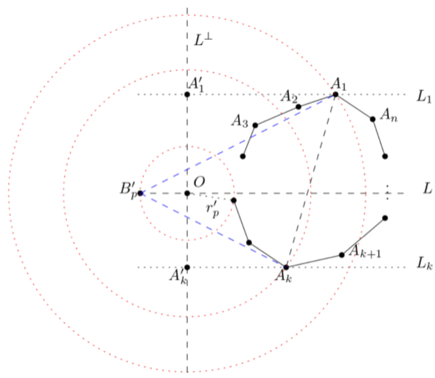

FIGURE 9. B'_p

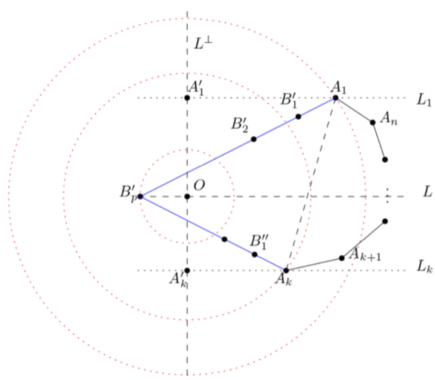

FIGURE 10. $A_1B'_1\cdots B'_pB''_q\cdots B''_1A_kA_{k+1}\cdots A_n$ is a convex configuration of $\mathcal{R}$, and contains O as an interior point

1. We define the set

$$\mathcal{R}'' = \{r \in \mathcal{R}' \mid \text{ there are } i, j \in \{2, \ldots, k-1\} \text{ such that } i \neq j \text{ and } r_i = r_j = r\}.$$

It is clear that $\mathcal{R}' \neq \emptyset$. Noting that $A_2, \ldots, A_{k-1}$ lie in the trapezoid $A_1 A_1' A_k' A_k$. By Lemma 3.3, we have that the multiset $\{r_2, \ldots, r_{k-1}\}$ is the sum of sets $\mathcal{R}'$ and $\mathcal{R}''$. Assume that

$$\mathcal{R}' = \{r_1', \ldots, r_p'\} \text{ and } \mathcal{R}'' = \{r_1'', \ldots, r_q''\}$$

where $p + q = k - 2$, $r_1' > \cdots > r_p'$ and $r_1'' > \cdots > r_q''$. From Lemma 3.3, we know that $r_1' < r_1$. Furthermore, if $\mathcal{R}'' \neq \emptyset$, then we have that $r_p' \leq r_q'' < \cdots < r_1'' < r_k$. Let B_p' be the intersection of the line L and the circle $\mathcal{C}(r_p')$ (see Figure 9). One can see that the open line segment $\overline{A_1 B_p'}$ intersects the circle $\mathcal{C}(r)$ for all $r_p' \leq r \leq r_1'$, and the open line segment $\overline{B_p' A_k}$ intersects the circle $\mathcal{C}(r)$ for all $r_q'' \leq r \leq r_1''$. For $i = 1, \ldots, p-1$, let B_i' be the intersection of the open line segment $\overline{A_1 B_p'}$ and the circle $\mathcal{C}(r_i')$. For $i = 1, \ldots, q$, let B_i'' be the intersection of the open line segment $\overline{B_p' A_k}$ and the circle $\mathcal{C}(r_i'')$. It is not hard to see that the polygon $A_1 B_1' \cdots B_p' B_q'' \cdots B_1'' A_k A_{k+1} \cdots A_n$ is a convex configuration of $\mathcal{R}$, and contains O as an interior point (see Figure 10).

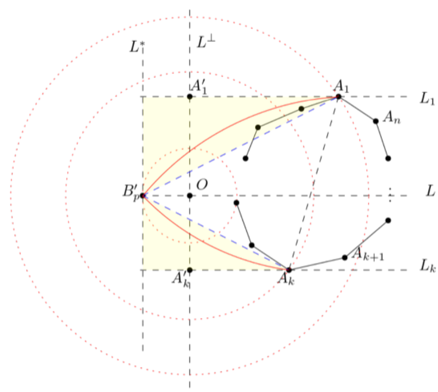

FIGURE 11. replacing $\overline{A_1 B_p'}$ and $\overline{B_p' A_k}$ with strictly convex curves

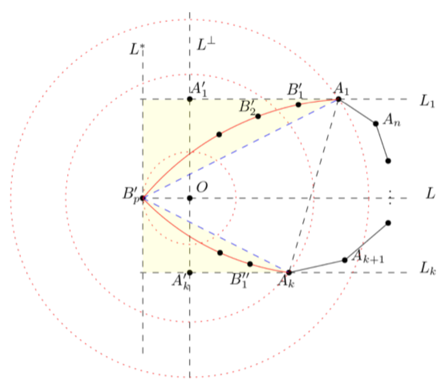

FIGURE 12. $A_1 B_1' \cdots B_p' B_q'' \cdots B_1'' A_k A_{k+1} \cdots A_n$ is a strictly convex configuration of $\mathcal{R}$, and contains O as an interior point

For the strictly convex case, if $k = 2$, then we can use the same argument to obtain a strictly convex configuration of $\mathcal{R}$ that contains O as an interior point. If $k \geq 3$, then we let L^* be the line perpendicular to L and passing through B_p'. We replace the line segment $\overline{A_1 B_p'}$ in the above argument with a strictly convex curve with endpoints A_1 and B_p', where the curve (exclude both endpoints) intersects the circle $\mathcal{C}(r)$ for all $r_p' \le r \le r_1'$, and lies inside the triangle formed by the lines $A_1 B_p'$, L_1 and L^*; and replace the line segment $\overline{B_p' A_k}$ with a strictly convex curve with endpoints B_p' and A_k, where the curve (exclude both endpoints) intersects the circle $\mathcal{C}(r)$ for all $r_q'' \le r \le r_1''$, and lies inside the triangle formed by the lines $B_p' A_k$, L_k and L^* (see Figure 11). One can define the points B_i' and B_i'' as above, and obtain that the polygon $A_1 B_1' \cdots B_p' B_q'' \cdots B_1'' A_k A_{k+1} \cdots A_n$ is a strictly convex configuration of $\mathcal{R}$, and contains O as an interior point (see Figure 12). $\square$

3.2. **Initial Sequences and Core Indexes.** It follows from Lemma 3.4 that, when we discuss the existence of convex configurations of a given multiset $\mathcal{R}$, one may consider only those convex configurations which contain O as an interior point. Let $A_1 A_2 \cdots A_n$ be a convex configuration of $\mathcal{R}$ that contains O as an interior point. We may assume, without loss of generality, that $|OA_1| = \max \mathcal{R}$, $|OA_n| \ne \max \mathcal{R}$ and $A_1, A_2, ..., A_n$ are arranged in counterclockwise order with respect to O. Let $r_i = |OA_i|$ where $i = 1, 2, \ldots, n$. It is clear that $(r_1, r_2, ..., r_n)$ is a permutation of $\mathcal{R}$ where $r_1 = \max \mathcal{R}$ and $r_n \ne \max \mathcal{R}$.

Definition 3.2. A permutation $(r_i)_{i=1}^n$ of $\mathcal{R}$ is called an *initial sequence* of $\mathcal{R}$, if $r_1 = \max \mathcal{R}$ and $r_n \ne \max \mathcal{R}$.

To show that a given multiset $\mathcal{R}$ cannot have a convex configuration, we only need to prove that the convex polygon containing O cannot be constructed for any initial sequence $(r_i)_{i=1}^n$ of $\mathcal{R}$. For convenience, we define $r_{n+i} = r_i$ for all integer i.

Definition 3.3. Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. For $i = 1, 2, \ldots, n$,

(1) if $r_{i-1} \le r_i$ and $r_i > r_{i+1}$, or $r_{i-1} < r_i$ and $r_i \ge r_{i+1}$, then we call r_i a *peak* of S;
(2) if $r_{i-1} \ge r_i$ and $r_i < r_{i+1}$, or $r_{i-1} > r_i$ and $r_i \le r_{i+1}$ then we call r_i a *bottom* of S.

We call i a peak index if r_i is a peak of S; and call i a bottom index, if r_i is a bottom.

Note that r_1 is always a peak of any initial sequence S. Furthermore, one can prove that if there are two peaks r_i and r_j with $i < j$ such that there is no any bottom between them, then $r_i = r_{i+1} = \cdots r_j$. Similarly, if there are two bottoms r_i and r_j with $i < j$ such that there is no any peak between them, then $r_i = r_{i+1} = \cdots r_j$.

Definition 3.4. Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. For $i = 1, 2, \ldots, n$. We call i a *main peak index* of S, if for any peak index j with $1 \le j < i$, there is a bottom index k such that $j < k < i$. We call i a *main bottom index* of S, if for any bottom index j with $1 \le j < i$, there is a peak index k such that $j < k < i$. We denote the core index set of S by $C(S)$ the set of all main peak indexes and all main bottom indexes.

To illustrate the terms related to *peak* and *bottom*, let us consider the following example which shows the intuition of Definition 3.3 and 3.4.

For the initial sequence $S_1 = (r_1, r_2, \ldots, r_{11})$, where

i	1	2	3	4	5	6	7	8	9	10	11
r_i	5	5	5	4	2	1	2	3	2	3	4

All peak indexes of S are $1, 3, 8$, and all bottom indexes of S are $6, 9$. All main peak indexes of S are $1, 8$, and all main bottom indexes of S are $6, 9$. Hence, the core index set $C(S) = \{1, 6, 8, 9\}$. The initial sequence in this example has a convex configuration as shown in Figure 13. From the figure, it is roughly to say that when i increases, the peak index is considered as the largest value of r_i before r_i decreases. Similarly, the bottom index can be seen as the smallest value of r_i before r_i increases when i increases. The main peak and main bottom are defined to be aware of the repeated values.

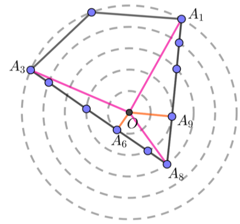

FIGURE 13. A convex configuration of the initial sequence S_1 defined above

Similar to the example of S_1 with a longer initial sequence,let consider $S_2 = (r_1, r_2, \ldots, r_{15})$, where

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
r_i	5	5	3	3	3	2	2	3	4	2	1	2	2	3	3

All peak indexes of S are $1, 2, 5, 9, 12, 14$, and all bottom indexes of S are $3, 6, 7, 11, 13, 15$. All main peak indexes of S are $1, 5, 9, 12, 14$, and all main bottom indexes of S are $3, 6, 11, 13, 15$. Hence, the core index set $C(S) = \{1, 3, 5, 6, 9, 11, 12, 13, 14, 15\}$.

From the definition of initial sequences S of $\mathcal{R}$, one can see that the index $i = 1$ is always a main peak index of S, and the largest index in $C(S)$ must be a main bottom index. Furthermore, we have that main peak indexes and main bottom indexes are arranged alternately. Therefore, if we let $C(S) = \{i_1, i_2, \ldots, i_m\}$ where $i_1 < i_2 < \cdots < i_m$, then m must be even, and for $k = 1, 2, \ldots, m$, the index i_k is a main peak index if and only if k is an odd number.

For convenience, when $S = (r_i)_{i=1}^n$ and $C(S) = \{i_1, i_2, \ldots, i_m\}$, we always assume that $1 = i_1 < i_2 < \cdots < i_m$, and define $i_{qm+k} = qn + i_k$ where q and k are integers.

Lemma 3.5. *Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. Given $k \in \{1, 2, \ldots, l\}$,*

(1) for $i_{2k-1} \le i < j \le i_{2k}$, if $r_i \ne r_j$, then $r_i > r_j$; if $r_i = r_j$, then $r_{i_{2k-1}} = r_{i_{2k-1}+1} = \cdots = r_i = r_{i+1} = \cdots = r_j$;

(2) for $i_{2k} \le i < j \le i_{2k+1}$, if $r_i \ne r_j$, then $r_i < r_j$; if $r_i = r_j$, then $r_{i_{2k}} = r_{i_{2k}+1} = \cdots = r_i = r_{i+1} = \cdots = r_j$.

Proof. Let $k \in \{1, 2, \ldots, l\}$. We will prove the first part by using an induction on j.

For $j = i_{2k-1} + 1$. Because i_{2k-1} is a peak index, so $r_{i_{2k-1}} \ge r_j$. Hence it is clear that the statement is true for $j = i_{2k-1} + 1$.

For any positive integer t with $i_{2k-1} < t < i_{2k}$, we suppose that the statement is true for all j with $i_{2k-1} < j \le t$. We will show that the statement is also true for $j = t + 1$.

For any positive integer i with $i_{2k-1} \le i < t + 1$. Suppose that $r_i < r_{t+1}$. Then there is a positive integer p with $i \le p \le t$ such that $r_p < r_{p+1}$. By the induction hypothesis, we have that $r_j \ge r_{j+1}$ for all $j = i_{2k-1}, \ldots, t - 1$. Hence $p = t$, i.e., $r_t < r_{t+1}$. Since $r_{t-1} \ge r_t$ we know that t is a bottom index of S. On the other hand, from the induction hypothesis, one can deduce that there is no any bottom index between i_{2k-1} and t. This implies that t must be a main bottom index which is impossible, since $i_{2k-1} < t < i_{2k}$. Therefore, we have $r_i \ge \cdots \ge r_t \ge r_{t+1}$. If $r_t > r_{t+1}$, then the statement is true for $j = t + 1$. If $r_t = r_{t+1}$, then we consider the following two cases:

Case 1: $r_{t-1} = r_t$. By the induction hypothesis, we have that $r_{i_{2k-1}} = \cdots = r_t = r_{t+1}$, and hence the statement is true for $j = t + 1$.

Case 2: $r_{t-1} > r_t$. Then t is a bottom index. But we have already shown that t cannot be a bottom index.

Since the concepts of peaks and bottoms are dual, we can use an analogous argument to prove the second part of the lemma. □

From Lemma 3.5, we know that for any $k \in \{1, 2, \ldots, l\}$,

$$\max\left\{\frac{r_{i_{2k}}}{r_{i_{2k-1}}}, \frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right\} \leq 1$$

and

$$\min\left\{\frac{r_{i_{2k}}}{r_{i_{2k-1}}}, \frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right\} < 1.$$

To consider whether the given initial sequence can generate a convex configuration, the sum of circular angles between points of two adjacent peaks should not exceed 2π. We remark that the orientation of the bottom affects those circular angles. To clarify it rigorously, we define the angle and the conditions for S to be convex as follows.

Definition 3.5. Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index of S. We define

$$\text{Angle}(S) = \sum_{k=1}^{l}\left(\arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right).$$

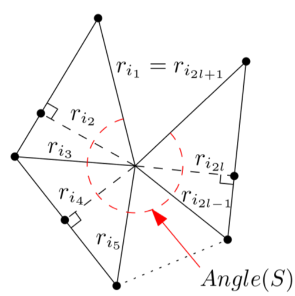

FIGURE 14. the geometric meaning of $\text{Angle}(S)$

Noting an alternative definition that

$$\text{Angle}(S) = \sum_{m=1}^{2l}\arccos\left(\min\left\{\frac{r_{i_m}}{r_{i_{m+1}}}, \frac{r_{i_{m+1}}}{r_{i_m}}\right\}\right).$$

Definition 3.6. Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. We call S a *convex sequence* of $\mathcal{R}$, if S satisfies one of the following conditions:

(1) $\text{Angle}(S) < 2\pi$;
(2) $\text{Angle}(S) = 2\pi$ and $r_i \neq r_{i+1}$ for all $i \geq 1$.

We call S a *strictly convex sequence* of $\mathcal{R}$, if $\text{Angle}(S) < 2\pi$.

3.3. **Lemmas about the angle of an initial sequence.** In this part, we provide the properties of the angle computed from an initial sequence S, $\text{Angle}(S)$. These properties will be applied in the main theorems of the next section.

Lemma 3.6. *Let $A_1A_2\cdots A_n$ be a convex configuration of $\mathcal{R}$ that contains O as an interior point. For any integers i, j where $1 \leq i < j \leq n$, if $\measuredangle A_iOA_j \leq \pi$, then $A_iA_kA_jO$ is convex, for all $i < k < j$.*

Proof. Since the polygon $A_1A_2\dots A_n$ is convex and A_i, A_k, A_j are the vertices of $A_1A_2\dots A_n$, we have that

$$\measuredangle A_kA_iO \le \pi,\ \measuredangle A_jA_kA_i \le \pi,\ \measuredangle OA_jA_k \le \pi.$$

Therefore, $A_iA_kA_jO$ is convex. $\square$

Lemma 3.7. *Let r_1, r_2, r_3 be positive real numbers where $r_1 \ge r_2$ and $r_2 \le r_3$. If the quadrilateral $A_1A_2A_3O$ is convex and $|OA_i| = r_i$ $(i = 1, 2, 3)$, then*

$$\measuredangle A_1OA_3 \ge \arccos\frac{r_2}{r_1} + \arccos\frac{r_2}{r_3},$$

where the equality holds if and only if A_1, A_2, A_3 are collinear and the line OA_2 is perpendicular to the line A_1A_3.

Proof. Since $A_1A_2A_3O$ is convex, A_3 and O must lie on the same side of the line A_1A_2 (the point A_3 could lie on the line A_1A_2). Note that $r_2 \le r_3$, hence when the positions of A_1 and A_2 are fixed, the measure of $\measuredangle A_1OA_3$ reaches its minimum when and only when A_3 lies on the line A_1A_2.

Now we assume that A_1, A_2, A_3 are collinear. Let θ be the radian measure of the angle between the vectors $\overrightarrow{OA_2}$ and $\overrightarrow{A_1A_3}$. Then $\theta \in [0, \pi]$ and

$$\measuredangle A_1OA_3 = \arccos\left(\frac{r_2}{r_1}\sin\theta\right) + \arccos\left(\frac{r_2}{r_3}\sin\theta\right).$$

Hence,

$$\measuredangle A_1OA_3 \ge \arccos\frac{r_2}{r_1} + \arccos\frac{r_2}{r_3},$$

where the equality holds if and only if $\theta = \frac{\pi}{2}$.

$\square$

Lemma 3.8. *Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. Let $C(S) = \{i_1, i_2, \dots, i_{2l}\}$ be the core index set of S. For any given positive integer $k \in \{1, 2, \dots, l\}$, if $r_i \ne r_{i+1}$ for all $i_{2k-1} \le i < i_{2k+1}$, then there exist $i_{2k+1} - i_{2k-1} + 1$ points $A_{i_{2k-1}}, A_{i_{2k-1}+1}, \dots, A_{i_{2k+1}}$ which are arranged in counterclockwise order with respect to O, and satisfy*

(1) $|OA_i| = r_i$, for $i = i_{2k-1}, i_{2k-1}+1, \dots, i_{2k+1}$;
(2) $A_{i_{2k-1}}, A_{i_{2k-1}+1}, \dots, A_{i_{2k+1}}$ are collinear;
(3) the line $OA_{i_{2k}}$ is perpendicular to the line $A_{i_{2k-1}}A_{i_{2k+1}}$, and hence

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} = \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}.$$

Proof. Since $r_i \ne r_{i+1}$ for all $i_{2k-1} \le i < i_{2k+1}$, by Lemma 3.5, we have that

$$r_{i_{2k-1}} > r_{i_{2k-1}+1} > \cdots > r_{i_{2k}} \text{ and } r_{i_{2k}} < \cdots < r_{i_{2k+1}-1} < r_{i_{2k+1}}.$$

Choose three points $A_{i_{2k-1}}, A_{i_{2k}}, A_{i_{2k+1}}$ in counterclockwise order with respect to O, such that

(1) $|OA_i| = r_i$ for $i = i_{2k-1}, i_{2k}, i_{2k+1}$,
(2) $A_{i_{2k-1}}, A_{i_{2k}}, A_{i_{2k+1}}$ are collinear,
(3) the line $OA_{i_{2k}}$ is perpendicular to $A_{i_{2k-1}}A_{i_{2k+1}}$.

It follows that $\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} = \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}$. Furthermore, there must be $i_{2k} - i_{2k-1} - 1$ points $A_{i_{2k-1}+1}, \dots, A_{i_{2k}-1}$ (arranged in order) on the line segment $\overline{A_{i_{2k-1}}A_{i_{2k}}}$ and $i_{2k+1} - i_{2k} - 1$ points $A_{i_{2k}+1}, \dots, A_{i_{2k+1}-1}$ (arranged in order) on the line segment $\overline{A_{i_{2k}}A_{i_{2k+1}}}$ such that $|OA_i| = r_i$ where $i = i_{2k-1}+1, \dots, i_{2k}-1, i_{2k}+1, \dots, i_{2k+1}-1$.

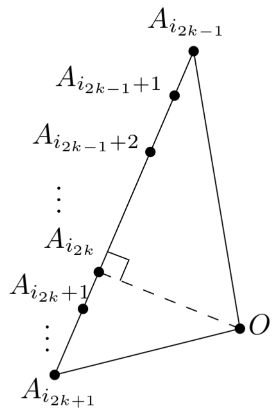

FIGURE 15. $A_{i_{2k-1}}, A_{i_{2k-1}+1}, \ldots, A_{i_{2k+1}}$ that satisfy the conditions in Lemma 3.8

$\square$

Lemma 3.9. *Let $A_1A_2...A_n$ be a convex configuration of $\mathcal{R}$ that contains O as an interior point, and suppose that $|OA_1| = \max \mathcal{R}$, $|OA_n| \neq \max \mathcal{R}$. Let $S = (r_i)_{i=1}^n = (|OA_i|)_{i=1}^n$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. Then for a given $k \in \{1, 2, \ldots, l\}$, we have*

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} \geq \arccos \frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \frac{r_{i_{2k}}}{r_{i_{2k+1}}},$$

where the equality holds if and only if $A_{i_{2k-1}}, \ldots, A_{i_{2k}}, \ldots, A_{i_{2k+1}}$ are collinear and the line $OA_{i_{2k}}$ is perpendicular to the line $A_{i_{2k-1}}A_{i_{2k+1}}$. In particular, if there is a positive integer i such that $i_{2k-1} \leq i < i_{2k+1}$ and $r_i = r_{i+1}$, then

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} > \arccos \frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \frac{r_{i_{2k}}}{r_{i_{2k+1}}}.$$

Proof. Since $\arccos \frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \frac{r_{i_{2k}}}{r_{i_{2k+1}}} < \pi$, if $\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} \geq \pi$, then

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} > \arccos \frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \frac{r_{i_{2k}}}{r_{i_{2k+1}}}.$$

Now we assume that $\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} < \pi$. By Lemma 3.6, we know that $A_{i_{2k-1}}A_{i_{2k}}A_{i_{2k+1}}O$ is convex. Note that $r_{i_{2k}} \leq r_{i_{2k-1}}$ and $r_{i_{2k}} \leq r_{i_{2k+1}}$. By Lemma 3.7, one obtains

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} \geq \arccos \frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \frac{r_{i_{2k}}}{r_{i_{2k+1}}},$$

where the equality holds if and only if $A_{i_{2k-1}}, A_{i_{2k}}, A_{i_{2k+1}}$ are collinear and the line $OA_{i_{2k}}$ is perpendicular to the line $A_{i_{2k-1}}A_{i_{2k+1}}$. Since $A_1A_2\ldots A_n$ is convex and the vertices are arranged in order, we have that $A_{i_{2k-1}}, A_{i_{2k}}, A_{i_{2k+1}}$ are collinear if and only if $A_{i_{2k-1}}, A_{i_{2k-1}+1}, \ldots, A_{i_{2k+1}}$ are collinear.

Furthermore, if $A_{i_{2k-1}}, \ldots, A_{i_{2k}}, \ldots, A_{i_{2k+1}}$ are collinear and the line $OA_{i_{2k}}$ is perpendicular to the line $A_{i_{2k-1}}A_{i_{2k+1}}$, then it is clear that $r_i \neq r_{i+1}$ for all $i_{2k-1} \leq i < i_{2k+1}$. $\square$

Lemma 3.10. *Let r_1, r_2 be positive real numbers where $r_2 \leq r_1$. Let A_1 and A_2 be two points. For any positive real number $\varepsilon < \pi - \arccos \frac{r_2}{r_1}$, if $|OA_1| = r_1$, $|OA_2| = r_2$ and $\measuredangle A_1OA_2 =$*

$\arccos \dfrac{r_2}{r_1} + \varepsilon$, *then*

$$\measuredangle A_2 A_1 O < \frac{\pi}{2}, \ \measuredangle O A_2 A_1 < \frac{\pi}{2}$$

Proof. If $\arccos \dfrac{r_2}{r_1} + \varepsilon \geq \dfrac{\pi}{2}$ then the result is obvious. Now we suppose that $\arccos \dfrac{r_2}{r_1} + \varepsilon < \dfrac{\pi}{2}$ Let $\measuredangle A_2 A_1 O = \theta_1$ and $\measuredangle O A_2 A_1 = \theta_2$, and let $|A_1 A_2| = r$. It suffices to show that $\cos\theta_1 > 0$ and $\cos\theta_2 > 0$. By the law of cosines, we have

$$r^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\left(\arccos \frac{r_2}{r_1} + \varepsilon\right) > r_1^2 + r_2^2 - 2r_1 r_2 \cos\left(\arccos \frac{r_2}{r_1}\right) = r_1^2 - r_2^2.$$

Therefore,

$$\cos\theta_1 = \frac{r^2 + r_1^2 - r_2^2}{2rr_1} > \frac{r_1^2 - r_2^2}{rr_1} \geq 0$$

and

$$\cos\theta_2 = \frac{r^2 + r_2^2 - r_1^2}{2rr_2} > 0.$$

□

Lemma 3.11. *Let $S = (r_i)_{i=1}^n$ be an initial sequence of $\mathcal{R}$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. For any given $k \in \{1, 2, \ldots, l\}$ and any arbitrary positive real number $\varepsilon < \pi - \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} - \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k+1}}}$, there are $i_{2k+1} - i_{2k-1} + 1$ points $A_{i_{2k-1}}, A_{i_{2k-1}+1}, \ldots, A_{i_{2k+1}}$ arranged in counterclockwise order such that the polygon $A_{i_{2k-1}} A_{i_{2k-1}+1} \cdots A_{i_{2k+1}} O$ is a strictly convex and satisfy the following conditions:*

(1) *$|OA_i| = r_i$ for $i = i_{2k-1}, i_{2k-1}+1, \ldots, i_{2k+1}$;*

(2) *$\measuredangle A_{i_{2k-1}+1} A_{i_{2k-1}} O < \dfrac{\pi}{2}$ and $\measuredangle O A_{i_{2k+1}} A_{i_{2k+1}-1} < \dfrac{\pi}{2}$;*

(3) *$\measuredangle A_{i_{2k-1}} O A_{i_{2k+1}} = \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k+1}}} + \varepsilon$.*

Proof. From Lemma 3.5, we know that there are indexes p_k and q_k with $i_{2k-1} \leq p_k \leq i_{2k}$ and $i_{2k} \leq q_k \leq i_{2k+1}$, such that

$$r_{i_{2k-1}} = \cdots = r_{p_k} > \cdots > r_{i_{2k}}$$

and

$$r_{i_{2k}} = \cdots = r_{q_k} < \cdots < r_{i_{2k+1}}.$$

Let $\varepsilon_1' = \frac{\varepsilon}{2(i_{2k} - i_{2k-1})}$ and $\varepsilon_1 = (i_{2k} - p_k)\varepsilon_1'$.

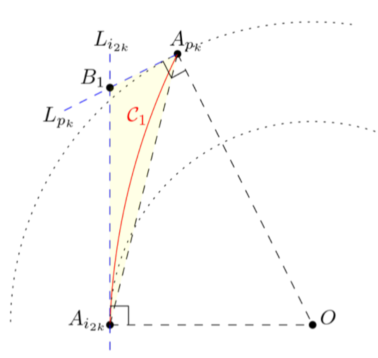

FIGURE 16. the curve $\mathcal{C}_1$

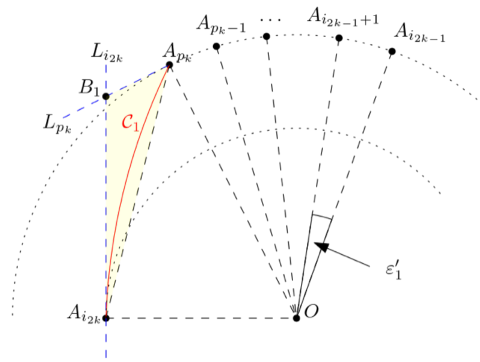

FIGURE 17. the points $A_{i_{2k-1}}, \ldots, A_{p_k-1}$

We firstly choose the points $A_{p_k}, A_{p_k+1}, \ldots, A_{i_{2k}}$. If $p_k = i_{2k}$, then one can pick a point A_{p_k} such that $|OA_{p_k}| = r_{p_k}$. If $p_k \neq i_{2k}$, then one can choose two points A_{p_k} and $A_{i_{2k}}$ such that $\measuredangle A_{p_k} O A_{i_{2k}} = \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} + \varepsilon_1$, and $|OA_{p_k}| = r_{p_k},\ |OA_{i_{2k}}| = r_{i_{2k}}$. Noting that $\arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} + \varepsilon_1 < \pi$. By Lemma 3.10, we have that

$$\measuredangle A_{i_{2k}} A_{p_k} O < \frac{\pi}{2},\ \measuredangle O A_{i_{2k}} A_{p_k} < \frac{\pi}{2}.$$

Denote by L_{p_k} the line perpendicular to the line OA_{p_k} and passing through A_{p_k}. Denote by $L_{i_{2k}}$ the line perpendicular to the line $OA_{i_{2k}}$ and passing through $A_{i_{2k}}$. Let B_1 be the intersection of the lines L_{p_k} and $L_{i_{2k}}$. There must be a strictly convex curve $\mathcal{C}_1$ with endpoints A_{p_k} and $A_{i_{2k}}$ such that the curve (exclude two endpoints) lies inside the triangle $A_{p_k} B_1 A_{i_{2k}}$ (see Figure 16). Since $r_{p_k} > r_{p_k+1} > \cdots > r_{i_{2k}}$, by the continuity of the curve $\mathcal{C}_1$, there exist points $A_{p_k+1}, \ldots, A_{i_{2k}-1}$ (arranged in order) on the curve $\mathcal{C}_1$ such that $|OA_i| = r_i$ for all $i = p_k + 1, \ldots, i_{2k} - 1$. Since the curve $\mathcal{C}_1$ is strictly convex and lies inside the triangle $A_{p_k} B_1 A_{i_{2k}}$, we have that the polygon $A_{p_k} A_{p_k+1} \cdots A_{i_{2k}} O$ is strictly convex and

$$\measuredangle A_{p_k+1} A_{p_k} O < \frac{\pi}{2},\ \measuredangle O A_{i_{2k}} A_{i_{2k}-1} < \frac{\pi}{2}.$$

After choosing the points $A_{p_k}, A_{p_k+1}, \ldots, A_{i_{2k}}$, then one can choose $p_k - i_{2k-1}$ points $A_{i_{2k-1}}, \ldots, A_{p_k-1}$ such that $|OA_i| = r_i = r_{i_{2k-1}}$ and $\measuredangle A_i O A_{i+1} = \varepsilon_1'$ for all $i = i_{2k-1}, \ldots, p_k - 1$. It is clear that for all $i = i_{2k-1}, \ldots, p_k - 1$, the triangle $A_i O A_{i+1}$ is isosceles triangle, and their two base angles are less than $\frac{\pi}{2}$. Therefore, we obtain that the polygon $A_{i_{2k-1}} A_{i_{2k-1}+1} \cdots A_{i_{2k}} O$ is a strictly convex polygon with $\measuredangle A_{i_{2k-1}} O A_{i_{2k}} = \measuredangle A_{i_{2k-1}} O A_{p_k} + \measuredangle A_{p_k} O A_{i_{2k}} = (p_k - i_{2k-1})\varepsilon_1' + \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} + \varepsilon_1 = \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k-1}}} + \frac{\varepsilon}{2}$ and

$$\measuredangle A_{i_{2k-1}+1} A_{i_{2k-1}} O < \frac{\pi}{2},\ \measuredangle O A_{i_{2k}} A_{i_{2k}-1} < \frac{\pi}{2}.$$

By symmetry, one can use an analogous argument to choose $i_{2k+1} - i_{2k}$ points $A_{i_{2k}+1}, \ldots, A_{i_{2k+1}}$ such that the polygon $A_{i_{2k}} A_{i_{2k}+1} \cdots A_{i_{2k+1}} O$ is a strictly convex polygon with $\measuredangle A_{i_{2k}} O A_{i_{2k+1}} = \arccos \dfrac{r_{i_{2k}}}{r_{i_{2k+1}}} + \frac{\varepsilon}{2}$ and

$$\measuredangle A_{i_{2k}+1} A_{i_{2k}} O < \frac{\pi}{2},\ \measuredangle O A_{i_{2k+1}} A_{i_{2k+1}-1} < \frac{\pi}{2}.$$

Therefore, we obtain the strictly convex polygon $A_{i_{2k-1}} A_{i_{2k-1}+1} \cdots A_{i_{2k+1}} O$ that satisfies the desired conditions. $\square$

4. Necessary and sufficient conditions of $\mathcal{R}$ which has a convex configuration

Now, we are ready to prove the necessary and sufficient conditions of $\mathcal{R}$ which has a convex configuration. We mainly use the angle properties mentioned in Section 3.2 to verify the following statements.

Theorem 4.1. *$\mathcal{R}$ has a strictly convex configuration if and only if there is a strictly convex sequence $(r_i)_{i=1}^n$ of $\mathcal{R}$.*

Proof. Assume that $\mathcal{R}$ has a strictly convex configuration $A_1A_2\cdots A_n$, where $|OA_1| = \max\mathcal{R}$ and $|OA_n| \neq \max\mathcal{R}$. By Lemma 3.4, we may assume that $A_1A_2\ldots A_n$ contains O as an interior point. Since the polygon $A_1A_2\cdots A_n$ is strictly convex, any three vertices cannot be collinear. By Lemma 3.9, we have

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} > \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}, \tag{4.1}$$

and hence

$$2\pi = \sum_{k=1}^{l}\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} > \sum_{k=1}^{l}\left(\arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right) = \mathrm{Angle}(S).$$

Therefore, S is a strictly convex sequence of $\mathcal{R}$.

Conversely, assume that $S = (r_i)_{i=1}^n$ is a strictly convex sequence of $\mathcal{R}$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. One can choose a positive real number ε such that $\varepsilon < \pi - \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} - \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}$ for all $k = 1, 2, \ldots, l$ and $\mathrm{Angle}(S) + l\varepsilon < 2\pi$. By Lemma 3.11, there are i_3 points $A_1, A_2, \ldots, A_{i_3}$ such that

(1) $|OA_i| = r_i$ for $i = 1, \ldots, i_3$;
(2) $\measuredangle A_2A_1O < \frac{\pi}{2}$ and $\measuredangle OA_{i_3}A_{i_3-1} < \frac{\pi}{2}$;
(3) $\measuredangle A_{i+1}A_iA_{i-1} < \pi$, for $i = 2, \ldots, i_3 - 1$;
(4) $\measuredangle A_1OA_{i_3} = \arccos\frac{r_{i_2}}{r_{i_1}} + \arccos\frac{r_{i_2}}{r_{i_3}} + \varepsilon$.

Using the same argument, there are $i_5 - i_3$ points $A_{i_3+1}, \ldots, A_{i_5}$ such that

(1) $|OA_i| = r_i$ for $i = i_3 + 1, \ldots, i_5$;
(2) $\measuredangle A_{i_3+1}A_{i_3}O < \frac{\pi}{2}$ and $\measuredangle OA_{i_5}A_{i_5-1} < \frac{\pi}{2}$;
(3) $\measuredangle A_{i+1}A_iA_{i-1} < \pi$, for $i = i_3 + 1, \ldots, i_5 - 1$;
(4) $\measuredangle A_{i_3}OA_{i_5} = \arccos\frac{r_{i_4}}{r_{i_3}} + \arccos\frac{r_{i_4}}{r_{i_5}} + \varepsilon$.

Noting that $i_{2l+1} = n + 1$. By repeating this construction, we obtain the points $A_1, A_2, \ldots, A_n, A_{n+1}$ such that satisfy the following conditions:

(1) $|OA_i| = r_i$ for $i = 1, \ldots, n$ and $|OA_{n+1}| = r_1$;
(2) $\measuredangle A_2A_1O < \frac{\pi}{2}$ and $\measuredangle OA_{n+1}A_n < \frac{\pi}{2}$;
(3) $\measuredangle A_{i+1}A_iA_{i-1} < \pi$ for all $i = 2, \ldots, n$;
(4) $\measuredangle A_1OA_{n+1} = \sum_{k=1}^{l}\left(\arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}} + \varepsilon\right) = \mathrm{Angle}(S) + l\varepsilon < 2\pi$.

Since $|OA_1| = |OA_{n+1}|$, we have that the polygon $A_1A_2\cdots A_nA_{n+1}$ is a strictly convex polygon (which may or may not contain O as an interior point), and hence the polygon $A_1A_2\cdots A_n$ is strictly convex (see Figure 18). It follows that the polygon $A_1A_2\cdots A_n$ is a strictly convex configuration of $\mathcal{R}$.

$\square$

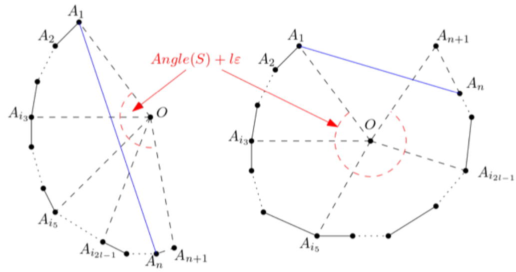

FIGURE 18. $A_1A_2\cdots A_n$ is strictly convex

Theorem 4.2. *$\mathcal{R}$ has a convex configuration if and only if there is a convex sequence $(r_i)_{i=1}^n$ of $\mathcal{R}$.*

Proof. Assume that $\mathcal{R}$ has a convex configuration $A_1A_2\cdots A_n$, where $|OA_1| = \max\mathcal{R}$ and $|OA_n| \neq \max\mathcal{R}$. By Lemma 3.4, we may assume that $A_1A_2\ldots A_n$ contains O as an interior point. Let $S = (r_i)_{i=1}^n = (|OA_i|)_{i=1}^n$. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. By Lemma 3.9, we have

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} \geq \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}, \tag{4.2}$$

and hence

$$2\pi = \sum_{k=1}^{l}\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} \geq \sum_{k=1}^{l}\left(\arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right) = \mathrm{Angle}(S)$$

If $\mathrm{Angle}(S) = 2\pi$, then (4.2) holds as an equality, for all $k = 1, 2, \ldots, l$. By Lemma 3.9, we know that $r_i \neq r_{i+1}$ for all $i \geq 1$. Therefore S is a convex sequence of $\mathcal{R}$.

Conversely, we assume that $S = (r_i)_{i=1}^n$ is a convex sequence of $\mathcal{R}$. If $\mathrm{Angle}(S) < 2\pi$, then by Theorem 4.1, $\mathcal{R}$ has a (strictly) convex configuration. Now we consider the case $\mathrm{Angle}(S) = 2\pi$ and $r_i \neq r_{i+1}$ for all $i \geq 1$. By applying Lemma 3.8, one can choose $n+1$ points $A_1, A_2, \ldots, A_n, A_{n+1}$ which are arranged in counterclockwise order with respect to O and satisfy the following conditions:

(1) $|OA_i| = r_i$ where $i = 1, 2, \ldots, n$;
(2) for all $k = 1, 2, \ldots, l$, the line $OA_{i_{2k}}$ is perpendicular to the line $A_{i_{2k-1}}A_{i_{2k+1}}$ and

$$\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} = \arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}};$$

(3) for $k = 1, 2, \ldots, l$ and positive integer i, if $i_{2k-1} \leq i \leq i_{2k+1}$, then A_i lies on the line segment $\overline{A_{i_{2k-1}}A_{i_{2k+1}}}$.

We have

$$\sum_{k=1}^{l}\measuredangle A_{i_{2k-1}}OA_{i_{2k+1}} = \sum_{k=1}^{l}\left(\arccos\frac{r_{i_{2k}}}{r_{i_{2k-1}}} + \arccos\frac{r_{i_{2k}}}{r_{i_{2k+1}}}\right) = \mathrm{Angle}(S) = 2\pi.$$

Therefore, A_{n+1} coincides with A_1, and $A_1A_2\ldots A_n$ forms a convex polygon (see Figure 19).

$\square$

From Theorem 4.1, one can easily obtain the following results.

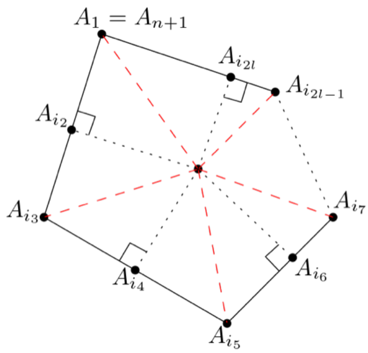

FIGURE 19. the case $\mathrm{Angle}(S) = 2\pi$

Corollary 4.1. [3] *For a given vertex-norm set*

$$\mathcal{R} = \{r_{(1,1)}, ..., r_{(1,m_1)}, r_{(2,1)}, ..., r_{(2,m_2)}, ..., r_{(k,1)}, ..., r_{(k,m_k)}\}$$

such that $r_{(i,1)} = ... = r_{(i,m_i)}$ *and* $1 \leq m_i \leq 4$, *for each* $i = 1, ..., k$. *Then* $\mathcal{R}$ *has a strictly convex configuration.*

Corollary 4.2. *If* $\mathcal{R}$ *consists of at most four different vertex-norms such that each vertex-norm has a finite multiplicity, then* $\mathcal{R}$ *has a strictly convex configuration.*

Corollary 4.3. *Let* $\mathcal{R}$ *be a multiset of positive real number and* $r^* = \max \mathcal{R}$ *and* $r_* = \min \mathcal{R}$. *Let* $\mathcal{S}$ *be a multiset containing only the elements* r^* *or* r_*. *If* $\mathcal{R}$ *has a strictly convex configuration and* $\mathcal{T}$ *is the sum of* $\mathcal{R}$ *and* $\mathcal{S}$, *then* $\mathcal{T}$ *has a strictly convex configuration.*

5. EXAMPLE OF $\mathcal{R}$ WHICH DOES NOT HAVE A CONVEX CONFIGURATION

Based on the necessary and sufficient conditions presented in Section 4, we can find the example of $\mathcal{R}$, which does not have a convex configuration. The observation of the example is derived from the reversed condition mentioned in Section 2 as a simple case. Remark that if a multiset $\mathcal{R}$ does not have a convex configuration, then a multiset $\mathcal{R}'$ containing $\mathcal{R}$ does not have a convex configuration, either.

Corollary 5.4. *There exist infinitely many multisets* $\mathcal{R}$ *such that* $\mathcal{R}$ *has no convex configuration.*

Proof. Let $\mathcal{R} = \{10, 10, 10, 10, 10, 10^2, 10^2, 10^2, 10^2, 10^2, \ldots, 10^5, 10^5, 10^5, 10^5, 10^5\}$. Then for any $r, r' \in \mathcal{R}$ with $r < r'$, we have $\arccos\left(\frac{r}{r'}\right) \geq \arccos\left(\frac{1}{10}\right) > \frac{2\pi}{5}$.

Let $S = (r_i)_{i=1}^{25}$ be an initial sequence of $\mathcal{R}$. We will show that S cannot be a convex sequence. Let $C(S) = \{i_1, i_2, \ldots, i_{2l}\}$ be the core index set of S. For $k = 1, 2, 3, 4, 5$, we define the subindex sets

$$I_k = \{i \in \{1, 2, \ldots, 25\} \mid r_i = 10^k\} = \{i_{(k,1)}, i_{(k,2)}, i_{(k,3)}, i_{(k,4)}, i_{(k,5)}\}$$

where $i_{(k,1)} < i_{(k,2)} < i_{(k,3)} < i_{(k,4)} < i_{(k,5)}$. We will consider the following two cases:

Case 1: there is a $k \in \{1, 2, 3, 4, 5\}$ such that I_k does not contain any two consecutive indexes. Noting that $r_{i_{(k,p)}} = r_{i_{(k,q)}}$ for all $p, q \in \{1, 2, 3, 4, 5\}$. If there are $p, q \in \{1, 2, 3, 4, 5\}$ and $m \in \{1, 2, \ldots, 2l\}$ such that $p \neq q$ and $i_m \leq i_{(k,p)} < i_{(k,q)} < i_{m+1}$, then by Lemma 3.5, we know that $r_{i_m} = \cdots = r_{i_{(k,q)}}$ which is impossible, since I_k does not contain any two consecutive indexes. Therefore, there are $m_1, m_2, m_3, m_4, m_5 \in \{1, 2, \ldots, 2l\}$ such that $m_1 < m_2 < m_3 < m_4 < m_5$ and $i_{m_j} \leq i_{(k,j)} < i_{m_j+1}$ for each $j \in \{1, 2, 3, 4, 5\}$.

Furthermore, we have that $r_{i_{m_j}} \neq r_{i_{m_{j+1}}}$, for all $j = 1, 2, 3, 4, 5$. Then we have

$$\text{Angle}(S) \geq \sum_{j=1}^{5} \arccos\left(\min\left\{\frac{r_{i_{m_j}}}{r_{i_{m_j+1}}}, \frac{r_{i_{m_j+1}}}{r_{i_{m_j}}}\right\}\right) \geq \sum_{j=1}^{5} \arccos\left(\frac{1}{10}\right) > 5 \cdot \frac{2\pi}{5} = 2\pi,$$

and hence S is not a convex sequence.

Case 2: the subindex set I_k contains two consecutive indexes, for all $k = 1, 2, 3, 4, 5$. For a given $k \in \{1, 2, 3, 4, 5\}$, assume that $j, j+1 \in I_k$. Since $r_j = r_{j+1}$, by Lemma 3.5, we know that there is an $m \in \{1, 2, \ldots, 2l\}$ such that $i_m \leq j < j+1 \leq i_{m+1}$ and $r_{i_m} = \cdots = r_j = r_{j+1}$. By Lemma 3.5, we consider the following two cases:

Case 2.1: there is a j' such that $i_{m+1} \leq j' < i_{m+2}$ and $r_{i_{m+1}} = \cdots = r_{j'} = r_{j'+1}$. Note that $r_{i_m}, \ldots, r_j, r_{j+1}, \cdots, r_{i_{m+1}}$ is monotone. If $r_{j'} = r_j$, then by Lemma 3.5, we have $r_{i_m} = \cdots = r_{j+1} = \cdots = r_{i_{m+1}} = \cdots = r_{j'+1}$ which is impossible, since i_{m+1} is a peak or a bottom index. Now suppose that $r_{j'} \neq r_j$. Then $r_{i_m} \neq r_{i_{m+1}}$. Let

$$m_k = m.$$

We have

$$\arccos\left(\min\left\{\frac{r_{i_{m_k}}}{r_{i_{m_k+1}}}, \frac{r_{i_{m_k+1}}}{r_{i_{m_k}}}\right\}\right) \geq \arccos\left(\frac{1}{10}\right) > \frac{2\pi}{5}.$$

Case 2.2: $r_{i_{m+1}}, \ldots, r_{i_{m+2}}$ is strictly monotone. Then $r_{i_{m+1}} \neq r_{i_{m+2}}$. Let

$$m_k = m + 1.$$

We get

$$\arccos\left(\min\left\{\frac{r_{i_{m_k}}}{r_{i_{m_k+1}}}, \frac{r_{i_{m_k+1}}}{r_{i_{m_k}}}\right\}\right) \geq \arccos\left(\frac{1}{10}\right) > \frac{2\pi}{5}.$$

According to the selection of m_k, one can verify that m_1, m_2, m_3, m_4, m_5 are pairwise distinct. It follows that

$$\text{Angle}(S) \geq \sum_{k=1}^{5} \arccos\left(\min\left\{\frac{r_{i_{m_k}}}{r_{i_{m_k+1}}}, \frac{r_{i_{m_k+1}}}{r_{i_{m_k}}}\right\}\right) \geq \sum_{k=1}^{5} \arccos\left(\frac{1}{10}\right) > 5 \cdot \frac{2\pi}{5} = 2\pi.$$

Therefore, S is not a convex sequence.

By Theorem 4.2, we obtain that $\mathcal{R}$ has no any convex configuration.

It is obvious that if $\mathcal{R}'$ is a multiset such that $\mathcal{R} \subseteq \mathcal{R}'$, then $\mathcal{R}'$ do not have any convex configuration. Hence we obtain the desired result. □

6. Concluding Remarks

The problem of the existence of convex configuration for a given set of vertex-norm was initially proposed in [3]. The answer in three-dimensional space was positive, while the answer in two-dimensional space was positive in limited cases and was left as a conjecture. In this study, we proved the conjecture of the non-existence of convex configuration of a given set of vertex-norm in two-dimensional space by providing the necessary and sufficient conditions on the vertex-norm set for such convex configuration to exist. The results in this study cover all cases in two-dimensional space. With these conditions, the complete examples of vertex-norm sets for which no convex configuration exists can be illustrated. The approach proposed in this study is different from the directions of previous studies, e.g. [1, 4, 5, 7, 10], in the sense that we focus on the sum of angles around the fixed point, while the previous literature concentrated on the fixed lengths of linkages, together with angles between adjacent linkages.

Although the conditions we provided can be used to verify the existence of a convex configuration, they are not yet practical. Therefore, it would be beneficial to derive explicit conditions that can be programmed for used on computer. Since the conditions are based on the *initial sequence*, a procedure for enumeration the combinatorial patterns of the vertex-norm set is worth studying to simplify the process. In addition, substituting the Euclidean norm with other norms may lead to new applications and mathematical insights.

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