CARPATHIAN J. MATH.
Volume **39** (2023), No. 3,
Pages 605 - 620

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.03.03

New results on the dynamic geometry generated by sequences of nested triangles

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ABSTRACT. Starting from an initial triangle, one may wish to check whether a sequence of iterations is convergent, or is convergent in some shape, and to find the limit. In this paper we first prove a general result for the convergence of a sequence of nested triangles (Theorem 2.2), then we study some properties of the power curve Γ of a triangle. These are used to prove that the sequence of nested triangles defined by a point $Q^{(s)}$ on the power curve converges to a point for every $s \in [0, 2]$ (Theorem 4.2). In particular, we obtain that the sequence of nested triangles defined by the incenter converges to a point, completing the main result in [14]. Finally, we present some numerical simulations which inspire open questions regarding the convergence of such iterations.

1. INTRODUCTION

Given a triangle $\mathcal{T}_0 = \Delta A_0B_0C_0$ in the Euclidean plane, a sequence of **nested triangles** $(\mathcal{T}_n)_{n\geq 0}$ is defined by the property that the vertices A_{n+1}, B_{n+1}, C_{n+1} are located on the segments (B_nC_n), (C_nA_n), (A_nB_n), respectively. For $n \geq 0$, the complex coordinates of the vertices A_n, B_n, C_n are denoted by a_n, b_n, c_n. The sequences of side lengths of triangle $A_nB_nC_n$ are denoted by α_n, β_n, γ_n, i.e., we have $B_nC_n = \alpha_n = |b_n - c_n|$, $C_nA_n = \beta_n = |c_n - a_n|$, $A_nB_n = \gamma_n = |a_n - b_n|$, as in [2]. Clearly, a_n, b_n and c_n are pairwise distinct.

Definition 1.1. The sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is **convergent** if the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ of vertices are convergent. If the limits are A, B and C, respectively, then we say that the limit of the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is the triangle ΔABC. If $A = B = C$, i.e., ΔABC is degenerated to a point, then we say that the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point.

Many sequences of nested triangles are defined by geometric elements associated such as the incircle, the circumcircle, the pedal triangle, the orthic triangle, the incentral triangle, the bisector triangle, or other triangles generated by remarkable points. Numerous such configurations have been investigated in the papers [1], [4], [6], [7], [17]. We kindly request you send us

Definition 1.2. The sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is **convergent in shape** if the sequences $(\widehat{A}_n)_{n\geq 0}$, $(\widehat{B}_n)_{n\geq 0}$, $(\widehat{C}_n)_{n\geq 0}$ are convergent, and the limits are not zero.

In particular, when

$$\lim_{n\to\infty} \widehat{A}_n = \lim_{n\to\infty} \widehat{B}_n = \lim_{n\to\infty} \widehat{C}_n = \frac{\pi}{3},$$

the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ **converges in the shape of an equilateral triangle**.

Received: 01.12.2022. In revised form: 09.05.2023. Accepted: 16.05.2023
2020 *Mathematics Subject Classification.* 51P99, 60A99.
Key words and phrases. *nested triangles, cevians, dynamic geometry, convergence, convergence in shape, complex coordinate, power curve.*
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Clearly, if the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a non-degenerated triangle ΔABC, then it is also convergent in shape of the triangle ΔABC. Therefore, convergence in shape is particularly interesting when the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point.

For the sequence defined by the feet of bisectors considered by Trimble [22], for which the iteration step is illustrated in Figure 5, Ismailescu and Jacobs showed that it converges in the shape of an equilateral triangle [14], without proving whether the limit was an equilateral triangle, or actually a point. The given proof was elementary, but laborious.

A classical result concerning the convergence to a point of the sequence defined by the feet of medians has been studied and extended by Kasner. Here the limit point of the sequence is the centroid of the initial triangle. As $\Delta A_{n+1}B_{n+1}C_{n+1}$ is similar to $\Delta A_nB_nC_n$ with the ratio $1/2$, the sequence is also convergent in the shape of the initial triangle.

Many such results have been extended to the dynamic geometry of polygons, as seen in the papers [8], [11], [12], [13], [21], or considering complex weights in [5].

In Section 2 we prove that a sequence of nested triangles converges to a point if and only if the sequence of perimeters converges to zero (Theorem 2.2). Section 3 is devoted to the study of the power curve of a triangle. This is used in Section 4 to prove that the sequence of triangles defined by the points $Q_n^{(s)}$ of coordinates $q_n^{(s)}$ given in formula (4.9) is convergent to a point for every $s \in [0,2]$ (Theorem 4.4). The points $Q_n^{(s)}$ are located on the power curve Γ_n of the triangle $A_nB_nC_n$, $n \geq 0$. As a corollary, we obtain that the limit of the nested triangles defined by the feet of bisectors is actually a point, completing the result proved in [14]. In Section 5 we provide an explicit formulation of the process described in Theorem 2.2, illustrating the complexity of the problem. Section 6 presents numerical simulations related to particular values of $s \in \mathbb{R}$, motivating the formulation of some open problems in Section 7.

2. A GENERAL CONVERGENCE RESULT

The following interesting result is proved in [8], in the more general case of simplexes.

Theorem 2.1. *Every interior point of a triangle is the limit of a sequence of nested triangles.*

While the property seems obvious, a rigorous proof requires advanced techniques. In the proof it is natural to consider the sequence of triangles generated by the cevians through the given point, followed by a fine analysis involving stochastic matrices [10].

A complementary problem is the following: given a sequence of nested triangles starting with the triangle $A_0B_0C_0$, find whether the sequence is convergent and determine its limit. The following result provides a necessary and sufficient condition for convergence to a point. For convenience we denote the perimeter of triangle $\mathcal{T}_n$ by $p_n = \alpha_n + \beta_n + \gamma_n$.

Theorem 2.2. *Let $(\Delta A_nB_nC_n)_{n\geq 0}$ be a sequence of nested triangles. Then $(\Delta A_nB_nC_n)_{n\geq 0}$ is convergent to a point if and only if the sequence of perimeters $(p_n)_{n\geq 0}$ converges to zero.*

Proof. Let us recall that the complex plane $(\mathbb{C}, |\cdot|)$ is a Banach space. The well known theorem of Cantor (Lemma 48.3, p. 297 in [20]) states that a metric space (X,d) is complete if and only if any sequence $(\mathcal{F}_n)_{n\geq 0}$ of closed nonempty subsets with $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$, $n \geq 0$, satisfying $\lim_{n\to\infty} \operatorname{diam}(\mathcal{F}_n) = 0$, has a nonempty intersection. In fact, the intersection is a singleton. Indeed, if two points x and y are in $\cap_{n\geq 0}\mathcal{F}_n$, then $0 \leq d(x,y) \leq \operatorname{diam}(\mathcal{F}_n) \to 0$, hence $d(x,y) = 0$, therefore $x = y$.

For the direct implication of our statement, let us consider the common limit

$$M = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n.$$

For every $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$ one has the relations $A_n M < \frac{\varepsilon}{6}$, $B_n M < \frac{\varepsilon}{6}$ and $C_n M < \frac{\varepsilon}{6}$. It follows that for all $n \geq n_0$ we get

$$\begin{aligned} 0 < p_n &= A_n B_n + B_n C_n + C_n A_n \\ &\leq A_n M + M B_n + B_n M + M C_n + C_n M + A B_n \leq \varepsilon, \end{aligned}$$

from where we deduce that $\lim_{n\to\infty} p_n = 0$.

For the converse implication, let us consider the sequence of sets $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n = \operatorname{conv}\{A_n, B_n, C_n\}$, for all $n \geq 0$. We shall prove the following:

(i) $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$. This statement clearly follows by the definition.

(ii) $\mathcal{F}_n$ is closed. By the definition of the convex hull, for x_1, x_2, $x_3 \in \mathbb{C}$ we have

$$\operatorname{conv}\{x_1, x_2, x_3\} = \{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \colon \lambda_1, \lambda_2, \lambda_3 \in [0,1],\ \lambda_1 + \lambda_2 + \lambda_3 = 1\},$$

that is $\operatorname{conv}\{x_1, x_2, x_3\}$ is the image of the standard simplex in $\mathbb{R}^3$ by the continuous function $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$.

(iii) $\lim_{n\to\infty} \operatorname{diam}(\mathcal{F}_n) = 0$. Indeed, we have

$$\operatorname{diam}(\mathcal{F}_n) \leq \max\{A_n B_n, B_n C_n, C_n A_n\} < p_n$$

and since $\lim_{n\to\infty} p_n = 0$, we deduce that $\lim_{n\to\infty} \operatorname{diam}(\mathcal{F}_n) = 0$.

Clearly, by Cantor's Theorem we deduce that $\cap_{n\geq 0} \mathcal{F}_n = \{M\}$. As A_n, B_n, $C_n \in \mathcal{F}_n$ we get $\lim_{n\to\infty} M A_n = \lim_{n\to\infty} M B_n = \lim_{n\to\infty} M C_n = 0$, therefore the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ are convergent to the same point M. □

Remark 2.1. The condition in Theorem 2.2 may not be fulfilled, even in the case of iterations defined by usual cevians. A striking example is that of the sequence of triangles defined by the feet of the altitudes of a triangle, proved to be chaotic by (see [17], [18], [19]).

3. THE POWER CURVE OF A TRIANGLE

In this section we review some properties of the power curve of a triangle and we establish some extremum properties of the points of this.

3.1. **The power curve.** In a triangle ABC, denote by a, b, c the complex coordinates of the vertices and by α, β, γ the side lengths of the triangle. For a real number s we consider the point $Q^{(s)}$ having the complex coordinate

$$q^{(s)} = \frac{\alpha^s a + \beta^s b + \gamma^s c}{\alpha^s + \beta^s + \gamma^s}. \tag{3.1}$$

The point $Q^{(s)}$ is situated in the interior of triangle ABC, having the barycentric coordinates $\left(\frac{\alpha^s}{\alpha^s+\beta^s+\gamma^s}, \frac{\beta^s}{\alpha^s+\beta^s+\gamma^s}, \frac{\gamma^s}{\alpha^s+\beta^s+\gamma^s}\right)$ and the trilinear coordinates

$$\frac{\alpha^{s-1}}{\alpha^s + \beta^s + \gamma^s} : \frac{\beta^{s-1}}{\alpha^s + \beta^s + \gamma^s} : \frac{\gamma^{s-1}}{\alpha^s + \beta^s + \gamma^s}. \tag{3.2}$$

It is known that for $s \in \mathbb{R}$ the points $Q^{(s)}$ describe the so-called **power curve** of the triangle, denoted by Γ (see [15]). If the lengths α, β, γ of the sides of the triangle are distinct, the curve Γ is tangent to the shortest and longest sides, in their opposite vertices. If the triangle ABC is isosceles, say $\beta = \gamma \neq \alpha$, then Γ is reduced to the median $[AA']$ from the vertex A. The points A and A' are obtained for $s \to \pm\infty$. When the triangle ABC is equilateral, we have $\alpha = \beta = \gamma$, hence Γ is reduced to the centroid G of the triangle.

Some remarkable points in triangle geometry are obtained for particular values of s. Specifically, $Q^{(0)} = G$ is the centroid, $Q^{(1)} = I$ is the incentre, while $Q^{(2)} = K$ represents the symmedian (or Lemoine) point of ΔABC, as illustrated in Figure 1. Moreover, for all real values s, $\left(Q^{(1+s)}, Q^{(1-s)}\right)$ is a pair of isogonal conjugate points [15].

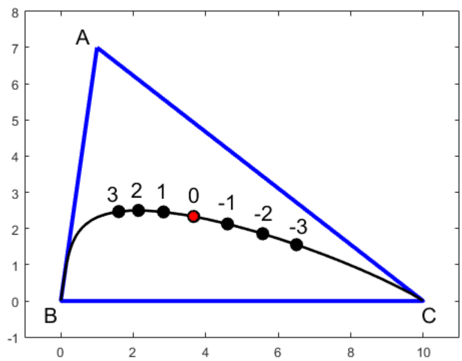

FIGURE 1. Selected points $Q^{(s)}$ on the power curve Γ, including centroid ($s = 0$), incentre ($s = 1$), and the symmedian point ($s = 2$).

In the current example plotted in Figure 1 we have $\gamma < \alpha < \beta$, and one can check that

$$\lim_{s\to-\infty} q^{(s)} = \lim_{s\to-\infty} \frac{\alpha^s a + \beta^s b + \gamma^s c}{\alpha^s + \beta^s + \gamma^s} = \lim_{s\to-\infty} \frac{\gamma^s \left(\left(\frac{\alpha}{\gamma}\right)^s a + \left(\frac{\beta}{\gamma}\right)^s b + c\right)}{\gamma^s \left(\left(\frac{\alpha}{\gamma}\right)^s + \left(\frac{\beta}{\gamma}\right)^s + 1\right)} = c,$$

$$\lim_{s\to\infty} q^{(s)} = \lim_{s\to\infty} \frac{\alpha^s a + \beta^s b + \gamma^s c}{\alpha^s + \beta^s + \gamma^s} = \lim_{s\to\infty} \frac{\beta^s \left(\left(\frac{\alpha}{\beta}\right)^s a + b + \left(\frac{\gamma}{\beta}\right)^s c\right)}{\beta^s \left(\left(\frac{\alpha}{\beta}\right)^s + 1 + \left(\frac{\gamma}{\beta}\right)^s\right)} = b.$$

3.2. **The function S_q.** For a point M situated in the interior or on the sides of triangle ABC, consider x, y, z the distances of M to the sides $[BC]$, $[CA]$ and $[AB]$, respectively. We consider the function $S_q : \mathrm{conv}\{A, B, C\} \to \mathbb{R}$, defined by $S_q(M) = x^q + y^q + z^q$, where $q \neq 0$ is a fixed number. Since $\mathrm{conv}\{A, B, C\}$ is compact and the function S_q is continuous, it follows that S_q is bounded and it attains its bounds. We aim to describe the extremum points of the function S_q. We first need a simple geometrical property.

Lemma 3.1. *Let d be a line and let U, V be distinct points in the plane, situated on the same half-plane defined by d. If the point $M \in [UV]$ satisfies $\frac{MV}{UV} = t$, then the the following relation holds:*

$$dist(M, d) = t \cdot dist(U, d) + (1 - t) \cdot dist(V, d).$$

This result helps us to understand the geometric nature of the function S_q.

Proposition 3.1. *(1) If $q \in (-\infty, 0) \cup (1, +\infty)$, then S_q is strictly convex.*
(2) If $q \in (0, 1)$, then S_q is strictly concave.

Proof. Let $U, V \in \mathrm{conv}\{A, B, C\}$ be two distinct points, and $M \in [U, V]$ such that $\frac{MV}{UV} = t$. According to the result in the previous Lemma we have

$$\mathrm{dist}(M, AB) = t \cdot \mathrm{dist}(U, AB) + (1 - t) \cdot \mathrm{dist}(V, AB).$$

(1) As the function $s \mapsto s^q$, $s > 0$, is strictly convex for $q \in (-\infty, 0) \cup (1, +\infty)$, for every $0 < t < 1$ we obtain the inequality

$$\mathrm{dist}^q(M, AB) < t \cdot \mathrm{dist}^q(U, AB) + (1-t) \cdot \mathrm{dist}^q(V, AB),$$

and other two similar inequalities corresponding to the sides BC and CA. Summing up these three inequalities it follows that for every $0 < t < 1$ we have

$$S_q(M) < tS_q(U) + (1-t)S_q(V),$$

that is S_q is strictly convex.

(2) The proof is analogous. □

By this theorem, it also follows that the function S_q has a unique minimum point if $q \in (-\infty, 0) \cup (1, +\infty)$, and a unique maximum point when $q \in (0, 1)$.

3.3. **Extremum properties of S_q.** If u, v, w, x, y, z are positive real numbers, then by the Hölder's inequality we have

$$ux + vy + wz \le (u^p + v^p + w^p)^{1/p} (x^q + y^q + z^q)^{1/q}, \tag{3.3}$$

for every real numbers p and q with $\frac{1}{p} + \frac{1}{q} = 1$ and $q \in (-\infty, 0) \cup (1, \infty)$. For $q \in (0, 1)$, the reverse inequality holds, that is

$$ux + vy + wz \ge (u^p + v^p + w^p)^{1/p} (x^q + y^q + z^q)^{1/q}. \tag{3.4}$$

Moreover, we have equality in (3.3) and (3.4) only when $\frac{x^q}{u^p} = \frac{y^q}{v^p} = \frac{z^q}{w^p}$.

Assume that the numbers u, v, w are fixed and x, y, z are positive real numbers such that $ux + vy + wz = K$ (constant). Then, from (3.3) and (3.4) it follows that

$$x^q + y^q + z^q \begin{cases} \ge \frac{K^q}{(u^p+v^p+w^p)^{q/p}} & \text{if } q \in (-\infty, 0) \cup (1, \infty) \\ \le \frac{K^q}{(u^p+v^p+w^p)^{q/p}} & \text{if } q \in (0, 1) \end{cases} \tag{3.5}$$

and $(x, y, z) = \left((hu^p)^{\frac{1}{q}}, (hv^p)^{\frac{1}{q}}, (hw^p)^{\frac{1}{q}}\right)$ is the minimum point of $x^q + y^q + z^q$ in the first case (and the maximum point in the second case), where we have $h = \frac{x^q}{u^p} = \frac{y^q}{v^p} = \frac{z^q}{w^p}$.

Consider now in (3.5) $u = \alpha$, $v = \beta$, $w = \gamma$. Clearly, for every point $M \in \mathrm{conv}\{A, B, C\}$ we have $\alpha x + \beta y + \gamma z = 2K[ABC]$ (constant), where $K[ABC]$ is the area of triangle ABC. Taking into account that the trilinear coordinates of a point are proportional to the distances of the point to the sides of the triangle, it follows that the point $Q^{(s)}$ is this unique extremum point of the function S_q if and only if

$$\frac{(\alpha^{s-1})^q}{\alpha^p} = \frac{(\beta^{s-1})^q}{\beta^p} = \frac{(\gamma^{s-1})^q}{\gamma^p}.$$

These relations are equivalent to $(s-1)q = p$, hence $s = \frac{p}{q} + 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, one deduces that $p = \frac{q}{q-1}$, therefore $s = \frac{q}{q-1}$.

Grouping the above results we obtain the following result.

Theorem 3.3. *(1) For every real number $q \notin \{0, 1\}$, the extremum point of S_q situated in the interior of the triangle ABC is located on the power curve Γ of ABC.*

(2) If $s \in (-\infty, 0) \cup (1, \infty)$, then the curve Γ contains the minimum points of function S_q.

(3) If $s \in (0, 1)$, then the curve Γ contains the maximum points of function S_q.

The function $S_1 : \mathrm{conv}\{A, B, C\} \to \mathbb{R}$, $S_1(M) = x + y + z$ is affine in the variables x, y and z, hence its extremum points are attained at two vertices of the triangle. From the relations $s = \frac{q}{q-1}$, $q \neq 1$, it follows that $\lim_{q \to 1, q > 1} s = \lim_{q \to 1, q > 1} \frac{q}{q-1} = +\infty$ and $\lim_{q \to 1, q < 1} s = \lim_{q \to 1, q < 1} \frac{q}{q-1} = -\infty$.

Hence, by Theorem 3.3, in the example in Figure 1 where $\gamma < \alpha < \beta$, the minimum point of S_1 is B, and the maximum point of S_1 is C. Therefore, in every triangle we have

$$h_{\min} \leq S_1(M) \leq h_{\max}, \tag{3.6}$$

where $h_{\min}$ and $h_{\max}$ are the shortest, and longest altitudes, respectively.

Indeed, one clearly has the relations

$$\beta h_{\min} = \gamma h_{\max} = \alpha x + \beta y + \gamma z = 2K[ABC].$$

Since

$$\gamma\left(x+y+z\right) \leq \alpha x + \beta y + \gamma z = \gamma h_{\max},$$

one deduces that $x+y+z \leq h_{\max}$, while from

$$\beta h_{\min} = \alpha x + \beta y + \gamma z \leq \beta\left(x+y+z\right),$$

we obtain $h_{\min} \leq x+y+z$. The argument is similar for other orderings of α, β, γ.

We present in various contexts the explicit formula for $S_q(M)$, where M is an interior point of ΔABC. We have

$$x = \frac{2K[MBC]}{\alpha}, \quad y = \frac{2K[MCA]}{\beta}, \quad z = \frac{2K[MAB]}{\gamma}. \tag{3.7}$$

3.4. **The explicit formula for S_q in terms of distances to the sides.** One can express the function S_q in terms of x and y only, using the relation $\alpha x + \beta y + \gamma z = 2K$, where K denotes the area of triangle ABC. We have

$$S_q(M) = x^q + y^q + \left(\frac{2K - \alpha x - \beta y}{\gamma}\right)^q.$$

Clearly, we get $0 \leq x \leq h_A = \frac{2K}{\alpha}$, and $0 \leq y \leq \left(1 - \frac{1}{h_A}x\right) h_B = \left(1 - \frac{\alpha}{2K}x\right)\frac{2K}{\beta}$, where we have denoted by h_A, h_B, h_C the altitudes from A, B and C, respectively.

3.5. **The explicit formula for S_q in complex coordinates.** Assuming that the triangle ABC is positively oriented, then the triangles MBC, MCA and MAB are also positively oriented, and by the formula for the area of a triangle determined by three points we get

$$K[MBC] = \frac{i}{4}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}, \quad K[MCA] = \frac{i}{4}\begin{vmatrix} m & \overline{m} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix}, \quad K[MAB] = \frac{i}{4}\begin{vmatrix} m & \overline{m} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}.$$

By (3.7) we get an explicit formula for $S_q(M)$ in terms of complex coordinates a, b, c, m.

$$S_q(M) = \left(\frac{i}{2\alpha}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}\right)^q + \left(\frac{i}{2\beta}\begin{vmatrix} m & \overline{m} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix}\right)^q + \left(\frac{i}{2\gamma}\begin{vmatrix} m & \overline{m} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}\right)^q.$$

Considering the vertices of ΔABC of complex coordinates $A(1+7i)$, $B(0)$ and $C(10)$, we plot configurations corresponding to the cases: strictly convex ($q = 0.5$), affine ($q = 1$) and strictly concave ($q = 1.3$). The properties in Theorem 3.3 are shown in Figures 2, 3, 4.

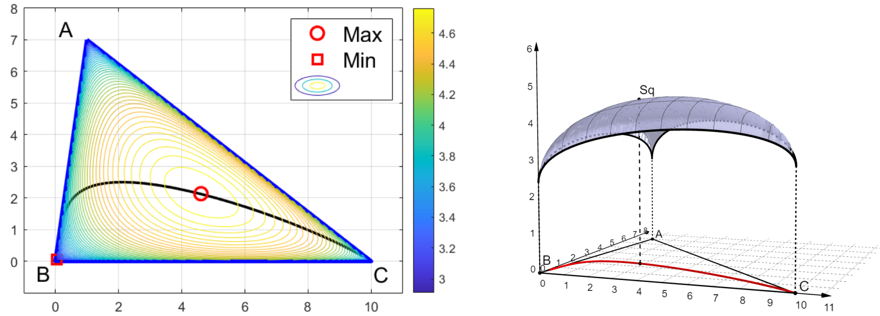

FIGURE 2. Contour and surface plots for S_q obtained for $q = 0.5$.

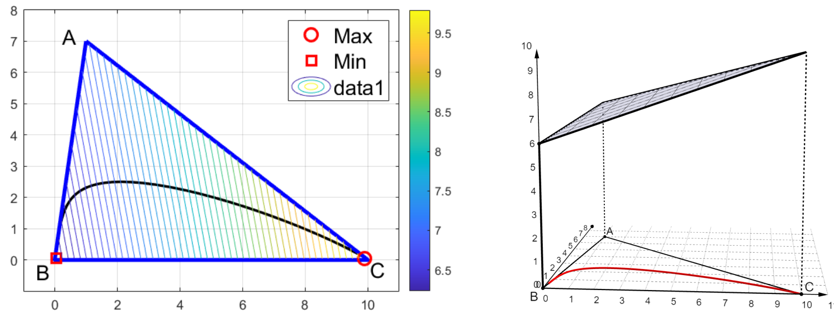

FIGURE 3. Contour and surface plots for S_q obtained for $q = 1$.

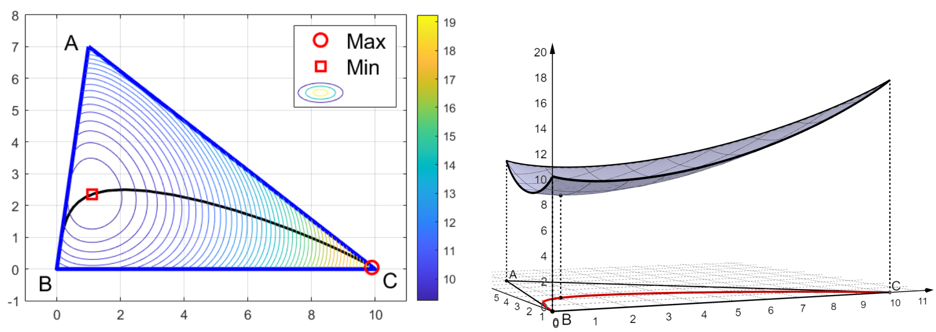

FIGURE 4. Contour and surface plots for S_q obtained for $q = 1.3$.

3.6. **The formula for S_q in cartesian coordinates.** Let us consider the plane of the triangle ABC endowed with the natural induced cartesian coordinate system. In this case we have $A(a', a'')$, $B(b', b'')$, $C(c', c'')$ and $M(u, v)$, and we assume again that the triangle ABC is positively oriented, with its area given by the well-known determinant formula

$$K[ABC] = \frac{1}{2}\begin{vmatrix} a' & a'' & 1 \\ b' & b'' & 1 \\ c' & c'' & 1 \end{vmatrix}.$$

Since the triangles MBC, MCA and MAB are also positively oriented, we have

$$K[MBC]=\frac{1}{2}\begin{vmatrix}u & v & 1\\ b' & b'' & 1\\ c' & c'' & 1\end{vmatrix},\quad K[MCA]=\frac{1}{2}\begin{vmatrix}u & v & 1\\ c' & c'' & 1\\ a' & a'' & 1\end{vmatrix},\quad K[MAB]=\frac{1}{2}\begin{vmatrix}u & v & 1\\ a' & a'' & 1\\ b' & b'' & 1\end{vmatrix}.$$

By (3.7) we obtain the following formula

$$S_q(M)=S_q(u,v)=\frac{1}{\alpha^q}\begin{vmatrix}u & v & 1\\ b' & b'' & 1\\ c' & c'' & 1\end{vmatrix}^q+\frac{1}{\beta^q}\begin{vmatrix}u & v & 1\\ c' & c'' & 1\\ a' & a'' & 1\end{vmatrix}^q+\frac{1}{\gamma^q}\begin{vmatrix}u & v & 1\\ a' & a'' & 1\\ b' & b'' & 1\end{vmatrix}^q.$$

When $q\in(0,1)$ or $q\in(-\infty,0)\cup(1,\infty)$, the unique extremum point of S_q in the interior of the triangle satisfies the system $\frac{\partial S_q}{\partial u}=0$ and $\frac{\partial S_q}{\partial v}=0$. This is equivalent to the fact that it satisfies the system

$$\begin{cases}\frac{b''-c''}{\alpha^q}\begin{vmatrix}u & v & 1\\ b' & b'' & 1\\ c' & c'' & 1\end{vmatrix}^{q-1}+\frac{c''-a''}{\beta^q}\begin{vmatrix}u & v & 1\\ c' & c'' & 1\\ a' & a'' & 1\end{vmatrix}^{q-1}+\frac{a''-b''}{\gamma^q}\begin{vmatrix}u & v & 1\\ a' & a'' & 1\\ b' & b'' & 1\end{vmatrix}^{q-1}=0\\ \frac{c'-b'}{\alpha^q}\begin{vmatrix}u & v & 1\\ b' & b'' & 1\\ c' & c'' & 1\end{vmatrix}^{q-1}+\frac{a'-c'}{\beta^q}\begin{vmatrix}u & v & 1\\ c' & c'' & 1\\ a' & a'' & 1\end{vmatrix}^{q-1}+\frac{b'-a'}{\gamma^q}\begin{vmatrix}u & v & 1\\ a' & a'' & 1\\ b' & b'' & 1\end{vmatrix}^{q-1}=0.\end{cases}$$

Moreover, by Theorem 3.3, we know that this point is located on the power curve Γ.

4. Nested triangles defined by points on the power curve

We first prove a useful auxiliary result.

Lemma 4.2. *Let ABC be a triangle and let $M\in(AB)$ and $N\in(AC)$ be points satisfying the relations $\frac{AM}{AB}=x$ and $\frac{AN}{AC}=y$. The following formula holds*

$$MN^2=xy\alpha^2-(x-y)\left(y\beta^2-x\gamma^2\right), \tag{4.8}$$

where α, β, γ are the lengths of the sides (BC), (CA), (AB) of triangle ABC.

Proof. From the Law of Cosines in the triangle ABC we get $\cos A=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}$, which applied in the triangle AMN gives the relation

$$\begin{aligned}MN^2&=y^2\beta^2+x^2\gamma^2-xy\left(\beta^2+\gamma^2-\alpha^2\right)\\ &=xy\alpha^2+y^2\beta^2+x^2\gamma^2-xy\left(\beta^2+\gamma^2\right)\\ &=xy\alpha^2-(x-y)\left(y\beta^2-x\gamma^2\right).\end{aligned}$$

□

Let $(\Delta A_nB_nC_n)_{n\geq 0}$ be the sequence of triangles defined as follows: for each positive integer n and each real number s consider the point $Q_n^{(s)}$ situated on the power curve Γ_n of the triangle $A_nB_nC_n$, with the complex coordinate

$$q_n^{(s)}=\frac{\alpha_n^s a_n+\beta_n^s b_n+\gamma_n^s c_n}{\alpha_n^s+\beta_n^s+\gamma_n^s}. \tag{4.9}$$

The points A_{n+1}, B_{n+1} and C_{n+1} are defined recursively by the intersections $(B_nC_n)\cap A_nQ_n^{(s)}$, $(C_nA_n)\cap B_nQ_n^{(s)}$, and $(A_nB_n)\cap C_nQ_n^{(s)}$, respectively. The following result holds for selected points on the power curve.

Theorem 4.4. *If $s\in[0,2]$, then the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is convergent to a point.*

Proof. For a fixed $n \geq 0$, by formula (4.9), the feet C_{n+1} and B_{n+1} of the cevians through $Q_n^{(s)}$ divide the segments (A_nB_n) and (A_nC_n) in the ratios

$$\frac{A_nC_{n+1}}{A_nB_n} = \frac{\beta_n^s}{\beta_n^s + \alpha_n^s}, \quad \frac{A_nB_{n+1}}{A_nC_n} = \frac{\gamma_n^s}{\gamma_n^s + \alpha_n^s}. \tag{4.10}$$

Using formula (4.8) in Lemma 4.2 we obtain

$$\begin{aligned}
\alpha_{n+1}^2 &= B_{n+1}C_{n+1}^2 \\
&= \frac{\beta_n^s}{\beta_n^s + \alpha_n^s}\frac{\gamma_n^s}{\gamma_n^s + \alpha_n^s}\alpha_n^2 - \left(\frac{\beta_n^s}{\beta_n^s + \alpha_n^s} - \frac{\gamma_n^s}{\gamma_n^s + \alpha_n^s}\right)\left(\frac{\gamma_n^s}{\gamma_n^s + \alpha_n^s}\beta_n^2 - \frac{\beta_n^s}{\beta_n^s + \alpha_n^s}\gamma_n^2\right) \\
&= \frac{\beta_n^s\gamma_n^s\alpha_n^2}{(\beta_n^s + \alpha_n^s)(\gamma_n^s + \alpha_n^s)} - \frac{\alpha_n^s(\beta_n^s - \gamma_n^s)}{(\beta_n^s + \alpha_n^s)(\gamma_n^s + \alpha_n^s)}\left(\frac{\gamma_n^s\beta_n^2}{\gamma_n^s + \alpha_n^s} - \frac{\beta_n^s\gamma_n^2}{\beta_n^s + \alpha_n^s}\right) \\
&= \frac{\beta_n^s\gamma_n^s\alpha_n^2}{(\beta_n^s + \alpha_n^s)(\gamma_n^s + \alpha_n^s)} \\
&\quad - \frac{\alpha_n^s(\beta_n^s - \gamma_n^s)\left[\beta_n^s\gamma_n^s\left(\beta_n^2 - \gamma_n^2\right) + \alpha_n^s\beta_n^s\gamma_n^s\left(\beta_n^{2-s} - \gamma_n^{2-s}\right)\right]}{(\beta_n^s + \alpha_n^s)^2(\gamma_n^s + \alpha_n^s)^2}.
\end{aligned}$$

The inequalities $(\beta_n^s - \gamma_n^s)\left(\beta_n^2 - \gamma_n^2\right) \geq 0$ and $(\beta_n^s - \gamma_n^s)\left(\beta_n^{2-s} - \gamma_n^{2-s}\right) \geq 0$ hold for every $0 \leq s \leq 2$, therefore we deduce that

$$\alpha_{n+1}^2 \leq \frac{\alpha_n^2\beta_n^s\gamma_n^s}{(\beta_n^s + \alpha_n^s)(\gamma_n^s + \alpha_n^s)} \leq \frac{\alpha_n^2\beta_n^s\gamma_n^s}{2\sqrt{\beta_n^s\alpha_n^s}\cdot 2\sqrt{\gamma_n^s\alpha_n^s}} = \frac{1}{4}\alpha_n^{2-s}\beta_n^{\frac{s}{2}}\gamma_n^{\frac{s}{2}}. \tag{4.11}$$

Finally, by the weighted AM-GM inequality, we conclude that

$$\alpha_{n+1} \leq \frac{1}{2}\alpha_n^{1-\frac{s}{2}}\beta_n^{\frac{s}{4}}\gamma_n^{\frac{s}{4}} \leq \frac{1}{2}\left[\left(1 - \frac{s}{2}\right)\alpha_n + \frac{s}{4}\beta_n + \frac{s}{4}\gamma_n\right].$$

Note that here it was essential that the weights satisfy $1 - \frac{s}{2} \geq 0$ and $\frac{s}{4} \geq 0$, i.e., $0 \leq s \leq 2$.

Repeating the argument for $\beta_{n+1} = C_{n+1}A_{n+1}$ and $\gamma_{n+1} = A_{n+1}B_{n+1}$ we obtain

$$\begin{aligned}
\alpha_{n+1} &\leq \frac{1}{2}\left[\left(1 - \frac{s}{2}\right)\alpha_n + \frac{s}{4}\beta_n + \frac{s}{4}\gamma_n\right] \\
\beta_{n+1} &\leq \frac{1}{2}\left[\left(1 - \frac{s}{2}\right)\beta_n + \frac{s}{4}\gamma_n + \frac{s}{4}\alpha_n\right] \\
\gamma_{n+1} &\leq \frac{1}{2}\left[\left(1 - \frac{s}{2}\right)\gamma_n + \frac{s}{4}\alpha_n + \frac{s}{4}\beta_n\right].
\end{aligned}$$

By summing down the above inequalities it follows that

$$p_{n+1} \leq \frac{1}{2}(\alpha_n + \beta_n + \gamma_n) = \frac{1}{2}p_n,$$

hence $p_{n+1} \leq \frac{1}{2^{n+1}}p_0$, so $\lim_{n\to\infty} p_n = 0$. The conclusion follows by Theorem 2.2. □

Since the point $Q_n^{(s)}$ is interior to the triangle $A_nB_nC_n$, $n \geq 0$, the following result holds.

Corollary 4.1. *If $s \in [0, 2]$, then the limit of the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ in the above theorem is equal to the limit of the sequence $(Q_n^{(s)})_{n\geq 0}$.*

Corollary 4.2. *For $s = 0$ in formula (3.1), we get the centroid G_0 with the complex coordinate $\frac{a_0+b_0+c_0}{3}$. The sequence $(\mathcal{T}_n)_{n\geq 0}$, where $\mathcal{T}_{n+1}$ is the median triangle of $\mathcal{T}_n$, converges to a point.*

In this case the sequence of perimeters verifies $p_{n+1} = \frac{1}{2}p_n$, hence $p_n = \frac{1}{2^n}p_0 \to 0$. The limit is G_0 since this point is interior to every triangle $A_nB_nC_n$, $n = 0, 1, \ldots$. This sequence is a special case of the so-called Kasner triangles studied in [6].

Corollary 4.3. *For $s = 1$ in* (3.1) *we get the incenter I_0 of triangle $A_0B_0C_0$, having the coordinate $\frac{\alpha_0 a_0+\beta_0 b_0+\gamma_0 c_0}{\alpha_0+\beta_0+\gamma_0}$. The sequence of triangles $(\mathcal{T}_n)_{n\geq 0}$, where $\mathcal{T}_{n+1}$ is defined by the feet of the internal bisectors of $\mathcal{T}_n$, converges to a point.*

The result in Corollary 4.3 completes the work in [14], where it was proved that the sequence $(\mathcal{T}_n)_{n\geq 0}$ converges in the shape of an equilateral triangle.

Corollary 4.4. *For $s = 2$ in* (3.1) *one gets the symmedian point K_0 of triangle $A_0B_0C_0$ having the coordinate $\frac{\alpha_0^2 a_0+\beta_0^2 b_0+\gamma_0^2 c_0}{\alpha_0^2+\beta_0^2+\gamma_0^2}$. The sequence $(\mathcal{T}_n)_{n\geq 0}$, where $\mathcal{T}_{n+1}$ is defined by the feet of the symmedians of $\mathcal{T}_n$ converges to a point.*

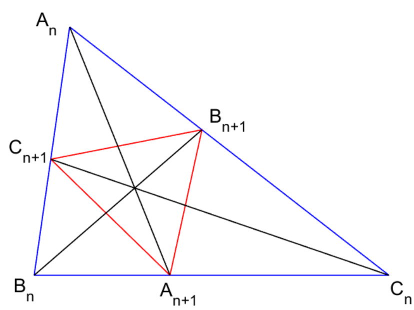

FIGURE 5. Triangles $\mathcal{T}_n = \Delta A_nB_nC_n$ and $\mathcal{T}_{n+1} = \Delta A_{n+1}B_{n+1}C_{n+1}$.

Remark 4.2. Denote by $K_n = K[A_nB_nC_n]$ the area of $\Delta A_nB_nC_n$, $n \geq 0$ and by Routh's theorem [2, p. 119] or [9, p. 211], we prove that the sequence $(K_n)_{n\geq 0}$ converges to zero.

Indeed, we have $K_n = \frac{1}{2}\beta_n\gamma_n \sin A_n = \frac{1}{2}\gamma_n\alpha_n \sin B_n = \frac{1}{2}\alpha_n\beta_n \sin C_n$. Using the notations in Figure 5 and the relations of type (4.10), we get

$$\frac{K[A_nC_{n+1}B_{n+1}]}{K[A_nB_nC_n]} = \frac{A_nB_{n+1}\cdot A_nC_{n+1}}{A_nC_n\cdot A_nB_n} = \frac{\gamma_n^s}{\alpha_n^s+\gamma_n^s}\cdot\frac{\beta_n^s}{\alpha_n^s+\beta_n^s},$$
$$\frac{K[B_nA_{n+1}C_{n+1}]}{K[A_nB_nC_n]} = \frac{B_nC_{n+1}\cdot B_nA_{n+1}}{B_nA_n\cdot B_nC_n} = \frac{\alpha_n^s}{\alpha_n^s+\beta_n^s}\cdot\frac{\gamma_n^s}{\gamma_n^s+\beta_n^s},$$
$$\frac{K[C_nA_{n+1}B_{n+1}]}{K[A_nB_nC_n]} = \frac{C_nA_{n+1}\cdot C_nB_{n+1}}{C_nB_n\cdot C_nA_n} = \frac{\beta_n^s}{\beta_n^s+\gamma_n^s}\cdot\frac{\alpha_n^s}{\alpha_n^s+\gamma_n^s}.$$

Therefore

$$\frac{K_{n+1}}{K_n} = 1-\frac{K[A_nC_{n+1}B_{n+1}]+K[B_nA_{n+1}C_{n+1}]+K[C_nA_{n+1}B_{n+1}]}{K[A_nB_nC_n]}$$
$$= \frac{2\alpha_n^s\beta_n^s\gamma_n^s}{(\alpha_n^s+\beta_n^s)(\beta_n^s+\gamma_n^s)(\gamma_n^s+\alpha_n^s)} \leq \frac{1}{4},$$

where we used that for any positive real numbers x, y and z one has

$$\frac{xyz}{(x+y)(y+z)(z+y)} \leq \frac{1}{8}.$$

Hence $K_n \leq \left(\frac{1}{4}\right)^n K_0$, $n \geq 0$, i.e., the sequence $(K_n)_{n\geq 0}$ converges to zero. However, we cannot infer that the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point.

Remark 4.3. If the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is convergent in shape, then by Remark 4.2, we can conclude that the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point.

5. Explicit formulation of the process in Theorem 4.4

For a fixed real number s, the sequences of complex coordinates $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ in Theorem 4.4, satisfy the recursive system:

$$\text{(5.12)}\qquad \begin{cases} a_{n+1} = \frac{|a_n-b_n|^s}{|c_n-a_n|^s+|a_n-b_n|^s}b_n + \frac{|c_n-a_n|^s}{|c_n-a_n|^s+|a_n-b_n|^s}c_n \\ b_{n+1} = \frac{|b_n-c_n|^s}{|a_n-b_n|^s+|b_n-c_n|^s}c_n + \frac{|a_n-b_n|^s}{|a_n-b_n|^s+|b_n-c_n|^s}a_n \\ c_{n+1} = \frac{|c_n-a_n|^s}{|b_n-c_n|^s+|c_n-a_n|^s}a_n + \frac{|b_n-c_n|^s}{|b_n-c_n|^s+|c_n-a_n|^s}b_n, \quad n\geq 0. \end{cases}$$

The system (5.12) is not a linearly recursive system (whose theory is well-known, see [4]) and the complexity of this form suggests that there is little chance to obtain results by analytic investigations. The matrix form of the system is

$$\text{(5.13)}\qquad X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\gamma_n^s}{\beta_n^s+\gamma_n^s} & \frac{\beta_n^s}{\beta_n^s+\gamma_n^s} \\ \frac{\gamma_n^s}{\gamma_n^s+\alpha_n^s} & 0 & \frac{\alpha_n^s}{\gamma_n^s+\alpha_n^s} \\ \frac{\beta_n^s}{\alpha_n^s+\beta_n^s} & \frac{\alpha_n^s}{\alpha_n^s+\beta_n^s} & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = T_n X_n, \quad n\geq 0,$$

where the matrix T_n is row-stochastic, $n \geq 0$, while α_n, β_n and γ_n are given by

$$\alpha_n = |b_n - c_n|, \quad \beta_n = |c_n - a_n|, \quad \gamma_n = |a_n - b_n|.$$

In this notation one can write

$$\text{(5.14)}\qquad X_{n+1} = (T_n T_{n-1} \cdots T_1 T_0) X_0.$$

By Theorem 4.4 it follows that for any complex numbers a_0, b_0, c_0 which are pairwise distinct and for any $s \in [0,2]$, the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ recursively defined by the system (5.12) are convergent to the same point x_s^*. Moreover, the point x_s^* can be written uniquely in terms of barycentric coordinates with respect to the vertices as $x_s^* = t_A a_0 + t_B b_0 + t_C c_0$, where $0 \leq t_A, t_B, t_C$ and $t_A + t_B + t_C = 1$. Considering the limit in the relation (5.14) one obtains

$$\prod_{n=0}^{\infty} \begin{pmatrix} 0 & \frac{\gamma_n^s}{\beta_n^s+\gamma_n^s} & \frac{\beta_n^s}{\beta_n^s+\gamma_n^s} \\ \frac{\gamma_n^s}{\gamma_n^s+\alpha_n^s} & 0 & \frac{\alpha_n^s}{\gamma_n^s+\alpha_n^s} \\ \frac{\beta_n^s}{\alpha_n^s+\beta_n^s} & \frac{\alpha_n^s}{\alpha_n^s+\beta_n^s} & 0 \end{pmatrix} = \begin{pmatrix} t_A & t_B & t_C \\ t_A & t_B & t_C \\ t_A & t_B & t_C \end{pmatrix}.$$

Note that each of the matrices on the left is non-singular, while the matrix on the right is row-stochastic and singular.

6. Numerical simulations

In this section we study the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ for different values of s, considering the initial triangle of complex coordinates $A_0(1+7i)$, $B_0(0)$ and $C_0(10)$. In particular, we will be computing the angles (in degrees), perimeter, area, and coordinates of the point $Q_n^{(s)}$ in each case. The numerical simulations have been implemented in Matlab.

1. For $s = 0$ the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ converges to the centroid G_0 of $A_0B_0C_0$. It is also convergnt in the shape of the original triangle, as one can infer from Table 1.

2. For $s = 1$ the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ converges to a point (Theorem 4.4), which is not currently known. It is also in the shape of an equilateral triangle, as proved by Jacobs and Ismailescu [14], and illustrated in the numerical simulations in Figure 6 and Table 2.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	3.6667 + 2.3333i
1	60.2551	81.8699	37.8750	14.2364	8.7500	3.6667 + 2.3333i
2	60.2551	81.8699	37.8750	7.1182	2.1875	3.6667 + 2.3333i
3	60.2551	81.8699	37.8750	3.5591	0.5469	3.6667 + 2.3333i
4	60.2551	81.8699	37.8750	1.7796	0.1367	3.6667 + 2.3333i
5	60.2551	81.8699	37.8750	0.8898	0.0342	3.6667 + 2.3333i
6	60.2551	81.8699	37.8750	0.4449	0.0085	3.6667 + 2.3333i
7	60.2551	81.8699	37.8750	0.2224	0.0021	3.6667 + 2.3333i
8	60.2551	81.8699	37.8750	0.1112	0.0005	3.6667 + 2.3333i
9	60.2551	81.8699	37.8750	0.0556	0.0001	3.6667 + 2.3333i

TABLE 1. Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 0$.

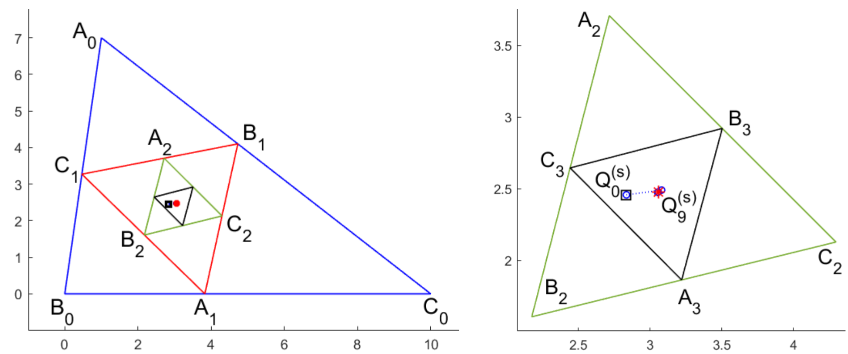

FIGURE 6. $\Delta A_n B_n C_n$, $n = 0, 1, 2, 3$, and $Q_n^{(s)}$, $n = 0, \ldots, 9$, for $s = 1$.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	2.8347 + 2.4585i
1	58.1565	66.5992	55.2442	13.2283	8.3620	3.0804 + 2.4916i
2	59.4254	61.7698	58.8047	6.5853	2.0855	3.0526 + 2.4760i
3	59.8490	60.4511	59.7000	3.2916	0.5213	3.0562 + 2.4784i
4	59.9618	60.1133	59.9249	1.6458	0.1303	3.0557 + 2.4781i
5	59.9904	60.0284	59.9812	0.8229	0.0326	3.0558 + 2.4781i
6	59.9976	60.0071	59.9953	0.4114	0.0081	3.0558 + 2.4781i
7	59.9994	60.0018	59.9988	0.2057	0.0020	3.0558 + 2.4781i
8	59.9999	60.0004	59.9997	0.1029	0.0005	3.0558 + 2.4781i
9	60.0000	60.0001	59.9999	0.0514	0.0001	3.0558 + 2.4781i

TABLE 2. Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 1$.

3. For $s = 2$ the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ convergences to a point, by Theorem 4.4. This point is not currently known. The numerical simulations in Figure 7 and Table 3 seem to

indicate that the sequence of triangles converges in the shape of an equilateral triangle, but at this moment do not know a proof of this property.

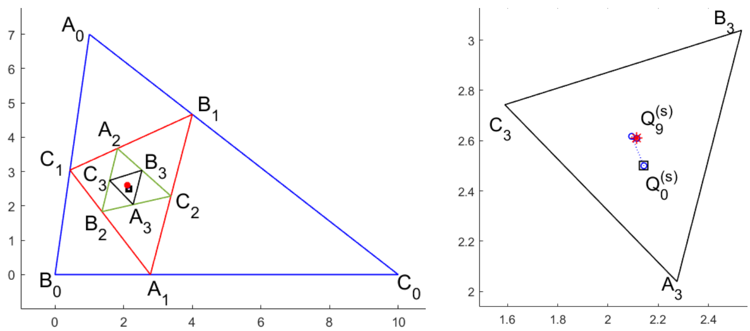

FIGURE 7. $\Delta A_nB_nC_n$, $n = 0, 1, 2, 3$, and $Q_n^{(s)}$, $n = 0, \ldots, 9$, for $s = 2$.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	2.1429 + 2.5000i
1	52.2670	50.8446	76.8885	12.5823	7.3269	2.0940 + 2.6171i
2	62.1740	63.4854	54.3406	6.0412	1.7468	2.1154 + 2.6088i
3	58.6416	58.1494	63.2090	3.0026	0.4331	2.1141 + 2.6095i
4	60.6049	60.8874	58.5077	1.4987	0.1080	2.1143 + 2.6094i
5	59.6801	59.5484	60.7715	0.7491	0.0270	2.1143 + 2.6094i
6	60.1555	60.2237	59.6209	0.3745	0.0067	2.1143 + 2.6094i
7	59.9212	59.8877	60.1912	0.1872	0.0017	2.1143 + 2.6094i
8	60.0391	60.0560	59.9048	0.0936	0.0004	2.1143 + 2.6094i
9	59.9804	59.9719	60.0477	0.0468	0.0001	2.1143 + 2.6094i

TABLE 3. Angles (in degrees), perimeter and area of triangles $\Delta A_nB_nC_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 2$.

4. For $s = 3$ the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ still seems to converge to a point, but does not seem to converge in the shape of an equilateral triangle, as suggested by the numerical simulations in Figure 7 and Table 4. In fact, the limit in shape seems to be degenerate.

7. CONCLUSIONS, OPEN PROBLEMS AND FUTURE WORK

In this paper we proved the convergence of nested iterations defined by the bisector triangle completing a result open in 2006 [14], and provided a framework in which more general problems can be investigated. Still, some open questions remain.

Problem 1. *By Theorem 4.4, for every $s \in [0, 2]$, the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point X_s^*. Further work is required to identify this point* (eventually in Kimberling's Encyclopedia of Triangle Centres [16]) *even in special cases. For example, while we know that $X_0^* = G$, we do not know even the coordinates of X_1^* (induced by the incenter I) or of X_2^* (induced by the symmedian point K).*

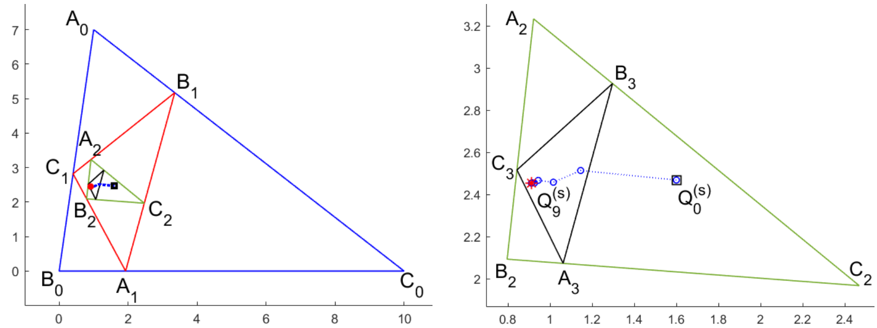

FIGURE 8. $\Delta A_nB_nC_n$, $n=0,1,2,3$, and $Q_n^{(s)}$, $n=0,\dots,9$, for $s=3$.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	1.5994 + 2.4685i
1	43.7770	36.0169	100.2061	12.3403	5.9474	1.1448 + 2.5137i
2	56.9816	88.0145	35.0038	4.8234	0.9616	1.0149 + 2.4581i
3	41.6992	32.5444	105.7564	1.9916	0.1456	0.9430 + 2.4667i
4	52.4537	94.4473	33.0990	0.7337	0.0213	0.9248 + 2.4542i
5	41.8269	30.7372	107.4359	0.2903	0.0030	0.9147 + 2.4557i
6	47.9812	100.6540	31.3648	0.1047	0.0004	0.9122 + 2.4535i
7	42.4043	29.7297	107.8660	0.0396	0.0001	0.9109 + 2.4538i
8	44.8259	105.1894	29.9847	0.0142	0.0000	0.9105 + 2.4535i
9	42.6407	29.2078	108.1515	0.0052	0.0000	0.9103 + 2.4535i

TABLE 4. Angles (in degrees), perimeter and area of triangles $\Delta A_nB_nC_n$ calculated for $n=0,\dots,9$, and sequence terms Q_n^s obtained for $s=3$.

Problem 2. *For $s=0$ the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges in the shape of the initial triangle, while for $s=1$ the convergence is in the shape of an equilateral triangle. Find the real parameters s for which the convergence in shape holds, and characterise the shape.*

Problem 3. *Find the values $s\notin[0,2]$, for which the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is convergent. Our numerical simulations seem to indicate that such values of s exist (e.g. $s=3$), but other methods need to be developed to investigate the dynamics in these cases.*

Problem 4. *Prove or disprove that any interior point of a triangle is the limit of a sequence of nested triangles which converge in shape.*

In our investigations we have also considered the framework involving set distances like Pompeiu-Hausdorff d_H. Let A, B, C be three distinct points in the complex plane, and consider the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ defined by

$$A_n=A,\quad B_n=\begin{cases}B, & \text{if } n \text{ is even}\\ C, & \text{if } n \text{ is odd}\end{cases},\quad C_n=\begin{cases}C, & \text{if } n \text{ is even}\\ B, & \text{if } n \text{ is odd}\end{cases},\quad n\geq 0.$$

If we consider the sets

$$\mathcal{F}=\{A,B,C\},\quad \mathcal{F}_n=\{A_n,B_n,C_n\},\quad n\geq 0,$$

it follows that $\lim_{n\to\infty} d_H(\mathcal{F}_n, \mathcal{F}) = 0$, but the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ is not convergent, in the sense of Definition 1.1. In our setup, the orientation of the triangles is important, especially when the convergence is not to a single point.

Further extensions of these results could be considered in more general settings, like arbitrary polygons, tetrahedrons or simplexes. For instance, as mentioned in Section 2, the result in Theorem 2.1 is proved for simplexes [8]. Also, in this direction we formulate another interesting open problem for quadrilaterals.

Problem 5. *Prove that every interior point of a convex quadrilateral is the limit of a sequence of nested quadrilaterals.*

However, triangles are still fascinating and worth investigating, being widely used in areas like engineering and architecture, as the most rigid of the polygons.

Acknowledgment. The authors wish to thank the referees for their useful feedback and comments which improved the clarity of the paper, and for suggestions of future work.

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