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Levitin−Polyak Well-Posedness for Parametric Set Optimization Problem

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ABSTRACT. The aim of this paper is to introduce two notions of Levitin−Polyak (LP in short) well-posedness for a parametric set optimization problem, a pointwise and a global notion. Necessary and sufficient conditions for a parametric set optimization problem to be LP well-posed are given. Characterizations of LP well-posedness for a parametric set optimization problem in terms of upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of approximate solution sets are also established.

1. INTRODUCTION

In recent years, many authors have worked on set optimization problems. The reason for this popularity is their applications in areas like game theory, mathematical economics, fuzzy optimization and many more; see [15] and references therein.

Kuroiwa [20, 21] proposed various set order relations for comparison of sets to define notions of minimal solution of a set optimization problem. For details, we refer the reader to the survey paper [2]. The study of the solution sets of a perturbed set optimization problem, perturbed with respect to the feasible set or the objective set-valued map is a fast growing topic and is studied under stability theory. Various authors have studied stability theory of a perturbed set optimization problem in different directions.

Xu and Li [29] derived the upper, lower semicontinuity and closedness of the minimal solution and weak minimal solution set mappings to a parametric set optimization problem under some strong assumptions. Later, Xu and Li in [30] weakened and modified the assumptions of [29] to study the continuity of the minimal solution set map to parametric set optimization problem. Karuna and Lalitha [14] studied stability in terms of Hausdorff and Painlevé−Kuratowski convergence of minimal and weak minimal solution sets in set optimization problems by perturbing the feasible set. Han and Huang [12] derived the Hausdorff upper semicontinuity of the minimal solution mapping to a parametric set optimization problem with perturbed feasible set map. Khoshkhabar-amiranloo [17] studied stability of the minimal solution mappings of parametric set optimization problems in terms of semi-continuity and compactness. Preechasilp and Wangkeeree [27] studied stability in terms of upper semicontinuity, lower semicontinuity, and closedness of the solution mapping to a parametric set optimization problem. Zhang and Huang [31] obtained the upper semi-continuity, lower semi-continuity and compactness of relaxed minimal and minimal solution mappings for parametric set optimization problems.

Well-posedness of optimization problems plays an important role in the study of the stability theory. Many authors have studied the well-posedness for set optimization problems under different conditions. In [32], the authors studied three types of well-posedness for set optimization problems with cone-bounded objective function values. Gutiérrez et

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al. [11] improved some results in [32] to obtain the well-posedness of set optimization problems, by relaxing the condition of cone-boundedness of the objective function values. Crespi et al. [5] obtained a notion of global well-posedness for set-optimization problems by generalizing one of the notions of global well-posedness in [32]. Dhingra and Lalitha [6] studied a notion of well-setness for set optimization problem using the excess function. Crespi et al. [4] obtained some characterizations for pointwise and global well-posedness in set optimization in terms of compactness and upper semicontinuity of solution set map.

Khoshkhabar-amiranloo and Khorram [18] studied LP well-posedness [24] for set optimization problems. They introduced global notions of metrically well-setness and metrically LP well-setness for set optimization problems and the pointwise notions of LP well-posedness for set optimization problems. They also obtained scalar characterizations of LP well-posedness and metrically well-setness of a set optimization problem using a scalarization function in terms of well-posedness and metrically well-setness of a corresponding scalar optimization problem, respectively. Khoshkhabar-amiranloo [16] derived characterizations of generalized LP well-posedness of set optimization problem in terms of the upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of sets of approximate solutions. Vui et al. [28] introduced different types of notions of LP well-posedness for set optimization problems using three types of set order relations. They also established characterizations of these notions using the Kuratowski measure of noncompactness. Recently, Ansari et al. [1] studied different notions of LP well-posedness for set optimization problem. They obtained characterizations LP well-posedness for set optimization problems using the Kuratowski measure of noncompactness. They also established the relationship between stability and LP well-posedness for set optimization problem. In [10], Gupta and Srivastava introduced a notion of LP well-posedness for set optimization problem and established its characterizations in terms of Hausdorff upper semicontinuity and compactness of an approximate solution map. Duy [8] studied various notions of LP well-posedness for set optimization problem with respect to the upper set less order relation, established relationships between them and gave sufficient conditions for them.

Well-posedness for perturbed optimization problems has been studied by various authors. Zolezzi [33, 34] introduced the notion of parametric well-posedness by embedding the original optimization problem in a family of perturbed optimization problems. Lignola and Morgan [25] studied parametric well-posedness for a family of variational inequalities. Lalitha and Bhatia [23] introduced the notion of well-posedness for parametric quasivariational inequality problems with set-valued maps. In [22], the authors studied LP well-posedness for parametric quasivariational inequality problem.

Motivated by these papers, we study LP well-posedness for parametric set optimization problem. In this paper, we introduce a pointwise and a global notion of LP well-posedness for parametric set optimization problem and obtain necessary and sufficient conditions for a parametric set optimization problem to be LP well-posed. We also establish upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of approximate solution sets of a LP well-posed parametric set optimization problem.

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries required in the sequel. In Section 3, we introduce a pointwise notion of LP well-posedness for parametric set optimization problem. We give Dontchev−Zolezzi measure and Furi−Vignoli measure for pointwise LP well-posed parametric set optimization problem. We then establish upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of approximate solution sets for a pointwise LP well-posed parametric set optimization problem. In Section 4, we define a global notion of LP well-posedness for parametric set optimization problem. We give necessary and sufficient conditions for global LP well-posed parametric set optimization problem. We also establish relationship between pointwise LP well-posedness and global LP well-posedness notions. We then establish upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of approximate solution set maps for a global LP well-posed parametric set optimization problem.

2. Preliminaries

Let Y be a real normed linear space and let K be a closed convex pointed cone with nonempty interior. Let K induce the following order relations in Y, for $y_1,\ y_2 \in Y$, we have

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K,$$
$$y_1 <_K y_2 \Longleftrightarrow y_2 - y_1 \in \text{int}K,$$

where $\text{int}K$ denotes the interior of K.

Let $\mathcal{P}[Y]$ denote the collection of all nonempty subsets of Y and A^c denote the complement of a set A in Y. We recall the following set order relations from [19], if $A, B \in \mathcal{P}[Y]$

$$A \leq_K^l B \Longleftrightarrow B \subseteq A + K$$

and

$$A <_K^l B \Longleftrightarrow B \subseteq A + \text{int}K.$$

Let X be a real normed linear space. We denote the open and closed ball in X centered at origin and radius r with $r > 0$ by $\mathcal{B}_X(r)$ and $\mathcal{B}_X[r]$ respectively and diameter of a set $A \subseteq X$ by diam $A := \sup\{\|x - y\| : x, y \in A\}$.

For two nonempty sets U and W of X, the excess function of U over W, denoted by $ex(U, W)$ is defined as

$$ex(U, W) := \sup_{u \in U} d(u, W), \text{ where } d(u, W) := \inf_{w \in W} \|u - w\|.$$

For a sequence of sets $\{U_n\} \subseteq X$, the sequence $\{U_n\}$ converges to a set U in X, in the sense of upper Hausdorff set convergence if

$$ex(U_n, U) \to 0 \text{ as } n \to \infty.$$

We now recall the notion of Painlevé−Kuratowski convergence (Definition 2.1, [9]). For a sequence of sets $\{U_n\}$ in X, we have

$$\text{Li}\, U_n := \{x \in X : x_n \to x, x_n \in U_n, \text{ for sufficiently large } n\},$$

$$\text{Ls}\, U_n := \{x \in X : x_{n_m} \to x, x_{n_m} \in U_{n_m},\ \{n_m\} \text{ is an increasing sequence of integers}\}.$$

The sequence $\{U_n\}$ converges to a set U in the sense of Painlevé−Kuratowski, if

$$\text{Ls}\, U_n \subseteq U \subseteq \text{Li}\, U_n.$$

The relation $\text{Ls}\, U_n \subseteq U$ is known as the upper part of the convergence and the relation $U \subseteq \text{Li}\, U_n$ is known as the lower part of the convergence.

We now recall the notions of upper continuous, lower continuous, continuous and compact set-valued map from [15]. For the sake of convenience, we refer the notions of upper continuous and lower continuous as upper semicontinuous and lower semicontinuous.

Definition 2.1. (Definition *3.1.1* and Definition *3.1.7*, [15]) Let $G : X \rightrightarrows Y$ be a set-valued map. Then G is

(i) upper semicontinuous at $\bar{x} \in X$ if for every open set W in Y containing $G(\bar{x})$, there exists a neighbourhood V of $\bar{x}$ such that $G(x) \subseteq W$ for all $x \in V$.

(ii) lower semicontinuous at $\bar{x} \in X$ if for every open set W in Y with $G(\bar{x}) \cap W \neq \emptyset$, there exists a neighbourhood V of $\bar{x}$ such that $G(x) \cap W \neq \emptyset$ for all $x \in V$.

(iii) continuous at $\bar{x} \in X$ if it is both upper semicontinuous and lower semicontinuous at $\bar{x}$.

(iv) compact at $\bar{x} \in X$ if for every sequence $\{x_n\}$ and $y_n \in G(x_n)$, with $x_n \to \bar{x}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \bar{y} \in G(\bar{x})$.

The map G is said to be upper semicontinuous (lower semicontinuous, continuous, compact) on a subset U of X if G is upper semicontinuous (lower semicontinuous, continuous, compact) at every point $\bar{x} \in U$.

In this paper, we require the following characterization of upper and lower semicontinuity.

Lemma 2.1. *Let $G : X \rightrightarrows Y$ be a set valued mapping.*

(i) *(Proposition* 3.1.6, [15]*) G is lower semicontinuous at $\bar{x} \in X$ if and only if for every sequence $\{x_n\}$ in X with $x_n \to \bar{x}$ and for any $\bar{y} \in G(\bar{x})$, there exist $y_n \in G(x_n)$ such that $y_n \to \bar{y}$.*

(ii) *(Proposition* 3.1.5, [15]*) If $G(\bar{x})$ is compact, then G is upper semicontinuous at $\bar{x}$ if and only if for any sequence $\{x_n\}$ in X with $x_n \to \bar{x}$ and for any $y_n \in G(x_n)$, there exist $\bar{y} \in G(\bar{x})$ and a subsequence $\{y_{n_k}\}$ of y_n such that $y_{n_k} \to \bar{y}$.*

Let Z be a normed space and T be a nonempty subset of Z. The parametric set optimization problem corresponding to a parameter $t \in T$ is defined as follows:

$$(\mathrm{P}(t)) \qquad \mathrm{Min} \quad F(x,t) \qquad \text{subject to} \quad x \in M(t),$$

where $M : T \rightrightarrows X$ and $F : X \times T \rightrightarrows Y$. We assume $M(t) \neq \emptyset$, compact set and $F(x,t) \neq \emptyset$, for every $t \in T$ and $x \in M(t)$.

A point $\bar{x} \in M(t)$ is said to be an l-weak minimal solution of (P(t)) if, for any $x \in M(t)$ such that $F(x,t) <^l_K F(\bar{x},t) \Rightarrow F(\bar{x},t) <^l_K F(x,t)$. We denote the set of all l-weak minimal solutions of the problem (P(t)) by l-WMin(F,t).

3. Pointwise LP well-posedness

In this section we introduce a notion of pointwise LP well-posedness for parametric set optimization problem (P(t)). Throughout the paper, we assume e to be a fixed element of int$K$.

Definition 3.2. Let $\bar{t} \in T$ and $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$. Let $\bar{x} \in l$-WMin$(F,\bar{t})$. A sequence $\{x_n\}$ in X is said to be a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$ if there exist $\varepsilon_n \downarrow 0$, $x_n \in M(t_n) + \mathcal{B}_X[\varepsilon_n]$ such that

$$F(x_n,t_n) \leq^l_K F(\bar{x},t_n) + \varepsilon_n e, \ \forall n.$$

Definition 3.3. Let $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F,\bar{t})$. (P($\bar{t}$)) is said to be pointwise LP well-posed at $\bar{x}$, if for any sequence $\{t_n\}$ in T converging to $\bar{t}$, every pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$ converges to $\bar{x}$.

Remark 3.1. Definition *3.2* and Definition *3.3* extend Definition *2.6* (iii) and Definition *2.7*(iii) of [16] respectively to the case of a parametric set optimization problem. In [16], the author defined these notions for l-minimal solutions of the set optimization problem.

The following example illustrates Definition 3.3.

Example 3.1. Consider parametric set optimization problem, where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [0, 1]$, $e = (1, 1)$ and $M : T \rightrightarrows X$ is defined as

$$M(t) := [-t, 1 + t]$$

and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x,t) := \begin{cases} [(x,x),(x,2)], & \text{if } 0 \le x \le 1, t = 0, \\ \{(1,1)\}, & \text{if } x < 0 \text{ or } x > 1, t = 0, \\ \{(t,t)\}, & \text{if } t \ne 0. \end{cases}$$

Let $\bar{t} = 0$. Then l-WMin$(F, \bar{t}) = \{0\}$. Clearly, (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x} \in l$-WMin$(F, \bar{t})$ where $\bar{x} = 0$.

Now, let $t' = 1$. Then l-WMin$(F, t') = M(t') = [-1, 2]$. Clearly, (P($t'$)) is not pointwise LP well-posed at $x' = 1$, since the sequence $\{x'_n\}$ where $x'_n = \frac{1}{n}$, is a pointwise LP minimizing sequence at $x' = 1$ corresponding to any $t'_n \to t'$ but $x'_n \to 0 \ne x'$.

Let $\bar{x} \in X$, then we define the approximate solution set map, $S^e(\bar{x}, \cdot, \cdot) : \{\bar{x}\} \times T \times \mathbb{R}_+ \rightrightarrows X$ as

$$S^e(\bar{x}, t, \varepsilon) := \{x \in X : x \in M(t) + \mathcal{B}_X[\varepsilon], F(x,t) \le^l_K F(\bar{x}, t) + \varepsilon e\}.$$

We observe that if $t_n \to \bar{t}$ and $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x} \in l$-WMin$(F, \bar{t})$, then $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$, for every n where $\{\varepsilon_n\}$ is a sequence such that $\varepsilon_n \downarrow 0$.

We now give some properties of the map $S^e(\bar{x}, \cdot, \cdot)$.

Proposition 3.1. *Let $t \in T$ and $\bar{x} \in l$-WMin(F, t). Then the following conditions hold*:

(i) $\bar{x} \in S^e(\bar{x}, t, \varepsilon)$, *for every* $\varepsilon \ge 0$.
(ii) *If* $\varepsilon_1 \le \varepsilon_2$, *then* $S^e(\bar{x}, t, \varepsilon_1) \subseteq S^e(\bar{x}, t, \varepsilon_2)$.
(iii) $\bigcap_{\varepsilon > 0} S^e(\bar{x}, t, \varepsilon) = S^e(\bar{x}, t, 0)$, *if F is compact-valued on* $M(t) \times \{t\}$.
(iv) $\bigcup_{\bar{x} \in l\text{-WMin}(F,\, t)} S^e(\bar{x}, t, 0) = l$-WMin$(F, t)$.

Proof. (i) Let $\varepsilon \ge 0$. Then $F(\bar{x}, t) + \varepsilon e \subseteq F(\bar{x}, t) + K$, which implies $\bar{x} \in S^e(\bar{x}, t, \varepsilon)$.

(ii) Let $\varepsilon_1 \le \varepsilon_2$ and $x \in S^e(\bar{x}, t, \varepsilon_1)$, then $x \in M(t) + \mathcal{B}_X[\varepsilon_1]$ and $F(\bar{x}, t) + \varepsilon_1 e \subseteq F(x, t) + K$. Since $\varepsilon_1 \le \varepsilon_2$, thus $F(x, t) \le^l_K F(\bar{x}, t) + \varepsilon_2 e$ and $x \in M(t) + \mathcal{B}_X[\varepsilon_2]$.

(iii) Let $x \in \bigcap_{\varepsilon > 0} S^e(\bar{x}, t, \varepsilon)$. Then for every $\varepsilon > 0$, we have

$$x \in M(t) + \mathcal{B}_X[\varepsilon],\ F(\bar{x}, t) + \varepsilon e \subseteq F(x, t) + K. \tag{3.1}$$

As F is compact-valued on $M(t) \times \{t\}$, therefore, taking $\varepsilon \to 0$ in (3.1), we obtain

$$x \in M(t),\ F(\bar{x}, t) \subseteq F(x, t) + K,$$

which implies

$$x \in S^e(\bar{x}, t, 0).$$

Conversely, suppose $x \in S^e(\bar{x}, t, 0)$, then by (ii) it follows that $x \in S^e(\bar{x}, t, \varepsilon)$, for every $\varepsilon > 0$. Therefore, $x \in \bigcap_{\varepsilon > 0} S^e(\bar{x}, t, \varepsilon)$.

(iv) Let $\bar{x} \in l$-WMin(F, t), then $\bar{x} \in S^e(\bar{x}, t, 0)$. Therefore $\bar{x} \in \bigcup_{\bar{x} \in l\text{-WMin}(F,\, t)} S^e(\bar{x}, t, 0)$.

Conversely, let $x \in \bigcup_{\bar{x} \in l\text{-WMin}(F,\, t)} S^e(\bar{x}, t, 0)$. Then $x \in S^e(\bar{x}, t, 0)$, for some $\bar{x} \in l$-WMin(F, t), which implies

$$x \in M(t) \text{ and } F(x, t) \le^l_K F(\bar{x}, t). \tag{3.2}$$

Let $x_1 \in M(t)$ be such that $F(x_1,t) <^l_K F(x,t)$, then using (3.2), we have

$$F(\bar{x},t) \subseteq F(x,t) + K \subseteq F(x_1,t) + \text{int}K + K,$$

which implies

$$F(x_1,t) <^l_K F(\bar{x},t).$$

As $\bar{x} \in l\text{-WMin}(F,t)$, therefore $F(\bar{x},t) <^l_K F(x_1,t)$. Using (3.2), we obtain $F(x,t) <^l_K F(x_1,t)$ and hence $x \in l\text{-WMin}(F,t)$.

□

We now give sufficient conditions for the approximate solution set to be closed.

Theorem 3.1. *Let $t \in T$, $M(t)$ be a compact set and $\bar{x} \in l\text{-WMin}(F,t)$. If $F(\cdot,t)$ is upper semicontinuous and compact-valued on X, then for every $\varepsilon \geq 0$, $S^e(\bar{x},t,\varepsilon)$ is closed.*

Proof. Let $\varepsilon \geq 0$ and $\{x_n\}$ be a sequence such that $x_n \in S^e(\bar{x},t,\varepsilon)$, for every n and $x_n \to x'$. Since $x_n \in S^e(\bar{x},t,\varepsilon)$, therefore $x_n \in M(t) + \mathcal{B}_X[\varepsilon]$ and

$$F(x_n,t) \leq^l_K F(\bar{x},t) + \varepsilon e,$$

that is

$$F(\bar{x},t) + \varepsilon e \subseteq F(x_n,t) + K, \ \forall n.$$

For each $\bar{y} \in F(\bar{x},t)$, there exists $y_n \in F(x_n,t)$ such that

$$\bar{y} + \varepsilon e - y_n \in K. \tag{3.3}$$

Now $F(x',t)$ is compact and $F(\cdot,t)$ is upper semicontinuous at x'. Since $x_n \to x'$ and $y_n \in F(x_n,t)$, therefore there exist $y' \in F(x',t)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y'$. As K is closed, therefore (3.3) gives

$$\bar{y} + \varepsilon e - y' \in K,$$

which implies

$$F(x',t) \leq^l_K F(\bar{x},t) + \varepsilon e$$

and hence

$$x' \in S^e(\bar{x},t,\varepsilon).$$

□

We now present Dontchev−Zolezzi measure (Proposition 36, [7]) for pointwise LP well-posed problem.

Theorem 3.2. *Let $\bar{t} \in T$ and $\bar{x} \in l\text{-WMin}(F,\bar{t})$.*

(i) *If (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, then $S^e(\bar{x},\cdot,\cdot)$ is upper semicontinuous at $(\bar{t},0)$.*
(ii) *If $S^e(\bar{x},\bar{t},0) = \{\bar{x}\}$ and $S^e(\bar{x},\cdot,\cdot)$ is upper semicontinuous at $(\bar{t},0)$ then (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$.*

Proof. (i) Suppose on the contrary $S^e(\bar{x},\cdot,\cdot)$ is not upper semicontinuous at $(\bar{t},0)$. Then there exist an open set W containing $S^e(\bar{x},\bar{t},0)$ and sequences $t_n \to \bar{t}$ and $\varepsilon_n \downarrow 0$ such that

$$S^e(\bar{x},t_n,\varepsilon_n) \nsubseteq W. \tag{3.4}$$

Thus there exists a sequence $\{x_n\}$ such that $x_n \in S^e(\bar{x},t_n,\varepsilon_n)$ but $x_n \notin W$. Clearly, $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Since (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, therefore $x_n \to \bar{x} \in S^e(\bar{x},\bar{t},0) \subseteq W$, which contradicts (3.4).

(ii) Let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Then there exists $\varepsilon_n \downarrow 0$ such that $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$. As $S^e(\bar{x}, \cdot, \cdot)$ is upper semicontinuous at $(\bar{t}, 0)$, therefore for every neighbourhood V of 0, there exists $n_0 \in \mathbb{N}$ such that

$$S^e(\bar{x}, t_n, \varepsilon_n) \subseteq S^e(\bar{x}, \bar{t}, 0) + V, \text{ for every } n \geq n_0.$$

Since $S^e(\bar{x}, \bar{t}, 0) = \{\bar{x}\}$, we have $x_n \to \bar{x}$. Hence, (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$.

□

Remark 3.2. (i) The above result extends Proposition 2.1 of [34].

(ii) The condition $S^e(\bar{x}, \bar{t}, 0) = \{\bar{x}\}$ cannot be dropped in Theorem *3.2*(ii). In Example *3.1*, it can be seen that for $\bar{t} = 1$, l-WMin$(F, \bar{t}) = [-1, 2]$ and for $\bar{x} = 1$ and $e = (1, 1)$, $S^e(\bar{x}, \bar{t}, 0) = M(\bar{t}) = [-1, 2] \neq \{\bar{x}\}$ and $S^e(\bar{x}, \cdot, \cdot)$ is upper semicontinuous at $(\bar{t}, 0)$ but (P($\bar{t}$)) is not pointwise LP well-posed at $\bar{x}$.

We now give Furi−Vignoli measure (Page 21, [7]) for pointwise LP well-posed problem.

Theorem 3.3. *Let* $\bar{t} \in T$ *and* $\bar{x} \in l$-WMin$(F, \bar{t})$. *Then* (P($\bar{t}$)) *is pointwise LP well-posed at* $\bar{x}$ *if and only if* diam $S^e(\bar{x}, t, \varepsilon) \to 0$ *as* $(t, \varepsilon) \to (\bar{t}, 0)$.

Proof. Suppose (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$ and diam $S^e(\bar{x}, t, \varepsilon) \nrightarrow 0$ as $(t, \varepsilon) \to (\bar{t}, 0)$. Then there exist $t_n \to \bar{t}$, $\varepsilon_n \downarrow 0$ and $\delta > 0$ such that

$$\text{diam}\, S^e(\bar{x}, t_n, \varepsilon_n) > \delta, \ \forall\, n.$$

Thus, there exist u_n, $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$ such that

$$d(u_n, x_n) > \delta, \ \forall n. \tag{3.5}$$

Clearly, $\{u_n\}$ and $\{x_n\}$ are pointwise LP minimizing sequences at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Since (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, therefore $u_n \to \bar{x}$ and $x_n \to \bar{x}$. Therefore $d(u_n, x_n) \to 0$, which contradicts (3.5). Hence, diam $S^e(\bar{x}, t, \varepsilon) \to 0$ as $(t, \varepsilon) \to (\bar{t}, 0)$.

Conversely, let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Then there exist $\varepsilon_n \downarrow 0$ such that $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$, for every n. Also $\bar{x} \in S^e(\bar{x}, t_n, \varepsilon_n)$, for every n. If $x_n \nrightarrow \bar{x}$, then $\exists\, \delta > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, \bar{x}) \geq \delta, \text{ for every } k,$$

hence

$$\text{diam}\, S^e(\bar{x}, t_{n_k}, \varepsilon_{n_k}) \geq \delta > 0, \text{ for every } k,$$

which is a contradiction to the fact that diam $S^e(\bar{x}, t_n, \varepsilon_n) \to 0$ as $t_n \to \bar{t}$ and $\varepsilon_n \to 0$. Therefore $x_n \to \bar{x}$ and hence (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$. □

We again consider Example 3.1. It may be verified that $S^e(\bar{x}, t, \varepsilon) = [0, \varepsilon]$ for $\bar{x} = 0$, $\varepsilon > 0$ and $t \neq 1$. Therefore, diam $S^e(\bar{x}, t, \varepsilon) \to 0$ as $t \to 0$ and $\varepsilon \to 0$. Hence, by above theorem, (P(0)) is pointwise LP well-posed at $\bar{x} = 0$.

The next two theorems give necessary conditions for pointwise LP well-posedness.

Theorem 3.4. *If* $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F, \bar{t})$, (P($\bar{t}$)) *is pointwise LP well-posed at* $\bar{x}$ *and* F *is compact at* $(\bar{x}, \bar{t})$. *Then for any* $t_n \to \bar{t}$ *and any pointwise LP minimizing sequence* $\{x_n\}$ *at* $\bar{x}$ *for* (P($\bar{t}$)) *corresponding to* $\{t_n\}$, *we have*

$$ex(F(x_n, t_n), F(l\text{-WMin}(F, \bar{t}), \bar{t})) \to 0.$$

Proof. Let $t_n \in T$ be such that $t_n \to \bar{t}$ and $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x} \in l$-WMin$(F, \bar{t})$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Since (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, therefore $x_n \to \bar{x}$. On the contrary, suppose

$$ex(F(x_n, t_n), F(l\text{-WMin}(F, \bar{t}), \bar{t})) \nrightarrow 0.$$

Then there exists $\delta > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$F(x_{n_k}, t_{n_k}) \nsubseteq F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta), \ \forall\, k,$$

where $\mathcal{B}_Y(\delta)$ is open ball in Y with center at origin and radius δ. Thus for every k, there exists $v_{n_k} \in F(x_{n_k}, t_{n_k})$ such that

$$v_{n_k} \notin F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta)$$

which implies

$$v_{n_k} \in [F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta)]^c. \tag{3.6}$$

Now, $v_{n_k} \in F(x_{n_k}, t_{n_k})$ with $(x_{n_k}, t_{n_k}) \to (\bar{x}, \bar{t})$ and F is compact at $(\bar{x}, \bar{t})$, therefore there exists a subsequence $\{v_{n_{k_l}}\}$ of $\{v_{n_k}\}$ such that $v_{n_{k_l}} \to \bar{v} \in F(\bar{x}, \bar{t})$. Using (3.6), we have

$$\bar{v} \in [F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta)]^c,$$

which is a contradiction as $\bar{v} \in F(\bar{x}, \bar{t}) \subseteq F(l\text{-WMin}(F, \bar{t}), \bar{t})$. □

Next results give characterizations of pointwise LP well-posedness in terms of upper Hausdorff convergence of sequences of approximate solution sets.

Theorem 3.5. *If* $\bar{t} \in T, \bar{x} \in l$-WMin$(F, \bar{t})$ *and* (P($\bar{t}$)) *is pointwise LP well-posed at* $\bar{x}, \{t_n\}$ *is a sequence in* T *with* $t_n \to \bar{t}$ *and* $\{\varepsilon_n\}$ *is a sequence of real numbers such that* $\varepsilon_n \downarrow 0$, *then* $ex(S^e(\bar{x}, t_n, \varepsilon_n), S^e(\bar{x}, \bar{t}, 0)) \to 0$.

Proof. Let $\bar{t} \in T$ and $\bar{x} \in l$-WMin$(F, \bar{t})$. Let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n \downarrow 0$. If possible, suppose

$$ex(S^e(\bar{x}, t_n, \varepsilon_n), S^e(\bar{x}, \bar{t}, 0)) \nrightarrow 0.$$

Then there exists a $\delta > 0$ and subsequences $\{t_{n_k}\}$ of $\{t_n\}$ and $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$S^e(\bar{x}, t_{n_k}, \varepsilon_{n_k}) \nsubseteq S^e(\bar{x}, \bar{t}, 0) + \mathcal{B}_X(\delta), \ \forall\, k.$$

Then, for each k, there exists $x_{n_k} \in S^e(\bar{x}, t_{n_k}, \varepsilon_{n_k})$ such that

$$x_{n_k} \notin S^e(\bar{x}, \bar{t}, 0) + \mathcal{B}_X(\delta). \tag{3.7}$$

Clearly, $\{x_{n_k}\}$ is a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_{n_k}\}$. Thus $x_{n_k} \to \bar{x}$, which contradicts (3.7) as $\bar{x} \in S^e(\bar{x}, \bar{t}, 0)$. □

Theorem 3.6. *If* $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F, \bar{t}), S^e(\bar{x}, \bar{t}, 0)$ *is compact and for every sequence* $\{t_n\}$ *in* T *such that* $t_n \to \bar{t}$ *and every sequence of real numbers* $\{\varepsilon_n\}$ *such that* $\varepsilon_n \downarrow 0$,

$$ex(S^e(\bar{x}, t_n, \varepsilon_n), S^e(\bar{x}, \bar{t}, 0)) \to 0,$$

then $S^e(\bar{x}, \cdot, \cdot)$ *is upper semicontinuous at* $(\bar{t}, 0)$.

Proof. Let $\bar{t} \in T$ and $\bar{x} \in l$-WMin$(F, \bar{t})$). If $S^e(\bar{x}, \cdot, \cdot)$ is not upper semicontinuous at $(\bar{t}, 0)$, then there exist a $\delta > 0$, a sequence $\{t_n\}$ in T with $t_n \to \bar{t}$ and a sequence $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$ such that

$$S^e(\bar{x}, t_n, \varepsilon_n) \nsubseteq S^e(\bar{x}, \bar{t}, 0) + \mathcal{B}_X(\delta),\ \forall\, n,$$

which implies for each n, there exists $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$ such that

$$x_n \notin S^e(\bar{x}, \bar{t}, 0) + \mathcal{B}_X(\delta).$$

This is a contradiction to the fact that $ex(S^e(\bar{x}, t_n, \varepsilon_n), S^e(\bar{x}, \bar{t}, 0)) \to 0$. □

Using Theorem 3.2(ii) we have the following corollary.

Corollary 3.1. *If $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F, \bar{t})$, $S^e(\bar{x}, \bar{t}, 0) = \{\bar{x}\}$ and $ex(S^e(\bar{x}, t_n, \varepsilon_n),\ S^e(\bar{x}, \bar{t}, 0))$ $\to 0$, for every sequence $\{t_n\}$ in T with $t_n \to \bar{t}$ and every sequence of real numbers $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$, then* (P($\bar{t}$)) *is pointwise LP well-posed at $\bar{x}$.*

Next two theorems give characterizations of pointwise LP well-posedness in terms of the Painlevé−Kuratowski convergence of sequences of approximate solution sets.

Theorem 3.7. *Let $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F, \bar{t})$ and* (P($\bar{t}$)) *be pointwise LP well-posed at $\bar{x}$. If $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$. Then*

$$\text{Ls } S^e(\bar{x}, t_n, \varepsilon_n) \subseteq S^e(\bar{x}, \bar{t}, 0).$$

Proof. Let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$, $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and let $x \in$ Ls $S^e(\bar{x}, t_n, \varepsilon_n)$. Then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \in S^e(\bar{x}, t_{n_k}, \varepsilon_{n_k})$ and $x_{n_k} \to x$, where $\{t_{n_k}\}$ is a subsequence of $\{t_n\}$ and $\{\varepsilon_{n_k}\}$ is a subsequence of $\{\varepsilon_n\}$. Therefore, $\{x_{n_k}\}$ is a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Also, (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, therefore $x_{n_k} \to \bar{x}$, which together with the fact that $x_{n_k} \to x$ implies that $x \in S^e(\bar{x}, \bar{t}, 0)$. Hence

$$\text{Ls } S^e(\bar{x}, t_n, \varepsilon_n) \subseteq S^e(\bar{x}, \bar{t}, 0).$$

□

Theorem 3.8. *Let $\bar{t} \in T$, $\bar{x} \in l$-WMin$(F, \bar{t})$ and* (P($\bar{t}$)) *be pointwise LP well-posed at $\bar{x}$. If $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and $S^e(\bar{x}, \bar{t}, 0)$ is singleton set. Then*

$$S^e(\bar{x}, \bar{t}, 0) \subseteq \text{Li } S^e(\bar{x}, t_n, \varepsilon_n).$$

Proof. Since $S^e(\bar{x}, \bar{t}, 0)$ is singleton, therefore $S^e(\bar{x}, \bar{t}, 0) = \{\bar{x}\}$. Let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and let $x_n \in S^e(\bar{x}, t_n, \varepsilon_n)$, then $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$. Therefore $x_n \to \bar{x}$ and hence $S^e(\bar{x}, \bar{t}, 0) \subseteq$ Li $S^e(\bar{x}, t_n, \varepsilon_n)$. □

4. Global LP well-posedness

In this section we introduce a notion of global LP well-posedness for parametric set optimization problem (P(t)).

Definition 4.4. Let $\bar{t} \in T$ and $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$. A sequence $\{x_n\}$ in X is said to be a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$, if there exist $\varepsilon_n \downarrow 0$, $x_n \in M(t_n) + \mathcal{B}_X[\varepsilon_n]$, $u_n \in l$-WMin(F, t_n) such that

$$F(x_n, t_n) \leq^l_K F(u_n, t_n) + \varepsilon_n e,\ \forall\, n.$$

Definition 4.5. Let $\bar{t} \in T$. (P($\bar{t}$)) is said to be globally LP well-posed, if for any sequence $\{t_n\}$ in T converging to $\bar{t}$ and every global LP minimizing sequence $\{x_n\}$ corresponding to $\{t_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in l$-WMin$(F, \bar{t})$ such that $x_{n_k} \to \bar{x}$.

Remark 4.3. (i) Let $\bar{t} \in T$. If (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x} \in l$-WMin$(F, \bar{t})$ and l-WMin$(F, \bar{t})$ is singleton, then (P($\bar{t}$)) is globally LP well-posed.

(ii) Definition *4.4* and Definition *4.5* extend Definition *2.6*(iv) and Definition *2.7*(iv) of [16] respectively to the case of a parametric set optimization problem. In [16], the author introduced the notion of generalized Levitin−Polyak well-posedness for l-minimal solutions of the set optimization problem.

(iii) Definition *4.4* and Definition *4.5* also extend Definition *3.1* and Definition *3.2* of [3] respectively. In [3], authors defined the notion of LP well-posed vector optimization problem using weak efficient solutions of a vector optimization problem.

The following example illustrates Definition 4.5.

Example 4.2. Consider parametric set optimization problem, where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [0, 1]$, $e = (1, 1)$ and $M : T \rightrightarrows X$ is defined as

$$M(t) := [-1 - t, 1 + t]$$

and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x,t) := \begin{cases} [0,t] \times [0,t], & \text{if } 0 \le t < 1, \\ ]0,1] \times]0,1], & \text{if } x < 0, t = 1, \\ [0,1] \times [0,1], & \text{if } x \ge 0, t = 1. \end{cases}$$

Let $0 \le \bar{t} < 1$. Then l-WMin$(F, \bar{t}) = [-1 - \bar{t}, 1 + \bar{t}]$. Then (P($\bar{t}$)) is globally LP well-posed. Let $t' = 1$. Then l-WMin$(F, t') = [0, 2]$. Clearly, $x_n = -1 - \frac{1}{n}$ is a global LP minimizing sequence for (P(t')) and $x_n \to -1 \notin l$-WMin(F, t'). Hence, (P(t')) is not globally LP well-posed.

Example 4.3. Consider parametric set optimization problem, where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [0, 1]$, $e = (1, 1)$ and $M : T \rightrightarrows X$ is defined as

$$M(t) := [-1 - t, 1 + t]$$

and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x,t) := [0,1] \times [0,1], \ \forall\, x, t.$$

Clearly, l-WMin$(F, t) = [-1 - t, 1 + t]$ and (P(t)) is globally LP well-posed for any $t \in [0, 1]$ but (P(t)) is not pointwise LP well-posed at any $\bar{x} \in l$-WMin(F, t). For instance, $x_n = 1 + \frac{1}{n}$ is a pointwise LP minimizing sequence at $\bar{x} = 0$ but $x_n \not\to 0$.

We define the approximate solution set map $S^e(\cdot, \cdot) : T \times \mathbb{R}_+ \rightrightarrows X$ as

$S^e(t, \varepsilon) = \{x \in X : x \in M(t) + \mathcal{B}_X[\varepsilon]$ and $\exists\, \bar{x} \in l$-WMin(F, t) such that $F(x,t) \le^l_K F(\bar{x}, t) + \varepsilon e\}$.

We observe that if $\bar{t} \in T$, $\{t_n\} \subseteq T$ such that $t_n \to \bar{t}$ and $\{x_n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$ then $x_n \in S^e(t_n, \varepsilon_n)$.

We now give some properties of the map $S^e(\cdot, \cdot)$.

Proposition 4.2. *Let $t \in T$. The following statements are true*:

(i) $\displaystyle\bigcup_{\bar{x} \in l\text{-WMin}(F,\, t)} S^e(\bar{x}, t, \varepsilon) = S^e(t, \varepsilon), \ \forall\, \varepsilon \ge 0.$

(ii) l-WMin$(F, t) \subseteq S^e(t, \varepsilon), \ \forall\, \varepsilon \ge 0.$

(iii) $S^e(t,0) = l\text{-WMin}(F,t)$.

Proof. (i) Let $\bar{x} \in l\text{-WMin}(F,t)$ and let $x \in S^e(\bar{x},t,\varepsilon)$, then $x \in M(t) + \mathcal{B}_X[\varepsilon]$ and $F(x,t) \leq_K^l F(\bar{x},t) + \varepsilon e$. Thus, $x \in S^e(t,\varepsilon)$. Conversely, suppose $x \in S^e(t,\varepsilon)$, then $\exists\, \bar{x} \in l\text{-WMin}(F,t)$ such that $x \in M(t) + \mathcal{B}_X[\varepsilon]$ and $F(x,t) \leq_K^l F(\bar{x},t) + \varepsilon e$. Thus $x \in S^e(\bar{x},t,\varepsilon)$ and hence $x \in \bigcup_{\bar{x}\in l\text{-WMin}(F,t)} S^e(\bar{x},t,\varepsilon)$.

(ii) Let $\bar{x} \in l\text{-WMin}(F,t)$ then $\bar{x} \in S^e(t,\varepsilon), \forall\, \varepsilon \geq 0$.

(iii) Using (ii), we have $l\text{-WMin}(F,t) \subseteq S^e(t,0)$. Let $x \in S^e(t,0)$, then $x \in M(t)$ and there exists $\bar{x} \in l\text{-WMin}(F,t)$ such that

$$F(x,t) \leq_K^l F(\bar{x},t). \tag{4.8}$$

Let $x_1 \in M(t)$ be such that $F(x_1,t) <_K^l F(x,t)$. As $F(x,t) \subseteq F(x_1,t) + \text{int}K$, then by using (4.8), we have

$$F(\bar{x},t) \subseteq F(x,t) + K \subseteq F(x_1,t) + \text{int}K + K = F(x_1,t) + \text{int}K,$$

which implies

$$F(x_1,t) <_K^l F(\bar{x},t).$$

Now $\bar{x} \in l\text{-WMin}(F,t)$, so $F(\bar{x},t) \not<_K^l F(x_1,t)$. Using (4.8), we have $F(x,t) \not<_K^l F(x_1,t)$. Hence, $x \in l\text{-WMin}(F,t)$.

□

We now give Dontchev−Zolezzi measure [7] for globally LP well-posed problem.

Theorem 4.9. *Let $\bar{t} \in T$. (P($\bar{t}$)) is globally LP well-posed if and only if $S^e(\cdot,\cdot)$ is upper semicontinuous at $(\bar{t},0)$ and l-WMin$(F,\bar{t})$ is compact.*

Proof. Suppose $S^e(\cdot,\cdot)$ is not upper semicontinuous at $(\bar{t},0)$ then there exists an open set W containing $S^e(\bar{t},0)$ and sequences $t_n \to \bar{t}$ and $\varepsilon_n \downarrow 0$ such that

$$S^e(t_n,\varepsilon_n) \nsubseteq W. \tag{4.9}$$

Thus, there exists a sequence $\{x_n\}$ such that $x_n \in S^e(t_n,\varepsilon_n)$ and $x_n \notin W$. Clearly, $\{x_n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$. Since (P($\bar{t}$)) is globally LP well-posed, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x} \in l\text{-WMin}(F,\bar{t})$. Thus $\bar{x} \in l\text{-WMin}(F,\bar{t}) = S^e(\bar{t},0) \subseteq W$, which is a contradiction. Hence $S^e(\cdot,\cdot)$ is upper semicontinuous at $(\bar{t},0)$.

We now claim that l-WMin$(F,\bar{t})$ is compact. Let $\{u_n\}$ be a sequence in $S^e(\bar{t},0) = l\text{-WMin}(F,\bar{t})$, then $u_n \in M(t)$ and there exist $w_n \in l\text{-WMin}(F,\bar{t})$ such that

$$F(u_n,\bar{t}) \leq_K^l F(w_n,\bar{t}),$$

for every n. Thus, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ we have

$$F(u_n,\bar{t}) \leq_K^l F(w_n,\bar{t}) + \varepsilon_n e,$$

which implies that $u_n \in S^e(\bar{t},\varepsilon_n)$. Since (P($\bar{t}$)) is globally LP well-posed, therefore there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to \bar{u} \in l\text{-WMin}(F,\bar{t})$. Hence, l-WMin$(F,\bar{t})$ is compact.

Conversely, let $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and let $\{x_n\}$ be a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$, then there exist $\varepsilon_n \downarrow 0$ such that $x_n \in S^e(t_n,\varepsilon_n)$, for every n. Since $S^e(\cdot,\cdot)$ is upper semicontinuous at $(\bar{t},0)$, therefore for every neighbourhood V of 0, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in S^e(\bar{t},0) + V,\ \forall\, n \geq n_0,$$

which implies $d(x_n, S^e(\bar{t}, 0)) \to 0$. Since l-WMin$(F, \bar{t}) = S^e(\bar{t}, 0)$ is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x} \in l$-WMin$(F, \bar{t})$. Hence, (P($\bar{t}$)) is globally LP well-posed. □

We now give an example to show that the condition l-WMin$(F, \bar{t})$ is compact cannot be relaxed.

Example 4.4. Consider parametric set optimization problem where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [0, 1]$ and $M : T \rightrightarrows X$ is defined as $M(t) := \mathbb{R}$ and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x, t) := [0, t] \times [0, t].$$

Then l-WMin$(F, t) = \mathbb{R}$, for every $t \in T$. Let $\bar{t} \in T$ and $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$. Clearly, $S^e(\cdot, \cdot)$ is upper semicontinuous at $(t, \varepsilon) = (\bar{t}, 0)$ but l-WMin$(F, \bar{t})$ is not compact. Let $x_n = n$, then $\{x_n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$, but it has no convergent subsequence. Hence, (P($\bar{t}$)) is not globally LP well-posed.

The following theorem establishes relationship between pointwise LP well-posedness and global LP well-posedness.

Theorem 4.10. *If $\bar{t} \in T$ and l-WMin$(F, \bar{t})$ is compact. If $\bar{x} \in l$-WMin$(F, \bar{t})$ and $U_{\bar{x}}$ is a neighbourhood of $(\bar{t}, 0)$ then the graph of the family $\{U_{\bar{x}}\}_{\bar{x} \in l\text{-WMin}}$, that is*

$$\bigcup_{\bar{x} \in l\text{-WMin}} (U_{\bar{x}} \times \bar{x}) = \{(u, \bar{x}) : u \in U_{\bar{x}},\ \bar{x} \in l\text{-WMin}\} \tag{4.10}$$

is open and (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, for every $\bar{x} \in l$-WMin$(F, \bar{t})$ then (P($\bar{t}$)) is globally LP well-posed.

Proof. Let $\bar{t} \in T$. By Theorem 4.9, it is sufficient to show that $S^e(\cdot, \cdot)$ is upper semicontinuous at $(t, \varepsilon) = (\bar{t}, 0)$. Let V be a neighbourhood of $S^e(\bar{t}, 0)$. For any $\bar{x} \in l$-WMin$(F, \bar{t})$, we have $S^e(\bar{x}, \bar{t}, 0) \subseteq S^e(\bar{t}, 0) \subseteq V$. Since (P($\bar{t}$)) is pointwise LP well-posed at $\bar{x}$, then $S^e(\bar{x}, \cdot, \cdot)$ is upper semicontinuous at $(t, \varepsilon) = (\bar{t}, 0)$. Therefore there exists a neighbourhood $U_{\bar{x}}$ of $(\bar{t}$,0) such that $S^e(\bar{x}, t, \varepsilon) \subseteq V$, $\forall\, (t, \varepsilon) \in U_{\bar{x}}$. Let $U = \bigcap\{U_{\bar{x}} : \bar{x} \in l\text{-WMin}(F, \bar{t})\}$. Using Proposition 3 of [26], we have U is an open set hence, U is a neighbourhood of $(\bar{t}, 0)$ and

$$\bigcup_{\bar{x} \in l\text{-WMin}(F, \bar{t})} S^e(\bar{x}, t, \varepsilon) \subseteq V,\ \forall\, (t, \varepsilon) \in U.$$

Thus, there exists a neighbourhood U of $(\bar{t}, 0)$ such that

$$S^e(t, \varepsilon) \subseteq V,\ \forall\, (t, \varepsilon) \in U.$$

Hence, $S^e(\cdot, \cdot)$ is upper semicontinuous at $(t, \varepsilon) = (\bar{t}, 0)$. □

The set l-WMin(F, t) can be considered as a set-valued map l-WMin$(F, \cdot) : T \rightrightarrows X$ as

$$l\text{-WMin}(F, t) = \{\bar{x} \in M(t) : F(x, t) <^l_K F(\bar{x}, t), x \in M(t) \Rightarrow F(\bar{x}, t) <^l_K F(x, t)\}.$$

We now recall a result from [12], which gives upper semicontinuity of the set-valued map l-WMin$(F, \cdot)$.

Theorem 4.11. (*Theorem* 4.1, [12]) *Let $\bar{t} \in T$. Suppose*

(i) *$M(\bar{t})$ is compact and $M(\cdot)$ is continuous at $\bar{t}$,*
(ii) *F is continuous on $M(\bar{t}) \times \{\bar{t}\}$ with compact values.*

Then l-WMin$(F,\cdot)$ *is upper semicontinuous at* $\bar{t}$.

We now give sufficient conditions for globally LP well-posed parametric set optimization problem.

Theorem 4.12. *Let* $\bar{t}\in T$. *If* F *is continuous on* $M(\bar{t})\times\{\bar{t}\}$ *with compact values,* $M(\bar{t})$ *is compact and* $M(\cdot)$ *is continuous at* $\bar{t}$. *Then* (P($\bar{t}$)) *is globally LP well-posed.*

Proof. Let $\{t_n\}$ be a sequence in T such that $t_n\to\bar{t}$ and $\{x_n\}$ be a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$, then there exist $\varepsilon_n\downarrow 0$, $x_n\in M(t_n)+\mathcal{B}_X[\varepsilon_n]$, $u_n\in l$-WMin(F,t_n) such that

$$F(x_n,t_n)\leq^l_K F(u_n,t_n)+\varepsilon_n e,\ \forall\, n.$$

Using Theorem 4.11, we have l-WMin$(F,\cdot)$ is upper semicontinuous at $\bar{t}$. As $t_n\to\bar{t}$ and $u_n\in l$-WMin(F,t_n), therefore there exists $\bar{u}\in l$-WMin$(F,\bar{t})$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$$u_{n_m}\to\bar{u}\in l\text{-WMin}(F,\bar{t}).$$

Also, $x_n\in M(t_n)+\mathcal{B}_X[\varepsilon_n]$, hence there exists $x'_n\in M(t_n)$ such that

$$\|x_n-x'_n\|\leq\varepsilon_n. \tag{4.11}$$

Now $t_n\to\bar{t}$, $x'_n\in M(t_n)$ and $M(\cdot)$ is upper semicontinuous at $\bar{t}$, therefore there exists a subsequence $\{x'_{n_m}\}$ of $\{x'_n\}$ and $\bar{x}\in M(\bar{t})$ such that $x'_{n_m}\to\bar{x}\in M(\bar{t})$. Using (4.11), we have $x_{n_m}\to\bar{x}\in M(\bar{t})$.

We claim that $F(\bar{x},\bar{t})\leq^l_K F(\bar{u},\bar{t})$. Let $\bar{w}\in F(\bar{u},\bar{t})$. Since F is lower semicontinuous at $(\bar{u},\bar{t})$, therefore there exists $w_{n_m}\in F(u_{n_m},t_{n_m})$ such that $w_{n_m}\to\bar{w}$. Now

$$w_{n_m}\in F(u_{n_m},t_{n_m})\subseteq F(x_{n_m},t_{n_m})-\varepsilon_{n_m}e+K,$$

therefore there exists $v_{n_m}\in F(x_{n_m},t_{n_m})$ such that

$$w_{n_m}\in v_{n_m}-\varepsilon_{n_m}e+K. \tag{4.12}$$

Since F is upper semicontinuous at $(\bar{x},\bar{t})$, therefore without loss of generality there exists a subsequence $\{v_{n_m}\}$ of $\{v_{n_m}\}$ and $\bar{v}\in F(\bar{x},\bar{t})$ such that $v_{n_m}\to\bar{v}$. Taking limit, (4.12) gives $\bar{w}-\bar{v}\in K$, which implies $F(\bar{x},\bar{t})\leq^l_K F(\bar{u},\bar{t})$. Let $z\in M(\bar{t})$ be such that $F(z,\bar{t})<^l_K F(\bar{x},\bar{t})$, then

$$F(\bar{u},\bar{t})\subseteq F(\bar{x},\bar{t})+K\subseteq F(z,\bar{t})+\text{int}K+K,$$

which implies $F(z,\bar{t})<^l_K F(\bar{u},\bar{t})$. Also $\bar{u}\in l$-WMin$(F,\bar{t})$, so $F(\bar{u},\bar{t})<^l_K F(z,\bar{t})$, which leads to $F(\bar{x},\bar{t})<^l_K F(z,\bar{t})$. Therefore, $\bar{x}\in l$-WMin$(F,\bar{t})$ and hence (P($\bar{t}$)) is globally LP well-posed. □

We now give examples to show that conditions assumed in Theorem 4.12 cannot be relaxed.

Example 4.5. (i) (F is compact-valued on $M(\bar{t})\times\{\bar{t}\}$ cannot be dropped)

Consider parametric set optimization problem where $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $Z=\mathbb{R}$, $T=[0,1]$, $e=(1,1)$ and $M:T\rightrightarrows X$ is defined as $M(t):=[-1-t,1+t]$ and $F:X\times T\rightrightarrows Y$ is defined as

$$F(x,t):=\begin{cases}]0,1]\times]0,1], & \text{if } x<0, t=0,\\ [0,1]\times[0,1], & \text{if } x\geq 0, t=0,\\ [0,t]\times[0,t], & \text{if } t\neq 0.\end{cases}$$

Let $\bar{t}=0$. Then $M(0)=[-1,1]$ and l-WMin$(F,\bar{t})=[0,1]$. Clearly, $M(0)$ is compact, $M(\cdot)$ is continuous at $\bar{t}=0$, F is continuous on $M(0)\times\{0\}$ but F is not compact-valued on $M(0) \times \{0\}$. Let $t_n = \frac{1}{n}$, $\{\varepsilon_n\} = \{\frac{1}{n}\}$, and $x_n = -1 - \frac{1}{n}$, then $\{x_n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$. Now $x_n \to -1 \notin l$-WMin$(F, \bar{t})$. Therefore, (P($\bar{t}$)) is not globally LP well-posed.

(ii) ($M(\bar{t})$ is compact cannot be dropped)

Consider parametric set optimization problem where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [-1, 1]$, $e = (1, 1)$ and $M(t) := \mathbb{N}$, $\forall t \in T$ and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x,t) := \begin{cases} \{(t,t)\}, & \text{if } x = 1, \\ \{(t + \frac{1}{x}, t + \frac{1}{x})\}, & \text{if } x \in \mathbb{N} \text{ and } x \neq 1, \\ \{(2,2)\}, & \text{otherwise.} \end{cases}$$

Let $\bar{t} = 1$. Then l-WMin$(F, \bar{t}) = \{1\}$. Clearly, $M(\cdot)$ is continuous at $\bar{t}$, F is continuous on $M(\bar{t}) \times \{\bar{t}\}$ with compact values but $M(\bar{t})$ is not compact. Clearly, (P($\bar{t}$)) is not globally LP well-posed as for $\{\varepsilon_n\} = \{\frac{1}{n}\}$, $\{x_n\} = \{n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) for any $\{t_n\}$ in T such that $t_n \to \bar{t}$ but $\{x_n\}$ has no convergent subsequence.

(iii) (F is continuous on $M(\bar{t}) \times \{\bar{t}\}$ cannot be dropped)

Consider parametric set optimization problem where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $T = [0, 1]$, $e = (1, 1)$ and $M : T \rightrightarrows X$ is defined as $M(t) := [-1 + t, 2 - t]$ and $F : X \times T \rightrightarrows Y$ is defined as

$$F(x,t) := \begin{cases} [0,1] \times [0,1], & \text{if } -1 \leq x < 1, \\ [1,2] \times [1,2] & \text{if } 1 \leq x \leq 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly, l-WMin$(F, t) = [-1 + t, 1[$. Let $\bar{t} = 0$. Then $M(\cdot)$ is continuous at $\bar{t} = 0$ and $M(\bar{t})$ is compact, F is compact-valued on $M(\bar{t}) \times \{\bar{t}\}$ but F is not continuous at $x = 1 \in M(\bar{t}) \times \{\bar{t}\}$. Let $\{t_n\}$ be any sequence in T such that $t_n \to \bar{t}$, and let $x_n = 1 - \frac{1}{n}$. Let $\{\varepsilon_n\} = \{\frac{1}{n}\}$. Then $\{x_n\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$ and $x_n \to 1 \notin l$-WMin$(F, \bar{t})$. Therefore (P($\bar{t}$)) is not globally LP well-posed.

We now present necessary conditions for globally LP well-posed parametric set optimization problem.

Theorem 4.13. *If $\bar{t} \in T$, $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$, F is compact on l-WMin$(F, \bar{t})$ and (P($\bar{t}$)) is globally LP well-posed, then for any global LP minimizing sequence $\{x_n\}$ for (P($\bar{t}$)) corresponding to $\{t_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$ex(F(x_{n_k}, t_{n_k}), F(l\text{-WMin}(F, \bar{t}), \bar{t})) \to 0.$$

Proof. Let $\bar{t} \in T$, $\{t_n\}$ be a sequence in T such that $t_n \to \bar{t}$ and $\{x_n\}$ be a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$. Since (P($\bar{t}$)) is globally LP well-posed, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$, where $\bar{x} \in l$-WMin$(F, \bar{t})$. Suppose $ex(F(x_{n_k}, t_{n_k}), F(l\text{-WMin}(F, \bar{t}), \bar{t})) \nrightarrow 0$. Then there exists $\delta > 0$ and without loss of generality a subsequence $\{x_{n_k}\}$ of $\{x_{n_k}\}$ such that

$$F(x_{n_k}, t_{n_k}) \nsubseteq F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta),$$

for every k. Proceeding as in Theorem 3.4, we obtain there exists $\bar{v} \in F(\bar{x}, \bar{t})$ such that

$$\bar{v} \in [F(l\text{-WMin}(F, \bar{t}), \bar{t}) + \mathcal{B}_Y(\delta)]^c,$$

which is a contradiction as $\bar{v} \in F(\bar{x}, \bar{t}) \subseteq F(l\text{-WMin}(F, \bar{t}), \bar{t})$. □

Next results present characterizations of globally LP well-posedness in terms of the upper Hausdorff convergence of sequences of approximate solution sets. Lalitha and Chatterjee [3] and Khoshkhabar-amiranloo [16] gave similar characterizations of LP well-posedness for vector and set optimization problems respectively.

Theorem 4.14. *If $\bar{t} \in T$, $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$, $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and* (P($\bar{t}$)) *is globally LP well-posed, then*

$$ex(S^e(t_n, \varepsilon_n), S^e(\bar{t}, 0)) \to 0.$$

Proof. If possible, suppose $ex(S^e(t_n, \varepsilon_n), S^e(\bar{t}, 0)) \nrightarrow 0$. Then there exist a $\delta > 0$ and a subsequence $\{\varepsilon_{n_k}\}$ such that

$$S^e(t_{n_k}, \varepsilon_{n_k}) \nsubseteq S^e(\bar{t}, 0) + \mathcal{B}_X(\delta),$$

which implies for every k, there exists $x_{n_k} \in S^e(t_{n_k}, \varepsilon_{n_k})$ such that

$$x_{n_k} \notin S^e(\bar{t}, 0) + \mathcal{B}_X(\delta),\ \forall k. \tag{4.13}$$

Therefore, $\{x_{n_k}\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_{n_k}\}$. Thus, there exists a subsequence of $\{x_{n_k}\}$ converging to some element of l-WMin$(F, \bar{t})$. Without loss of generality, we assume $x_{n_k} \to x$, where $x \in l$-WMin$(F, \bar{t}) = S^e(\bar{t}, 0)$, which is a contradiction to (4.13). □

Theorem 4.15. *If $\bar{t} \in T$, $S^e(\bar{t}, 0)$ is compact and $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$, $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and $ex(S^e(t_n, \varepsilon_n), S^e(\bar{t}, 0)) \to 0$, then $S^e(\cdot, \cdot)$ is upper semicontinuous at $(\bar{t}, 0)$.*

Proof. Proof is similar to Theorem 3.6. □

Using Theorem 4.9 we have the following corollary.

Corollary 4.2. *If $\bar{t} \in T$, l-WMin$(F, \bar{t})$ is compact, $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$, $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$ and $ex(S^e(t_n, \varepsilon_n), S^e(\bar{t}, 0)) \to 0$, then* (P($\bar{t}$)) *is globally LP well-posed.*

Next two theorems give characterizations of globally LP well-posedness in terms of the Painlevé−Kuratowski convergence of sequences of approximate solution sets.

Theorem 4.16. *If $\bar{t} \in T$, $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$,* (P($\bar{t}$)) *is globally LP well-posed. Then*

$$\text{Ls } S^e(t_n, \varepsilon_n) \subseteq S^e(\bar{t}, 0).$$

Converse holds, if $M(\bar{t})$ is compact and $M(\cdot)$ is upper semicontinuous at $\bar{t}$.

Proof. Let $\bar{x} \in \text{Ls } S^e(t_n, \varepsilon_n)$. Then, there exist $x_{n_k} \in S^e(t_{n_k}, \varepsilon_{n_k})$ such that $x_{n_k} \to \bar{x}$, where $\{t_{n_k}\}$ and $\{\varepsilon_{n_k}\}$ are subsequences of $\{t_n\}$ and $\{\varepsilon_n\}$ respectively. Therefore, $\{x_{n_k}\}$ is a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_{n_k}\}$. Since (P($\bar{t}$)) is globally LP well-posed, therefore $\{x_{n_k}\}$ has a subsequence that converges to some element in l-WMin$(F, \bar{t})$. Also, $x_{n_k} \to \bar{x}$, therefore every subsequence of $\{x_{n_k}\}$ converges to $\bar{x}$. Hence, $\bar{x} \in S^e(\bar{t}, 0)$.

Conversely, let $\{x_n\}$ be a global LP minimizing sequence for (P($\bar{t}$)) corresponding to $\{t_n\}$. Then $x_n \in S^e(t_n, \varepsilon_n)$. Now $x_n \in M(t_n) + \mathcal{B}_X[\varepsilon_n]$ implies there exists $x'_n \in M(t_n)$ such that $\| x_n - x'_n \| \leq \varepsilon_n$. Since $M(\cdot)$ is upper semicontinuous at $\bar{t}$, therefore there exists subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ and $\bar{x} \in M(\bar{t})$ such that $x'_{n_k} \to \bar{x}$. Hence, $x_{n_k} \to \bar{x} \in M(\bar{t})$. Thus, $\bar{x} \in \text{Ls } S^e(t_{n_k}, \varepsilon_{n_k})$, which implies $\bar{x} \in S^e(\bar{t}, 0)$.

□

Theorem 4.17. *If $\bar{t} \in T$, $\{t_n\}$ is a sequence in T such that $t_n \to \bar{t}$ and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \downarrow 0$, $S^e(\bar{t}, 0)$ is singleton and* $(\mathrm{P}(\bar{t}))$ *is globally LP well-posed. Then*

$$S^e(\bar{t}, 0) \subseteq \mathrm{Li}\ S^e(t_n, \varepsilon_n).$$

Proof. Let $\bar{x} \in S^e(\bar{t}, 0)$ and $\{x_n\}$ be a sequence such that $x_n \in S^e(t_n, \varepsilon_n)$. Clearly, $\{x_n\}$ is a global LP minimizing sequence for $(\mathrm{P}(\bar{t}))$ corresponding to $\{t_n\}$. Since $(\mathrm{P}(\bar{t}))$ is globally LP well-posed, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in l\text{-WMin}(F, \bar{t}) = S^e(\bar{t}, 0)$. Since $S^e(\bar{t}, 0)$ is singleton, therefore $x_{n_k} \to \bar{x}$ and hence $\bar{x} \in \mathrm{Li}\ S^e(t_n, \varepsilon_n)$. □

Theorem 4.16 and Theorem 4.17 extend Theorem 3.4 of [16] to the case of parametric set optimization problem.

5. Conclusions

In this paper we introduce a pointwise and a global notion of Levitin−Polyak well-posedness for a parametric set optimization problem. These notions are characterized in terms of upper Hausdorff convergence and Painlevé−Kuratowski convergence of sequences of approximate solution sets.

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